Problem Books in Mathematics

Pedro M. Gadea
Jaime Muñoz Masqué
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Analysis and Algebra on Differentiable Manifolds

A Workbook for Students and Teachers

Second Edition

Springer

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ISSN 0941-3502 Problem Books in Mathematics
ISBN 978-94-007-5951-0 ISBN 978-94-007-5952-7 (eBook)
DOI 10.1007/978-94-007-5952-7
Springer London Heidelberg New York Dordrecht

Library of Congress Control Number: 2012955855

Mathematics Subject Classification: 53-01, 53A04, 53A05, 53A07, 53A45, 53C05, 53C07, 53C22, 53C30, 53C35, 53C43, 53C50, 53C55, 53C80, 53D05, 53D22, 55R10, 57R20, 58-01, 58A05, 58A10, 58A12, 58A14, 58A15, 58A17, 58A20, 58A25, 58A30

Printed on acid-free paper

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To Mary

Foreword

Geometry has come a long way from the time of Euclid, Pythagoras and Archimedes. Many developments in the subject have accompanied revolutions in ways of thinking about and describing our world. The creation of the calculus by Newton and Leibniz was married to analytic descriptions of geometric forms, gravitational forces and motions of planets. The realisation of non-Euclidean geometries by Lobachevsky, Bolyai and Gauss lead to abstract descriptions of geometries of surfaces and Riemann's introduction of the concept of manifold. The consequent diminishing of the central role of coordinates accompanied Einstein's relativity theories. Similar ideas from Hamilton's reformulation of mechanics lead to Poincaré's fundamental ideas in dynamics.

Formalisation and extension of such concepts result in the fascinating interplay between tensorial geometry and symmetries. This provides foundational building blocks for theoretical models in physics, but has also become an essential part of the modern treatment of statistical models, engineering design and a significant arena for analysis.

This is a mathematical subject that is under continual development and spectacular advances based on these ideas are still being made. Of most popular recent note is Perelman's resolution of the Poincaré conjecture and the resulting proof of Thurston's geometrisation conjecture for three-manifolds.

For those wishing to make good use of these ideas and concepts, there are a number of excellent texts. However, an exposition of theory is often not enough and there is always limited space for demonstrating hands-on computations. In contrast to many others, this book is centred around providing a useful set of worked examples, carefully designed to help develop the reader's skills and intuition in a systematic way.

This new edition adds fresh examples and extends the reference material. It stays within the general scope of the first edition, but also provides welcome material on new topics, detailed in the Preface, most notably in the area of symplectic geometry and Hamiltonian dynamics. The student or teacher of a course in modern differential geometry will find this a valuable resource.

Aarhus University Andrew Swann

Foreword to the First Edition

A famous Swiss professor gave in Basel a student's course on "Riemann surfaces." After a couple of lectures a student asked him, "Professor, until now you did not give an exact definition of a Riemann surface." The professor answered, "Concerning Riemann surfaces, the main thing is to UNDERSTAND them, not to define them."

This episode really happened. The student's objection was necessary and reasonable. From a formal viewpoint, it is, of course, necessary to start as soon as possible with strict definitions. But the answer of the professor also has a substantial background. Namely, the pure definition of a Riemann surface—as a complex 1-dimensional complex analytic manifold—does absolutely nothing contribute to a real understanding. It takes really a long time to understand what a Riemann surface is.

This example is typical for the objects of global analysis—manifolds with structures. On the one hand, there are complex concrete definitions, but on the other hand, these do not automatically exhibit what they really mean, what we can do with them, which operations they really admit, how rigid this all is. Hence there arises the natural question: How to submit a deeper understanding, what is the best—or at least a good—way to do this?

One well-known way for this is to underpin the definitions, theorems and constructions by hierarchies of examples, counterexamples and exercises. Their choice, construction and logical order is for any teacher in global analysis an interesting, important and fun creating task.

This workbook is a really succeeded attempt to submit to the reader by very clever composed series of exercises and examples covering the whole area of manifolds, Lie groups, fibre bundles and Riemannian geometry a deep understanding and feeling.

The choice and order of the examples and exercises will be extraordinarily helpful and useful for any student or teacher of manifolds and differential geometry.

Greifswald University — Jürgen Eichhorn

Preface

As stated in the Preface to the first edition, this book intends to provide material for the practical side of standard courses on analysis and algebra on differentiable manifolds at a middle level, corresponding to advanced undergraduate and graduate years. The exercises focus on Lie groups, fibre bundles, and Riemannian geometry. Aims, approach and structure of the book remain largely the same as in the first edition. In the present edition, the number of figures is 68.

The prerequisites are linear and multilinear algebra, calculus of several variables, various concepts of point–set topology, and some familiarity with linear algebraic groups, the topology of fibre bundles, and manifold theory.

We would like to express our appreciation to the authors of some excellent books as those which appear in the references in chapters. These books have served us as a source of ideas, inspiration, statements and sometimes of results. We strongly recommend these books to the reader.

We introduce now a brief overview of the contents.

Chapters 1 to 6 contain 412 solved problems, sorted according to the aforementioned topics and in almost the same vein, notations, etc., as in the first edition, but 39 problems of the first edition have been deleted and 76 new problems have been added in the present edition. The first section of each chapter gives a selection of those definitions and theorems whose terminology, with ample use throughout the book, could be misleading due to the lack of universal acceptance. However, *we should like to insist on the fact that we do not claim that this is any kind or part of a book on the theory of differentiable manifolds*.

We now underline some of the changes in this edition.

Unlike the first edition of the book, in the present edition the Einstein summation convention is not used.

We consider in Sect. 1.3 (and only there) differentiable structures defined on sets, analysing what happens when one of the properties of being Hausdorff or second countable fails to hold. We thus try to elicit in the reader a better understanding of the meaning and importance of these two properties.

In Chap. 1 of the present edition, we have added, as an instructive example, a problem where we prove in detail that the manifold of affine straight lines of the plane, the 2-dimensional real projective space $\mathbb{R}P^2$ minus a point, and the infinite Möbius strip are diffeomorphic.

In Chap. 4, two new problems have been added in the section concerning the exponential map, where the simply connected Lie group corresponding to a given Lie algebra is obtained. The section devoted to the adjoint representation, contains six new problems concerning topics such as Weyl group, Cartan matrix, Dynkin diagrams, etc. Similarly, the section devoted to Lie groups of transformations has been increased in ten new application problems in symplectic geometry, Hamiltonian mechanics, and other related topics. Finally, we have added in the section concerning homogeneous spaces two problems on homogeneous spaces related to the exceptional Lie group G_2.

The section on characteristic classes in Chap. 5 includes two new problems on the Godbillon–Vey class in the present edition. Moreover, the last section, devoted to almost symplectic manifolds, Hamilton's equations, and the relation with principal $U(1)$-bundles, contains five new problems, including topics as Hamiltonian vector fields.

In the present edition, the section of Chap. 6 concerning Riemannian connections has been enlarged, including six new problems on almost complex structures. The section on Riemannian geodesics also includes four new problems on special metrics. Moreover, a completely new section is devoted to a generalisation of Gauss' Lemma. The section on homogeneous Riemannian and Riemannian symmetric spaces contains two new problems about general properties of homogeneous Riemannian manifolds and two new problems on specific three-dimensional Riemannian spaces. Furthermore, a short novel section deals with some properties of the energy of Hopf vector fields. The section on left-invariant metrics on Lie groups contains in particular two new problems: One gives in a detailed way the structure of the Kodaira–Thurston manifold; and the other furnishes the de Rham cohomology of a specific nilmanifold.

Chapter 7 offers an expanded 56-page long collection of formulae and tables concerning frequent spaces and groups in differential geometry. *Many of them do not actually appear in the problems, but having them collected together may prove useful as an* aide-mémoire*, even to teachers and researchers.*

At the end of the references to each chapter, several books (or papers) appear that have not been explicitly cited, but such that they have inspired several ideas of the chapter and/or are very useful references.

All in all, we hope that this new edition of the book will again render a good service to practitioners of differential geometry and related topics.

We acknowledge the anonymous referees for their thorough, enlightening and suggestive reports; their invaluable suggestions and corrections have contributed to improve several aspects of contents as well as presentation of the book.

Our hearty thanks to Professor José A. Oubiña for his generous help and wise advices. We are also indebted to Mrs. Dava Sobel and Professors William M. Boothby, José C. González-Dávila, Sigurdur Helgason, A. Montesinos Amilibia, Kent E. Morrison, John O'Connor, Peter Petersen, Edmund F. Robertson, Waldyr

A. Rodrigues, Jr., Chris M. Wood, and John C. Wood who kindly granted us permission to reproduce here some of their nice and interesting texts, results and constructions.

We are indebted to Donatas Akmanavičius and his team for their careful editing.

We are also indebted to José Ignacio Sánchez García for his excellent work on the graphics.

Our special thanks to Andrew Swann, who kindly accepted our invitation to write the Foreword.

Madrid, Spain — Pedro M. Gadea, Jaime Muñoz Masqué

L'viv, Ukraine — Ihor V. Mykytyuk

From the Preface to the First Edition

This book is intended to cover the exercises of standard courses on analysis and algebra on differentiable manifolds at a middle level, corresponding to advanced undergraduate and graduate years, with specific focus on Lie groups, fibre bundles and Riemannian geometry. It will prove of interest for students in mathematics and theoretical physics, and in some branches of engineering.

It is not intended as a handbook on those topics, in the form of problems, but merely as a practical complement to the courses, often found on excellent books, like those cited in the bibliography.

The prerequisites are linear and multilinear algebra, calculus on several variables and various concepts of point–set topology.

The first six chapters contain 375 solved problems sorted according to the aforementioned topics. These problems fall, "grosso modo," into four classes:

(i) Those consisting of mere calculations, mostly elementary, aiming at checking a number of notions on the subjects.
(ii) Problems dedicated to checking some specific properties introduced in the development of the theory.
(iii) A class of somewhat more difficult problems devoted to focusing the attention on some particular topics.
(iv) A few problems introducing the reader to certain questions not usually explained. The level of these problems is quite different, ranging from those handling simple properties to others that need sophisticated tools.

Throughout the book, differentiable manifolds, functions, and tensors fields are assumed to be of class C^∞, mainly to simplify the exposition. We call them, indiscriminately, either differentiable or C^∞.

Similarly, manifolds are supposed to be Hausdorff and second countable, though a section is included analysing what happens when these properties fail, aimed at a better understanding of the meaning of such properties.

The Einstein summation convention is used.

Chapter 7 provides a selection of the theorems and definitions used throughout the book, but restricted to those whose terminology could be misleading for the lack of universal acceptance. Moreover, to solve some types of problems, certain definitions and notations should be precisely fixed; recalling the exact statement of some theorems is often convenient in practice as well. However, this chapter has by no means the intention of being either a development or a digest of the theory.

Chapter 8 offers a 42-page long collection of formulae and tables concerning spaces and groups frequent in differential geometry. Many of them are used throughout the book; others are not, but *they have been included since such a collection should be useful as an aide-mémoire, even for teachers and researchers*. As in Chap. 7, no effort to be exhaustive has been attempted.

We hope the book will render a good service to teachers and students of differential geometry and related topics. While no reasonable effort has been spared to ensure accuracy and precision, the attempt of writing such a book necessarily will contain misprints, and probably some errors. (...) In the corrected reprint published by Springer in 2009, we corrected a couple of dozens of typos, slightly modified the statement of Problem 1.1.13, and changed the proof of Problem 5.3.6(2).

Madrid, Spain Pedro M. Gadea
Jaime Muñoz Masqué
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Acknowledgements for the First Edition

Our hearty thanks to the referees and to Professors Marco Castrillón López, Raúl Durán Díaz, Alberto Elduque, Fernando Etayo, Olga Gil-Medrano, Carolyn Gordon, José A. Oubiña, Encarna Reyes and Gustavo Santos García for their valuable help, corrections, and suggestions.

We are indebted to Arturo Quijada Arteaga for his excellent work in the graphics.

Our special thanks to Professor Jürgen Eichhorn, who kindly accepted our invitation to write the Foreword.

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Chapter 1
Differentiable Manifolds

Abstract After recalling some definitions and results on the basics of smooth manifolds, this chapter is devoted to solve problems including (but not limited to) the following topics: Smooth mappings, critical points and critical values, immersions, submersions and quotient manifolds, construction of manifolds by inverse image, tangent bundles and vector fields, with integral curves and flows. Functions and other objects are assumed to be of class C^∞ (also referred to either as 'differentiable' or 'smooth'), essentially for the sake of simplicity. Similarly, manifolds are assumed to be Hausdorff and second countable, though we have included a section that analyses what happens when one of these properties fails to hold. We thus try to elicit in the reader a better understanding of the meaning and importance of such properties. On purpose, we have sprinkled this first chapter with many examples and figures. As an instructive example, we prove in detail that the manifold of affine straight lines of the plane, the 2-dimensional real projective space $\mathbb{R}\mathrm{P}^2$ minus a point, and the infinite Möbius strip are diffeomorphic. As important and non-trivial examples of differentiable manifolds, the real projective space $\mathbb{R}\mathrm{P}^n$ and the real Grassmannian $G_k(\mathbb{R}^n)$ are studied in detail.

> Lines of latitude and longitude began crisscrossing our worldview in ancient times, at least three centuries before the birth of Christ. By A. D. 150, the cartographer Ptolemy had plotted them on the twenty-seven maps of his first world atlas.
>
> DAVA SOBEL, *Longitude*, Walker & Company, New York, 2007, pp. 2–3. (With kind permission from the author and from Walker & Company publishers.)
>
> A differentiable manifold is generally defined in one of two ways; as a point set with neighborhoods homeomorphic with Euclidean space E_n, coördinates in overlapping neighborhoods being related by a differentiable transformation (...) or as a subset of E_n, defined near each point by expressing some of the coördinates in terms of the others by differentiable functions (...). The first fundamental theorem is that the first definition is no more general than the second (...)

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_1,

HASSLER WHITNEY, "*Differentiable Manifolds*," Ann. of Math. *37* (1936), no. 3, p. 645. (With kind permission from the Annals of Mathematics.)

1.1 Some Definitions and Theorems on Differentiable Manifolds

Definitions 1.1 Let M be a topological space. A *covering* of M is a collection of open subsets of M whose union is M. A covering $\{U_\alpha\}_{\alpha\in A}$ of M is said to be *locally finite* if each $p \in M$ has a neighbourhood (an open subset of M containing p) which intersects only finitely many of the sets U_α.

A Hausdorff space M is called *paracompact* if for each covering $\{U_\alpha\}_{\alpha\in A}$ of M there exists a locally finite covering $\{V_\beta\}_{\beta\in B}$ which is *a refinement* of $\{U_\alpha\}_{\alpha\in A}$ (that is, each V_β is contained in some U_α). It is known that a locally compact Hausdorff space which has a countable base is paracompact.

A *locally Euclidean space* is a topological space M such that each point has a neighbourhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$. In particular, such a space is locally compact and paracompact. If φ is a homeomorphism of a connected open subset $U \subset M$ onto an open subset of $\mathbb{R}^n$, then U is called a *coordinate neighbourhood*; φ is called a *coordinate map*; the functions $x^i = t^i \circ \varphi$, where t^i denotes the ith canonical coordinate function on $\mathbb{R}^n$, are called the *coordinate functions*; and the pair (U, φ) (or the set $(U, x^1, \dots, x^n)$) is called a *coordinate system* or a (local) *chart*. An *atlas* $\mathscr{A}$ of class C^∞ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying the following two properties:

(i) $\bigcup_{\alpha\in A} U_\alpha = M$;
(ii) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^∞ on $\varphi_\beta(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in A$.

A *differentiable structure* (or *maximal atlas*) $\mathscr{F}$ on a locally Euclidean space M is an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of class C^∞, satisfying the above two properties (i) and (ii) and moreover the condition:

(iii) The collection $\mathscr{F}$ is maximal with respect to (ii), that is, if (U, φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^∞ on $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi(U \cap U_\alpha)$, respectively, then $(U, \varphi) \in \mathscr{F}$.

A *topological manifold* of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n. A *differentiable manifold of class C^∞ of dimension n* (or simply *differentiable manifold of dimension n*, or C^∞ *manifold*, or *smooth n-manifold*) is a pair $(M, \mathscr{F})$ consisting of a topological manifold M of dimension n, together with a differentiable structure $\mathscr{F}$ of class C^∞ on M.

The differentiable manifold $(M, \mathscr{F})$ is usually denoted by M, with the understanding that when one speaks of "the differentiable manifold" M one is considering the locally Euclidean space M with some given differentiable structure $\mathscr{F}$.

Let M and N be differentiable manifolds, of respective dimensions m and n. A map $\Phi\colon M \to N$ is said to be C^∞ provided that for every coordinate system (U, φ) on M and (V, ψ) on N, the composite map $\psi \circ \Phi \circ \varphi^{-1}$ is C^∞.

A *diffeomorphism* $\Phi\colon M \to N$ is a bijective C^∞ map such that the inverse map Φ^{-1} is also C^∞.

The *tangent space* T_pM to M at $p \in M$ is the space of real derivations of the local algebra $C_p^\infty M$ of germs of C^∞ functions at p, i.e. the $\mathbb{R}$-linear functions $X\colon C_p^\infty M \to \mathbb{R}$ such that

$$X(fg) = (Xf)g(p) + f(p)Xg, \quad f, g \in C_p^\infty M.$$

Let $C^\infty M$ denote the algebra of differentiable functions of class C^∞ on M. The *differential map at* p of the C^∞ map $\Phi\colon M \to N$ is the map

$$\Phi_{*p}\colon T_pM \to T_{\Phi(p)}N, \qquad (\Phi_{*p}X)(f) = X(f \circ \Phi), \quad f \in C^\infty N.$$

Theorem 1.2 (Partition of Unity) *Let M be a manifold and let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite covering of M. Assume that each closure $\overline{U}_\alpha$ is compact. Then there exists a system $\{\psi_\alpha\}_{\alpha\in A}$ of differentiable functions on M such that*

(a) *Each ψ_α has compact support contained in U_α;*
(b) $\psi_\alpha \geqslant 0$ *and* $\sum_{\alpha\in A}\psi_\alpha = 1$.

Definition 1.3 A *parametrisation of a surface S in* $\mathbb{R}^3$ is a homeomorphism

$$\mathbf{x}\colon U \subset \mathbb{R}^2 \longrightarrow V \cap S,$$

where U is an open subset of $\mathbb{R}^2$ and V stands for an open subset of $\mathbb{R}^3$, such that $\mathbf{x}_{*p}\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $p \in U$.

Remark 1.4 In the present book, we use parametrisations of surfaces from open subsets of $\mathbb{R}^2$ (or other $\mathbb{R}^n$ to parametrise n-dimensional smooth manifolds) in order to make them consistent with the fact that the coordinate neighbourhoods defined in Definitions 1.1 above are open subsets of the relevant surface (or smooth n-manifold).

Moreover, for the sake of simplicity, we usually give only a parametrisation, although it is necessary almost always to give more parametrisations to cover the surface (or other spaces). This should be understood in each case.

Definitions 1.5 The *stereographic projection* from the north pole $N = (0, \dots, 0, 1)$ (resp., south pole $S = (0, \dots, 0, -1)$) of the sphere

$$S^n = \left\{ \left(x^1, \dots, x^{n+1}\right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \left(x^i\right)^2 = 1 \right\}$$

onto the equatorial plane $x^{n+1} = 0$ is the map sending $p \in S^n \setminus \{N\}$ (resp., $p \in S^n \setminus \{S\}$) to the point where the straight line through N (resp., S) and p intersects the plane $x^{n+1} = 0$.

The *inverse of the stereographic projection* is the map from $x^{n+1} = 0$ to $S^n \setminus \{N\}$ (resp., $p \in S^n \setminus \{S\}$) sending the point q in the plane $x^{n+1} = 0$ to the point where the straight line through q and N (resp., S) intersects S^n.

Other stereographic projections can be defined. For instance, that defined as above but for the sphere

$$S^n = \left\{ \left(x^1, \dots, x^{n+1}\right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} \left(x^i\right)^2 + \left(x^{n+1} - 1\right)^2 = 1 \right\},$$

from the north pole $N = (0, \dots, 0, 2)$ onto the plane $x^{n+1} = 0$. The inverse map is defined analogously to the previous case.

Definitions 1.6 Let $\Phi \colon M \to N$ be a C^∞ map. A point $p \in M$ is said to be a *critical point of* Φ if $\Phi_* \colon T_p M \to T_{\Phi(p)} N$ is not surjective. A point $q \in N$ is said to be a *critical value of* Φ if the set $\Phi^{-1}(q)$ contains a critical point of Φ.

Let $f \in C^\infty M$. A point $p \in M$ is called a *critical point of* f if $f_{*p} = 0$. If we choose a coordinate system $(U, x^1, \dots, x^n)$ around $p \in M$, this means that

$$\frac{\partial f}{\partial x^1}(p) = \dots = \frac{\partial f}{\partial x^n}(p) = 0.$$

The real number $f(p)$ is then called a *critical value of* f. A critical point is called *non-degenerate* if the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right)$$

is non-singular. Non-degeneracy does not depend on the choice of coordinate system.

If p is a critical point of f, then the *Hessian* H^f *of* f *at* p is a bilinear function on $T_p M$ defined as follows. If $u, w \in T_p M$ and a vector field $X \in \mathfrak{X}(M)$ (see Definitions 1.17) satisfies $X_p = u$, then

$$H^f(u, w) = w(Xf).$$

The index of f at a critical point p is the index of its Hessian H^f_p.

Definitions 1.7 A subset S of $\mathbb{R}^n$ is said to have *measure zero* if for every $\varepsilon > 0$, there is a covering of S by a countable number of open cubes $C_1, C_2, \dots$, such that the Euclidean volume $\sum_{i=1}^\infty v(C_i) < \varepsilon$.

A subset S of a differentiable n-manifold M has *measure zero* if there exists a countable family $(U_1, \varphi_1), (U_2, \varphi_2), \dots$ of charts in the differentiable structure of M such that $\varphi_i(U_i \cap S)$ has measure zero in $\mathbb{R}^n$ for every $i = 1, 2, \dots$.

Theorem 1.8 (Sard's Theorem) *Let* $\Phi \colon M \to N$ *be a* C^∞ *map. Then the set of critical values of* Φ *has measure zero.*

Definitions 1.9 Let $\Phi\colon M \to N$ be a C^∞ map. Then:

(i) Φ is an *immersion* if Φ_{*p} is injective for each $p \in M$.
(ii) The pair (M, Φ) is a *submanifold* of N if Φ is a one-to-one immersion. If M is a subset of N and the inclusion map of M in N is a one-to-one immersion, then it is said that M is a submanifold of N.
(iii) Φ is an *embedding* if Φ is a one-to-one immersion which is also a homeomorphism into, that is, the induced map $\Phi\colon M \to \Phi(M)$ is open when $\Phi(M)$ is endowed with the topology inherited from that of N.
(iv) Φ is an *submersion* if Φ_{*p} is surjective for all $p \in M$.

Definition 1.10 Let $\Phi\colon M \to N$ be a C^∞ map, with $\dim M = m$, $\dim N = n$, and let $p \in M$. If (U, φ), (V, ψ) are coordinate systems around p and $\Phi(p)$, respectively, and $\Phi(U) \subset V$, then one has a corresponding expression for Φ in local coordinates, i.e.

$$\widetilde{\Phi} = \psi \circ \Phi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V).$$

The *rank of* Φ *at* p is defined to be the rank of Φ_{*p}, which is equal to the rank of the Jacobian matrix

$$\left(\frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big)\right), \quad i = 1, \dots, n,\ j = 1, \dots, m,$$

of the map

$$\widetilde{\Phi}\big(x^1, \dots, x^m\big) = \big(f^1\big(x^1, \dots, x^m\big), \dots, f^n\big(x^1, \dots, x^m\big)\big),$$

expressing Φ in local coordinates.

Theorem 1.11 (Theorem of the Rank) *Let $\Phi\colon M \to N$ be a C^∞ map, with* $\dim M = m$, $\dim N = n$, *and* $\operatorname{rank} \Phi = r$ *at every point of M. If $p \in M$, then there exist coordinate systems (U, φ), (V, ψ) as above such that $\varphi(p) = (0, \dots, 0) \in \mathbb{R}^m$, $\psi(\Phi(p)) = (0, \dots, 0) \in \mathbb{R}^n$, and $\widetilde{\Phi} = \psi \circ \Phi \circ \varphi^{-1}$ is given by*

$$\widetilde{\Phi}\big(x^1, \dots, x^m\big) = \big(x^1, \dots, x^r, 0, \dots, 0\big).$$

Moreover, we can assume $\varphi(U) = C^m_\varepsilon(0)$, $\varphi(V) = C^n_\varepsilon(0)$, with the same ε, where $C^n_\varepsilon(0)$ denotes the cubic neighbourhood centred at $0 \in \mathbb{R}^n$ of edge 2ε.

Theorem 1.12 (Inverse Map Theorem) *Let*

$$f = \big(f^1, \dots, f^n\big)\colon U \to \mathbb{R}^n$$

be a C^∞ map defined on an open subset $U \subseteq \mathbb{R}^n$. Given a point $x_0 \in U$, assume

$$\frac{\partial(f^1, \dots, f^n)}{\partial(x^1, \dots, x^n)}(x_0) \neq 0.$$

Then there exists an open neighbourhood $V \subseteq U$ of x_0 such that:

(i) *$f(V)$ is an open subset of $\mathbb{R}^n$*;
(ii) *$f: V \to f(V)$ is one-to-one*;
(iii) *$f^{-1}: f(V) \to V$ is C^∞*.

Theorem 1.13 (Implicit Map Theorem) *Denote the coordinates on $\mathbb{R}^n \times \mathbb{R}^m$ by $(x^1, \dots, x^n, y^1, \dots, y^m)$. Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open subset, and let*

$$f = \left(f^1, \dots, f^m\right): U \to \mathbb{R}^m$$

be a C^∞ map. Given a point $(x_0, y_0) \in U$, assume:

(i) $f(x_0, y_0) = 0$;
(ii)

$$\frac{\partial(f^1, \dots, f^m)}{\partial(y^1, \dots, y^m)}(x_0, y_0) \neq 0.$$

Then, there exist an open neighbourhood V of x_0 in $\mathbb{R}^n$ and an open neighbourhood W of y_0 in $\mathbb{R}^m$ such that $V \times W \subset U$, and there exists a unique C^∞ map $g: V \to \mathbb{R}^m$, such that for each $(x, y) \in V \times W$:

$$f(x, y) = 0 \quad \Leftrightarrow \quad y = g(x).$$

Theorem 1.14 (Implicit Map Theorem for Submersions) *Consider a submersion $\pi: M \to N$. Then, for every $q \in \operatorname{im}\pi$, the fibre $\pi^{-1}(q)$ is a closed submanifold of M and* $\dim \pi^{-1}(q) = \dim M - \dim N$.

Definition 1.15 Let $\sim$ be an equivalence relation in M, and let $\pi: M \to M/\sim$ be the quotient map. Endow $M/\sim$ with the quotient topology τ, i.e.

$$U \in \tau \quad \Leftrightarrow \quad \pi^{-1}(U) \text{ is open in the topology of } M.$$

The *quotient manifold* of M modulo $\sim$ is said to exist if there is a (necessarily unique) differentiable manifold structure on $M/\sim$ such that π is a submersion.

The following criterion is often used to construct quotient manifolds:

Theorem 1.16 (Theorem of the Closed Graph) *Let $\sim$ be an equivalence relation in M and let $N \subset M \times M$ be the graph of $\sim$, that is,*

$$N = \left\{(p, q) \in M \times M : p \sim q\right\}.$$

The quotient manifold $M/\sim$ exists if and only if the following two conditions hold true:

(i) *N is a closed embedded submanifold of $M \times M$.*
(ii) *The restriction $\pi: N \to M$ to N of the canonical projection* $\mathrm{pr}_1: M \times M \to M$ *onto the first factor is a submersion.*

Definitions 1.17 Let M be a differentiable n-manifold with differentiable structure $\mathscr{F}$. Let

$$TM = \bigcup_{p \in M} T_p M.$$

There is a natural projection $\pi : TM \to M$, given by $\pi(v) = p$ for any $v \in T_p M$. Let $(U, \varphi) = (U, x^1, \ldots, x^n) \in \mathscr{F}$. Define $\widetilde{\varphi} \colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(v) = \big(\big(x^1 \circ \pi\big)(v), \ldots, \big(x^n \circ \pi\big)(v), \mathrm{d}x^1(v), \ldots, \mathrm{d}x^n(v)\big),$$

for all $v \in \pi^{-1}(U)$. Then the collection of such $(\pi^{-1}(U), \widetilde{\varphi})$ determines on TM a differentiable structure $\widetilde{\mathscr{F}}$ with which TM is called the *tangent bundle* over M.

A *vector field along a curve* $\gamma \colon [a, b] \to M$ in the differentiable manifold M is a C^∞ map $X \colon [a, b] \to TM$ satisfying $\pi \circ X = \gamma$. A *vector field X on M* is a C^∞ section $X \colon M \to TM$. If $f \in C^\infty U$, then Xf is the function on U whose value at $p \in M$ is $X_p f$. The vector fields on M are usually identified to the derivations of C^∞ functions, that is, to the $\mathbb{R}$-linear maps $X \colon C^\infty M \to C^\infty M$ such that $X(fg) = (Xf)g + f(Xg)$. The $(C^\infty M)$-module of vector fields on M is denoted by $\mathfrak{X}(M)$.

If X and Y are vector fields on M, the *Lie bracket* $[X, Y]$ of X and Y is the vector field on M defined by

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf), \quad p \in M.$$

One has the following geometric interpretation of the bracket of two vector fields [4, vol. I, Proposition 1.9].

Proposition 1.18 *Let X and Y vector fields on the differentiable manifold M. If X generates a local one-parameter group of local transformations φ_t, then*

$$[X, Y]_p = \lim_{t \to 0} \frac{1}{t}\big(Y_p - (\varphi_{t*} Y)_p\big), \quad p \in M.$$

Let $X \in \mathfrak{X}(M)$. A C^∞ curve γ in M is said to be an *integral curve of* X if

$$\gamma'(t_0) = \gamma_* \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t_0} \right) = X_{\gamma(t_0)}.$$

Definitions 1.19 A vector field is said to be *complete* if each of its maximal integral curves is defined on the entire real line $\mathbb{R}$.

The *flow* or 1*-parameter group* of a complete vector field X on M is the map

$$\begin{aligned} \varphi \colon M \times \mathbb{R} &\to M \\ (p, t) &\mapsto \varphi_t(p), \end{aligned}$$

where $t \mapsto \varphi_t(p)$ is the maximal integral curve of X with initial point p.

Definitions 1.20 Let $\Phi\colon M\to N$ be a C^∞ map. The vector fields $X\in\mathfrak{X}(M)$, $Y\in\mathfrak{X}(N)$, are said to be *Φ-related* if

$$\Phi_* X_p = Y_{\Phi(p)},\qquad p\in M,$$

i.e.

$$\Phi_*\circ X = Y\circ\Phi.$$

Let $\Phi\colon M\to N$ be a diffeomorphism. Given $X\in\mathfrak{X}(M)$, the *vector field image* $\Phi\cdot X$ of X is defined by

$$(\Phi\cdot X)_p = \Phi_*(X_{\Phi^{-1}(p)}).$$

That is, $\Phi\cdot X$ is a shortening for the section $\Phi_*\circ X\circ\Phi^{-1}$ of TN.

Proposition 1.21 *Let vector fields $X_j\in\mathfrak{X}(M)$ and $Y_j\in\mathfrak{X}(N)$, $j=1,2$, be Φ-related with respect the map $\Phi\colon M\to N$. Then their brackets $[X_1,X_2]\in\mathfrak{X}(M)$ and $[Y_1,Y_2]\in\mathfrak{X}(N)$ are also Φ-related vector fields, i.e.*

$$\Phi_*\circ[X_1,X_2] = [Y_1,Y_2]\circ\Phi.$$

1.2 C^∞ Manifolds

Problem 1.22 Prove that the function $\varphi\colon\mathbb{R}\to\mathbb{R}$, $\varphi(s)=s^3$, defines a C^∞ differentiable structure on $\mathbb{R}$ different from the usual one (that of the atlas $\{(\mathbb{R},\mathrm{id}_\mathbb{R})\}$).

Solution Since $\varphi^{-1}(s)=\sqrt[3]{s}$, φ is a homeomorphism, so that $\{(\mathbb{R},\varphi)\}$ is trivially an atlas for $\mathbb{R}$, with only one chart.

To see that the differentiable structure defined by $\{(\mathbb{R},\varphi)\}$ is not the usual one, we must see that the atlases $\{(\mathbb{R},\varphi)\}$ and $\{(\mathbb{R},\mathrm{id}_\mathbb{R})\}$ are not equivalent, i.e. that $\{(\mathbb{R},\varphi),(\mathbb{R},\mathrm{id}_\mathbb{R})\}$ is not a C^∞ atlas on $\mathbb{R}$. In fact, although $\varphi\circ\mathrm{id}_\mathbb{R}^{-1}=\varphi$ is C^∞, the map $\mathrm{id}_\mathbb{R}\circ\varphi^{-1}=\varphi^{-1}$ is not differentiable at 0.

Let $\mathbb{R}_\varphi$ (resp., $\mathbb{R}_{\mathrm{id}}$) be the topological manifold $\mathbb{R}$ with the differentiable structure defined by the atlas $\{(\mathbb{R},\varphi)\}$ (resp., $\{(\mathbb{R},\mathrm{id}_\mathbb{R})\}$). Then, the map $\varphi\colon\mathbb{R}_\varphi\to\mathbb{R}_{\mathrm{id}}$ is a diffeomorphism. In fact, its representative map $\mathrm{id}\circ\varphi\circ\varphi^{-1}\colon\mathbb{R}\to\mathbb{R}$ is the identity map.

Problem 1.23 Prove that if $h\colon\mathbb{R}^n\to\mathbb{R}^n$ is a homeomorphism, then the atlas $\{(\mathbb{R}^n,h)\}$ defines the usual differentiable structure on $\mathbb{R}^n$ (that defined by the atlas $\{(\mathbb{R}^n,\mathrm{id}_{\mathbb{R}^n})\}$) if and only if h and h^{-1} are differentiable.

Solution If $h\colon\mathbb{R}^n\to\mathbb{R}^n$ is a homeomorphism such that the atlas $\{(\mathbb{R}^n,h)\}$ defines the usual differentiable structure on $\mathbb{R}^n$, then $h=h\circ\mathrm{id}_{\mathbb{R}^n}^{-1}$ and $h^{-1}=\mathrm{id}_{\mathbb{R}^n}\circ h^{-1}$ are differentiable. And conversely.

Problem 1.24 For each real number $r>0$, consider the map $\varphi_r\colon \mathbb{R}\to\mathbb{R}$, where $\varphi_r(t)=t$ if $t\leqslant 0$ and $\varphi_r(t)=rt$ if $t\geqslant 0$. Prove that the atlases $\{(\mathbb{R},\varphi_r)\}_{r>0}$ define an uncountable family of differentiable structures on $\mathbb{R}$. Are the corresponding differentiable manifolds diffeomorphic?

Solution For each $r>0$, φ_r is a homeomorphism, but φ_r and φ_r^{-1} are differentiable only when $r=1$ ($\varphi_1=\mathrm{id}_{\mathbb{R}}$). Thus $\{(\mathbb{R},\varphi_r)\}$, for fixed $r\neq 1$, is an atlas defining a differentiable structure different from the usual one. Moreover we have

$$\left(\varphi_r\circ\varphi_s^{-1}\right)(t)=\begin{cases} t & \text{if } t\leqslant 0,\\ (r/s)t & \text{if } t\geqslant 0.\end{cases}$$

So, if $r\neq s$, then $\varphi_r\circ\varphi_s^{-1}$ is not differentiable. Consequently, the atlases $\{(\mathbb{R},\varphi_r)\}$ and $\{(\mathbb{R},\varphi_s)\}$ define different differentiable structures and thus $\{(\mathbb{R},\varphi_r)\}_{r>0}$ defines a family of different differentiable structures on $\mathbb{R}$.

All of them are diffeomorphic, though. In fact, given two differentiable manifolds $\mathbb{R}_{\varphi_{r_1}}$ and $\mathbb{R}_{\varphi_{r_2}}$ defined from the differentiable structures obtained from the atlases $\{(\mathbb{R},\varphi_{r_1})\}$ and $\{(\mathbb{R},\varphi_{r_2})\}$, respectively, a diffeomorphism $\varphi\colon \mathbb{R}_{\varphi_{r_1}}\to\mathbb{R}_{\varphi_{r_2}}$ is given by the identity map for $t\leqslant 0$ and by $t\mapsto (r_1/r_2)t$ for $t\geqslant 0$. Indeed, the representative map $\varphi_{r_2}\circ\varphi\circ\varphi_{r_1}^{-1}$ is the identity map.

Problem 1.25 Consider the open subsets U and V of the unit circle S^1 of $\mathbb{R}^2$ given by

$$U=\left\{(\cos\alpha,\sin\alpha):\alpha\in(0,2\pi)\right\},\qquad V=\left\{(\cos\alpha,\sin\alpha):\alpha\in(-\pi,\pi)\right\}.$$

Prove that $\mathscr{A}=\{(U,\varphi),(V,\psi)\}$, where

$$\begin{aligned} &\varphi\colon U\to\mathbb{R}, &&\varphi(\cos\alpha,\sin\alpha)=\alpha, &&\alpha\in(0,2\pi),\\ &\psi\colon V\to\mathbb{R}, &&\psi(\cos\alpha,\sin\alpha)=\alpha, &&\alpha\in(-\pi,\pi),\end{aligned}$$

is an atlas on S^1.

Solution One has $U\cup V=S^1$ (see Fig. 1.1). The maps φ and ψ are homeomorphisms onto the open subsets $(0,2\pi)$ and $(-\pi,\pi)$ of $\mathbb{R}$, respectively, hence (U,φ) and (V,ψ) are local charts on S^1.

The change of coordinates $\psi\circ\varphi^{-1}$, given by

$$\begin{aligned} \varphi(U\cap V) &\overset{\varphi^{-1}}{\to} \quad U\cap V \quad \overset{\psi}{\to}\ \psi(U\cap V)\\ \alpha &\mapsto\ (\cos\alpha,\sin\alpha)\mapsto \begin{cases}\alpha & \text{if } \alpha\in(0,\pi),\\ \alpha-2\pi & \text{if } \alpha\in(\pi,2\pi),\end{cases}\end{aligned}$$

is obviously a diffeomorphism. Thus $\mathscr{A}$ is an atlas on S^1.

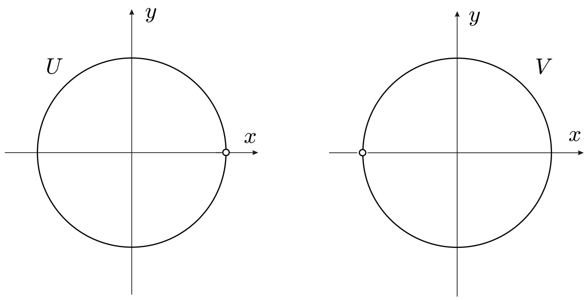


Fig. 1.1 An atlas on S^1 with two charts

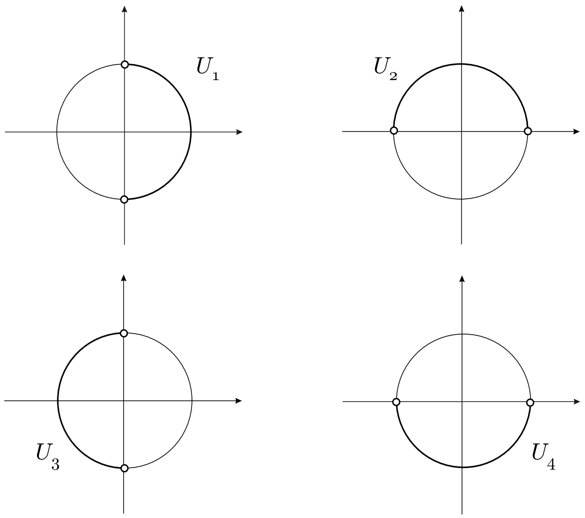


Fig. 1.2 An atlas on S^1 with four charts

Problem 1.26 Prove:

(i) $\mathscr{A} = \{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$, where

$$\begin{aligned}
U_1 &= \left\{(x, y) \in S^1 : x > 0\right\}, & \varphi_1 &: U_1 \to \mathbb{R}, & \varphi_1(x, y) &= y,\\
U_2 &= \left\{(x, y) \in S^1 : y > 0\right\}, & \varphi_2 &: U_2 \to \mathbb{R}, & \varphi_2(x, y) &= x,\\
U_3 &= \left\{(x, y) \in S^1 : x < 0\right\}, & \varphi_3 &: U_3 \to \mathbb{R}, & \varphi_3(x, y) &= y,\\
U_4 &= \left\{(x, y) \in S^1 : y < 0\right\}, & \varphi_4 &: U_4 \to \mathbb{R}, & \varphi_4(x, y) &= x,
\end{aligned}$$

is an atlas on the unit circle S^1 in $\mathbb{R}^2$.

(ii) $\mathscr{A}$ is equivalent to the atlas given in Problem 1.25.

Solution

(i) We have $S^1 = \bigcup_i U_i$, $i = 1, 2, 3, 4$ (see Fig. 1.2), and each φ_i is a homeomorphism onto the open subset $(-1, 1)$ of $\mathbb{R}$, thus each (U_i, φ_i) is a chart on S^1.

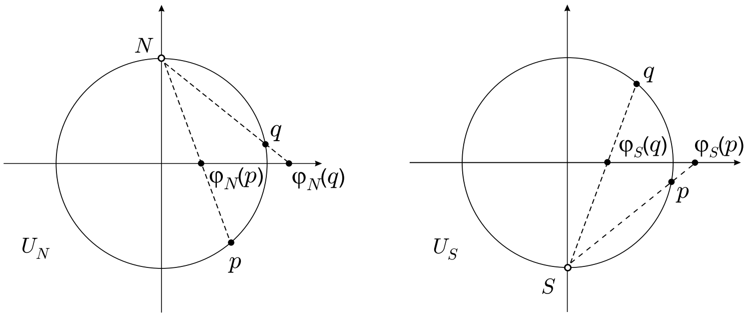


Fig. 1.3 Stereographic projections of S^1

The change of coordinates $\varphi_1 \circ \varphi_2^{-1}$, given by

$$\begin{aligned} \varphi_2(U_1 \cap U_2) = (0,1) \to \quad U_1 \cap U_2 \quad &\to \varphi_1(U_1 \cap U_2) = (0,1) \\ t \mapsto \left(t, \sqrt{1-t^2}\right) &\mapsto \sqrt{1-t^2}, \end{aligned}$$

is a C^∞ map, since $1 - t^2 > 0$. Actually it is a diffeomorphism. The other changes of coordinates are also C^∞, as it is easily proved, thus $\mathscr{A}$ is a C^∞ atlas on S^1.

(ii) To prove that the two atlases are equivalent, one must consider the changes of coordinates whose charts belong to different atlases. For example, for $\varphi \circ \varphi_1^{-1}$ we have $\varphi_1(U \cap U_1) = (-1,0) \cup (0,1)$, and the change of coordinates is given by

$$\begin{aligned} \varphi_1(U \cap U_1) \to \quad U \cap U_1 \quad &\to \varphi(U \cap U_1) = \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \\ t \mapsto \left(\sqrt{1-t^2}, t\right) &\mapsto \alpha = \arcsin t, \end{aligned}$$

which is a diffeomorphism of these intervals.

One can prove the similar results for the other cases.

Problem 1.27 Consider the set $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$, where

$$U_N = \left\{(x,y) \in S^1 : y \neq 1\right\}, \qquad U_S = \left\{(x,y) \in S^1 : y \neq -1\right\},$$

φ_N and φ_S being the stereographic projection (with the x-axis as image) from the north pole N and the south pole S of the sphere S^1, respectively (see Fig. 1.3).

(i) Prove that $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a C^∞ atlas on S^1.

(ii) Prove that the corresponding differentiable structure coincides with the differentiable structure on S^1 obtained in Problem 1.26.

Solution

(i) The maps $\varphi_N\colon U_N \to \mathbb{R}$ and $\varphi_S\colon U_S \to \mathbb{R}$, given by

$$\varphi_N(x,y) = \frac{x}{1-y}, \qquad \varphi_S(x,y) = \frac{x}{1+y},$$

respectively, are homeomorphisms. The inverse map φ_N^{-1} is given by

$$\varphi_N^{-1}(x') = (x,y) = \left(\frac{2x'}{1+x'^2}, \frac{x'^2-1}{1+x'^2}\right).$$

As for the change of coordinates

$$\varphi_S \circ \varphi_N^{-1}\colon \varphi_N(U_N \cap U_S) = \mathbb{R}\setminus\{0\} \to \varphi_S(U_N \cap U_S) = \mathbb{R}\setminus\{0\},$$

one has $(\varphi_S \circ \varphi_N^{-1})(t) = 1/t$, which is a C^∞ function on its domain. The inverse map is also C^∞. Thus, $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a C^∞ atlas on S^1.

(ii) Consider, for instance,

$$U_2 = \{(x,y) \in S^1 : y > 0\}, \qquad \varphi_2\colon U_2 \to (-1,1), \qquad \varphi_2(x,y) = x.$$

We have

$$\begin{aligned} \varphi_N \circ \varphi_2^{-1}\colon (-1,0)\cup(0,1) &\to (-\infty,-1)\cup(1,\infty) \\ t &\mapsto t/\bigl(1-\sqrt{1-t^2}\bigr), \end{aligned}$$

which is C^∞ on its domain. Similarly, the inverse map $\varphi_2 \circ \varphi_N^{-1}$, defined by

$$\begin{aligned} \varphi_N(U_N \cap U_2) = (-\infty,-1)\cup(1,\infty) &\to \varphi_2(U_N \cap U_2) = (-1,0)\cup(0,1) \\ s &\mapsto 2s/\bigl(1+s^2\bigr), \end{aligned}$$

is also C^∞. As one has a similar result for the other charts, we conclude.

Problem 1.28

(i) Define an atlas for the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

using the stereographic projection with the equatorial plane as image plane.

(ii) Generalise this construction to S^n, $n \geqslant 3$.

Solution

(i) Let us cover the sphere S^2 with the open subsets

$$U_N = \{(x,y,z) \in S^2 : z < a\}, \qquad U_S = \{(x,y,z) \in S^2 : z > -a\},$$

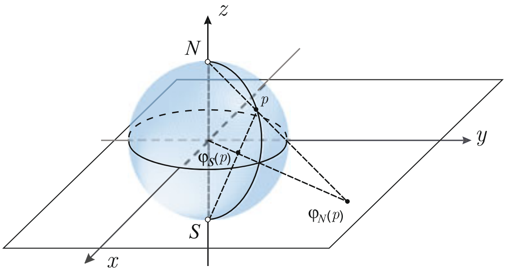


Fig. 1.4 Stereographic projections of S^2 onto the equatorial plane

for $0 < a < 1$. One can consider the equatorial plane as the image plane of the charts of the sphere (see Fig. 1.4).

We define $\varphi_N : U_N \to \mathbb{R}^2$ as the stereographic projection from the north pole $N = (0, 0, 1)$ and $\varphi_S : U_S \to \mathbb{R}^2$ as the stereographic projection from the south pole $S = (0, 0, -1)$. If x', y' are the coordinates of $\varphi_N(p)$, with $p = (x, y, z)$, we have:

$$\begin{aligned} \varphi_N\colon \qquad & U_N \to \mathbb{R}^2 \\ & (x, y, z) \mapsto \left(x', y'\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \varphi_S\colon \qquad & U_S \to \mathbb{R}^2 \\ & (x, y, z) \mapsto \left(x'', y''\right) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right). \end{aligned}$$

One has

$$\varphi_N(U_N) = \varphi_S(U_S) = B\big(0, 1/(1-a)\big) \subset \mathbb{R}^2.$$

Since the cases $z = 1$ or $z = -1$, respectively, have been dropped, φ_N and φ_S are one-to-one functions onto an open subset of $\mathbb{R}^2$. As a calculation shows, φ_N^{-1} is given by

$$\varphi_N^{-1}\left(x', y'\right) = \left(\frac{2x'}{1 + x'^2 + y'^2}, \frac{2y'}{1 + x'^2 + y'^2}, \frac{x'^2 + y'^2 - 1}{1 + x'^2 + y'^2}\right).$$

If $p \in U_N \cap U_S$, $p' = \varphi_N(p)$, and $p'' = \varphi_S(p)$, denoting by x', y' the coordinates of p' and by x'', y'' the coordinates of p'', we deduce that

$$\left(x'', y''\right) = \left(\varphi_S \circ \varphi_N{}^{-1}\right)\left(x', y'\right) = \left(\frac{x'}{x'^2 + y'^2}, \frac{y'}{x'^2 + y'^2}\right).$$

Hence $\varphi_S \circ \varphi_N{}^{-1}$ is a diffeomorphism.

(ii) For arbitrary n, with the conditions similar to the ones for S^2, we have

$$S^n = \left\{ \left(x^1, \dots, x^{n+1}\right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \left(x^i\right)^2 = 1 \right\},$$
$$U_N = \left\{ \left(x^1, \dots, x^{n+1}\right) \in S^{n+1} : x^{n+1} \neq 1 \right\},$$
$$U_S = \left\{ \left(x^1, \dots, x^{n+1}\right) \in S^{n+1} : x^{n+1} \neq -1 \right\},$$
$$\varphi_N \colon \quad U_N \quad \to \mathbb{R}^n$$
$$\left(x^1, \dots, x^{n+1}\right) \mapsto \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}} \right),$$
$$\varphi_S \colon \quad U_S \quad \to \mathbb{R}^n$$
$$\left(x^1, \dots, x^{n+1}\right) \mapsto \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}} \right),$$
$$\begin{aligned} \varphi_N^{-1}\left(y^1, \dots, y^n\right) &= \left(x^1, \dots, x^{n+1}\right) \\ &= \left(\frac{2y^1}{1 + \sum_i (y^i)^2}, \dots, \frac{2y^n}{1 + \sum_i (y^i)^2}, \frac{\sum_i (y^i)^2 - 1}{1 + \sum_i (y^i)^2} \right). \end{aligned}$$

So

$$\left(\varphi_S \circ \varphi_N^{-1}\right)\left(y^1, \dots, y^n\right) = \left(\frac{y^1}{\sum_i (y^i)^2}, \dots, \frac{y^n}{\sum_i (y^i)^2} \right) = \frac{y}{|y|^2},$$

and similarly

$$\left(\varphi_N \circ \varphi_S^{-1}\right)\left(y^1, \dots, y^n\right) = \frac{y}{|y|^2},$$

which are C^∞ in $\varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$.

Notice that with the stereographic projections, the number of charts is equal to two, which is the lowest possible figure, since S^n is compact.

Problem 1.29 Define an atlas on the *cylindrical surface*

$$M = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, 0 < z < h \right\},$$

where $h, r \in \mathbb{R}^+$.

Solution We only need to endow the circle $S^1(r) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ with an atlas. In fact, let (U, φ), (V, ψ) be an atlas as in Problem 1.25. This means that U, V are open subsets of $S^1(r) \subset \mathbb{R}^2$ such that $S^1(r) = U \cup V$, and $\varphi \colon U \to \mathbb{R}$, $\psi \colon V \to \mathbb{R}$ are diffeomorphisms. Then $U \times (0, h)$, $V \times (0, h)$ are open subsets of M and one defines an atlas on M by

$$\mathscr{A} = \left\{ \left(U \times (0, h), \varphi \times \mathrm{id}\right), \left(V \times (0, h), \psi \times \mathrm{id}\right) \right\}.$$

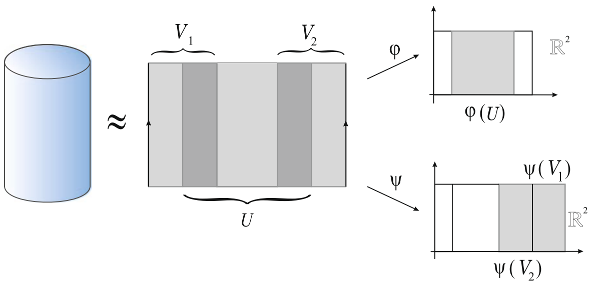


Fig. 1.5 Charts for the cylindrical surface

In fact, the map

$$(\psi \times \mathrm{id}) \circ (\varphi \times \mathrm{id})^{-1} : \varphi(U \cap V) \times (0, h) \to \psi(U \cap V) \times (0, h)$$

is a diffeomorphism, as it follows from the obvious formula

$$(\psi \times \mathrm{id}) \circ (\varphi \times \mathrm{id})^{-1} = \left(\psi \circ \varphi^{-1}\right) \times \mathrm{id}.$$

This construction is only an example of the general way of endowing a product of two manifolds with a differentiable structure. In fact, one can view M as the Cartesian product of $S^1(r)$ by an open interval.

Problem 1.30

(i) Define an atlas on the *cylindrical surface* defined as the quotient space $A/\!\sim$, where A denotes the rectangle $[0, a] \times (0, h) \subset \mathbb{R}^2$, $a > 0$, $h > 0$, with the topology inherited from the usual one of $\mathbb{R}^2$, and $\sim$ stands for the equivalence relation $(0, y) \sim (a, y)$, where $(0, y), (a, y) \in A$.
(ii) Relate this construction to the one in Problem 1.29.

Remark As for the fact that every subset of a topological space can be equipped with the *subspace topology* in which the open subsets are the intersections of the open subsets of the larger space with the given subset, see, for instance, [3, p. 55, 10].

Solution

(i) Denote by $[(x, y)]$ the equivalence class of (x, y) modulo $\sim$. Let $c, d, e, f \in \mathbb{R}$ be such that $0 < c < e < f < d < a$. We define (see Fig. 1.5) the charts (U, φ), (V, ψ) taking $U = \{[(x, y)] : c < x < d\}$, $V = V_1 \cup V_2$, where

$$V_1 = \left\{\left[(x, y)\right] : 0 \leqslant x < e\right\}, \qquad V_2 = \left\{\left[(x, y)\right] : f < x \leqslant a\right\},$$

$\varphi : U \to \mathbb{R}^2$, $\varphi([(x, y)]) = (x, y)$, and

$$\begin{aligned} \psi : \quad & V \to \mathbb{R}^2 \\ \left[(x, y)\right] & \mapsto \begin{cases} (x + a, y) & \text{if } (x, y) \in V_1, \\ (x, y) & \text{if } (x, y) \in V_2. \end{cases} \end{aligned}$$

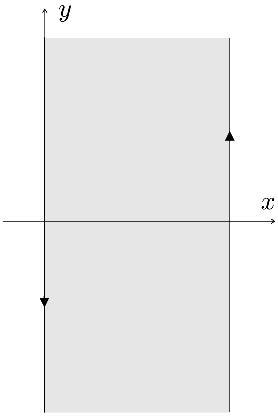


Fig. 1.6 The infinite Möbius strip

It is obvious that $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms. The change of coordinates $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is given by

$$\left(\psi \circ \varphi^{-1}\right)(x, y) = \begin{cases} (x + a, y) & \text{if } (x, y) \in \varphi(U \cap V_1), \\ (x, y) & \text{if } (x, y) \in \varphi(U \cap V_2), \end{cases}$$

which is trivially a diffeomorphism.

(ii) Let

$$\varphi\colon A = [0, a] \times (0, h) \to \mathbb{R}^3, \qquad \varphi(\alpha, z) = \left(r \cos \frac{2\pi\alpha}{a}, r \sin \frac{2\pi\alpha}{a}, z\right),$$
$$0 \leqslant \alpha \leqslant a,\ 0 < z < h,\ r = a/2\pi.$$

From the very definition of φ it follows that $\varphi(A) = M$, where $M \subset \mathbb{R}^3$ is the submanifold defined in Problem 1.29. Then it is easily checked that $\varphi(\alpha, z) = \varphi(\alpha', z')$ if and only if $(\alpha, z) \sim (\alpha', z')$. Hence φ induces a unique homeomorphism $\widehat{\varphi}\colon A/\sim\, \to M$ such that $\widehat{\varphi} \circ p = \varphi$, where $p\colon A \to A/\sim$ is the quotient map.

Problem 1.31 Define the *infinite Möbius strip* M as the topological quotient of $[0, 1] \times \mathbb{R}$ by the equivalence relation $\sim$ which identifies the pairs $(0, y)$ and $(1, -y)$ (see Fig. 1.6), with the topology inherited from the usual one of $\mathbb{R}^2$. Show that M admits a structure of C^∞ manifold consistent with its topology.

Solution Let $p\colon [0, 1] \times \mathbb{R} \to M = ([0, 1] \times \mathbb{R})/\sim$ be the quotient map. Consider the two open subsets of M given by

$$U = \big((0, 1) \times \mathbb{R}\big)/\sim, \qquad V = \Big(\big([0, 1/2) \cup (1/2, 1]\big) \times \mathbb{R}\Big)/\sim.$$

Every point $z \in U$ can be uniquely written as $z = p(x, y)$, with $(x, y) \in (0, 1) \times \mathbb{R}$ and we can define a homeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^2$ by setting $\varphi(z) = (x, y)$.

We also define $\psi\colon V \to \psi(V) \subset \mathbb{R}^2$ as follows: Set $z = p(x, y)$ with $(x, y) \in ([0, 1/2) \cup (1/2, 1]) \times \mathbb{R}$. Then,

$$\psi(z) = \begin{cases} (x+1, -y) & \text{if } x < 1/2, \\ (x, y) & \text{if } x > 1/2. \end{cases}$$

The definition makes sense as $\psi(p(0, y)) = \psi(p(1, -y)) = (1, -y)$, for all $y \in \mathbb{R}$. It is easily checked that ψ induces a homeomorphism between V and the open subset $(1/2, 3/2) \times \mathbb{R} \subset \mathbb{R}^2$. The change of coordinates $\varphi \circ \psi^{-1}\colon \psi(U \cap V) \to \varphi(U \cap V)$, that is,

$$\varphi \circ \psi^{-1}\colon \big((1/2, 1) \cup (1, 3/2)\big) \times \mathbb{R} \to \big((0, 1/2) \cup (1/2, 1)\big) \times \mathbb{R},$$

is given by

$$\big(\varphi \circ \psi^{-1}\big)(x, y) = \begin{cases} (x, y) & \text{if } 1/2 < x < 1, \\ (x-1, -y) & \text{if } 1 < x < 3/2, \end{cases}$$

which is a C^∞ map. Similarly, the change of coordinates

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V),$$

that is,

$$\psi \circ \varphi^{-1}\colon \big((0, 1/2) \cup (1/2, 1)\big) \times \mathbb{R} \to \big((1/2, 1) \cup (1, 3/2)\big) \times \mathbb{R},$$

is given by

$$\big(\psi \circ \varphi^{-1}\big)(x, y) = \begin{cases} (x+1, -y) & \text{if } 0 < x < 1/2, \\ (x, y) & \text{if } 1/2 < x < 1, \end{cases}$$

which also is a C^∞ map.

Problem 1.32

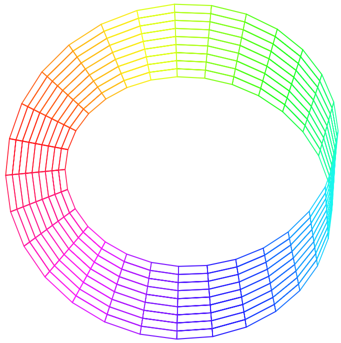
(i) Consider the circle in $\mathbb{R}^3$ given by $x^2 + y^2 = 4$, $z = 0$, and the open segment PQ in the yz-plane in $\mathbb{R}^3$ given by $y = 2$, $|z| < 1$. Move the centre C of PQ along the circle and rotate PQ around C in the plane Cz, so that when C goes through an angle u, PQ has rotated an angle $u/2$. When C completes a course around the circle, PQ returns to its initial position, but with its ends changed (see Fig. 1.7).

The surface so described is called the *Möbius strip*.

Consider the two parametrisations

$$\begin{aligned} \mathbf{x}(u, v) &= \big(x(u, v), y(u, v), z(u, v)\big) \\ &= \left(\left(2 - v \sin\frac{u}{2}\right)\sin u, \left(2 - v \sin\frac{u}{2}\right)\cos u, v \cos\frac{u}{2}\right), \\ &\quad 0 < u < 2\pi, \ -1 < v < 1, \end{aligned}$$

Fig. 1.7 The Möbius strip



$$
\begin{aligned}
\mathbf{x}'(u,v) &= \big(x'(u,v),\, y'(u,v),\, z'(u,v)\big) \\
&= \left(\left(2 - v' \sin\left(\frac{\pi}{4} + \frac{u'}{2}\right)\right) \cos u', \right. \\
&\quad \left. -\left(2 - v' \sin\left(\frac{\pi}{4} + \frac{u'}{2}\right)\right) \sin u',\ v' \cos\left(\frac{\pi}{4} + \frac{u'}{2}\right)\right), \\
&\quad \pi/2 < u' < 5\pi/2,\ -1 < v' < 1.
\end{aligned}
$$

Prove that the Möbius strip with these parametrisations is a 2-dimensional manifold.

(ii) Relate this manifold to the one given in Problem 1.31.

The relevant theory is developed, for instance, in do Carmo [2].

Solution

(i) The coordinate neighbourhoods corresponding to the parametrisations cover the Möbius strip. The intersection of these coordinate neighbourhoods has the two connected components

$$
U_1 = \big\{\mathbf{x}(u,v) : \pi < u < 2\pi\big\}, \qquad U_2 = \big\{\mathbf{x}(u,v) : 0 < u < \pi\big\},
$$

and the changes of coordinates are given on U_1 and U_2, respectively, by

$$
\begin{cases} u' = u - \dfrac{\pi}{2}, \\ v' = v, \end{cases} \qquad \begin{cases} u' = u + \dfrac{3\pi}{2}, \\ v' = -v, \end{cases}
$$

which are obviously C^∞.

(ii) Let $\varphi\colon [0,2\pi] \times (-1,1) \to \mathbb{R}^3$ be the map given by

$$
\varphi(u,v) = \left(\left(2 - v \sin\frac{u}{2}\right) \sin u,\ \left(2 - v \sin\frac{u}{2}\right) \cos u,\ v \cos\frac{u}{2}\right).
$$

Note that the restriction of φ to $(0, 2\pi) \times (-1, 1)$ coincides with the first parametrisation. Moreover, it is easy to see that $\operatorname{im}\varphi$ coincides with the Möbius strip.

Let $\alpha\colon [0, 1] \times \mathbb{R} \to [0, 2\pi] \times (-1, 1)$ be the homeomorphism given by $\alpha(s, t) = (2\pi s, (2/\pi)\arctan t)$. Set $\psi = \varphi \circ \alpha$. Let $(s_1, t_1), (s_2, t_2) \in [0, 1] \times \mathbb{R}$ be two distinct points such that $\psi(s_1, t_1) = \psi(s_2, t_2)$. As φ is a parametrisation when restricted to $(0, 2\pi) \times (-1, 1)$, the assumption implies that

$$(s_1, t_1), (s_2, t_2) \in \partial\big([0, 1] \times \mathbb{R}\big) = \big(\{0\} \times \mathbb{R}\big) \cup \big(\{1\} \times \mathbb{R}\big).$$

As $(s_1, t_1) \neq (s_2, t_2)$, either $(s_1, t_1) \in \{0\} \times \mathbb{R}$ and $(s_2, t_2) \in \{1\} \times \mathbb{R}$, or vice versa. In the first case, $\psi(0, t_1) = \psi(1, t_2)$ means

$$\varphi\big(0, (2/\pi)\arctan t_1\big) = \varphi\big(2\pi, (2/\pi)\arctan t_2\big).$$

So

$$\big(0, 2, (2/\pi)\arctan t_1\big) = \big(0, 2, -(2/\pi)\arctan t_2\big),$$

that is, $t_1 + t_2 = 0$. The other case is similar. This proves that the equivalence relation associated to ψ coincides with the equivalence relation $\sim$ defined in Problem 1.31.

Problem 1.33 Let T^2 be a torus of revolution in $\mathbb{R}^3$ with centre at $(0, 0, 0) \in \mathbb{R}^3$ and let $a\colon T^2 \to T^2$ be defined by $a(x, y, z) = (-x, -y, -z)$. Let K be the quotient space under the equivalence relation $p \sim a(p)$, $p \in T^2$, and let $\pi : T^2 \to K$ denote the map $\pi(p) = \{p, a(p)\}$. Assume T^2 is covered by parametrisations $\mathbf{x}_\alpha : U_\alpha \to T^2$ such that

$$\mathbf{x}_\alpha(U_\alpha) \cap (a \circ \mathbf{x}_\alpha)(U_\alpha) = \emptyset,$$

where each U_α is an open subset of $\mathbb{R}^2$.

Prove that K is covered by the parametrisations $(U_\alpha, \pi \circ \mathbf{x}_\alpha)$ and that the corresponding changes of coordinates are C^∞.

K is called the *Klein bottle* (see Fig. 1.8).

The relevant theory is developed, for instance, in do Carmo [2].

Solution The subsets $(\pi \circ \mathbf{x}_\alpha)(U_\alpha)$ cover K by assumption. Each of them is open in K as

$$\pi^{-1}\big((\pi \circ \mathbf{x}_\alpha)(U_\alpha)\big) = \mathbf{x}_\alpha(U_\alpha) \cup a\big(\mathbf{x}_\alpha(U_\alpha)\big)$$

and $\mathbf{x}_\alpha(U_\alpha)$, $a(\mathbf{x}_\alpha(U_\alpha))$ are open subsets of T^2. Moreover, each map $\pi \circ \mathbf{x}_\alpha : U_\alpha \to K$ is a parametrisation (that is, it is one-to-one) by virtue of the condition $\mathbf{x}_\alpha(U_\alpha) \cap (a \circ \mathbf{x}_\alpha)(U_\alpha) = \emptyset$. Finally, the changes of coordinates are C^∞. In fact, let

$$p \in \operatorname{domain}\big((\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha)\big).$$

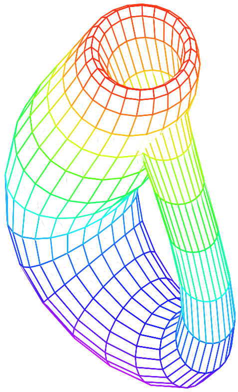


Fig. 1.8 The Klein bottle

Then $p \in U_\alpha$ and $(\pi \circ \mathbf{x}_\alpha)(p) \in (\pi \circ \mathbf{x}_\beta)(U_\beta)$; hence either $\mathbf{x}_\alpha(p) \in \mathbf{x}_\beta(U_\beta)$ or $\mathbf{x}_\alpha(p) \in (a \circ \mathbf{x}_\beta)(U_\beta)$. In the first case, one has

$$(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha) = \mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$$

on a neighbourhood of p; and in the second one, we have

$$(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha) = (a \circ \mathbf{x}_\beta)^{-1} \circ \mathbf{x}_\alpha,$$

on a neighbourhood of p. Since a is a diffeomorphism, $(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha)$ is a diffeomorphism on a neighbourhood of p. Thus, it is C^∞.

Problem 1.34 Define an atlas on the topological space $M(r \times s, \mathbb{R})$ of all the real matrices of order $r \times s$.

Solution The map $\varphi \colon M(r \times s, \mathbb{R}) \to \mathbb{R}^{rs}$ defined by

$$\varphi(a_{ij}) = (a_{11}, \dots, a_{1s}, \dots, a_{r1}, \dots, a_{rs}),$$

is one-to-one and surjective. Now endow $M(r \times s, \mathbb{R})$ with the topology for which φ is a homeomorphism. So, $(M(r \times s, \mathbb{R}), \varphi)$ is a chart on $M(r \times s, \mathbb{R})$, whose domain is all of $M(r \times s, \mathbb{R})$. The change of coordinates is the identity, hence it is a diffeomorphism. So, $\mathscr{A} = \{(M(r \times s, \mathbb{R}), \varphi)\}$ is an atlas on $M(r \times s, \mathbb{R})$.

1.3 Differentiable Structures Defined on Sets

In the present section, and only here, we consider differentiable structures defined on sets.

Let S be a set. An n-dimensional chart on S is an injection of a subset of S onto an open subset of $\mathbb{R}^n$. A C^∞ atlas on S is a collection of charts whose domains

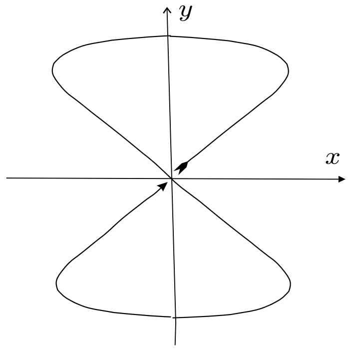


Fig. 1.9 The "Figure Eight" defined by (E, φ)

cover S, and such that if the domains of two charts φ, ψ overlap, then the change of coordinates $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$.

Hence, the manifold is not supposed to be a priori a topological space. It has the topology induced by the differentiable structure defined by the C^∞ atlas (see [1, 2.2]).

Problem 1.35 Consider $E = \{(\sin 2t, \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ (the *Figure Eight*).

(i) Prove that $\{(E, \varphi)\}$, where $\varphi\colon E \to \mathbb{R}$ is the injection of E onto an open interval of $\mathbb{R}$, defined by $\varphi(\sin 2t, \sin t) = t$, $t \in (0, 2\pi)$ (see Fig. 1.9), is an atlas on the set E. Here E has the topology inherited from its injection in $\mathbb{R}$.
(ii) Prove that, similarly, $\{(E, \psi)\}$, where $\psi\colon E \to \mathbb{R}$, $\psi(\sin 2t, \sin t) = t$, $t \in (-\pi, \pi)$, is an atlas on the set E.
(iii) Do the two atlases define the same differentiable structure on E?

Remark Notice that the "Figure Eight" is not endowed with the topology inherited from $\mathbb{R}^2$ as, in this case, it would not be a differentiable manifold. Instead, we endow it with the topology corresponding to its differentiable structure obtained from the atlases above. (Notice that the arguments here are similar to those given in studying the sets in Problems 1.43, 1.44 below).

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution

(i) φ is an injective map from E onto the open interval $(0, 2\pi)$ of $\mathbb{R}$, whose domain is all of E. Consequently, $\{(E, \varphi)\}$ is an atlas on E.
(ii) Similar to (E, φ).
(iii) The two atlases define the same differentiable structure if (E, φ) belongs to the structure defined by (E, ψ) and conversely. That is, the maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ must be C^∞. We have

$$\begin{array}{rcccl} \psi \circ \varphi^{-1}\colon \varphi(E) = (0, 2\pi) & \to & E & \to & \psi(E) = (-\pi, \pi) \\ t & \mapsto & (\sin 2t, \sin t) & \mapsto & \widetilde{\psi}(\sin 2t, \sin t), \end{array}$$

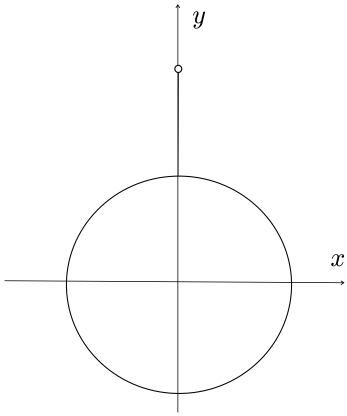


Fig. 1.10 The Noose

where

$$\widetilde{\psi}(\sin 2t, \sin t) = \begin{cases} t, & t \in (0, \pi), \\ 0, & t = \pi, \\ \psi(\sin(2t - 4\pi), \sin(t - 2\pi)) = t - 2\pi, & t \in (\pi, 2\pi). \end{cases}$$

Thus, $\psi \circ \varphi^{-1}$ is not even continuous and the differentiable structures defined by these atlases are different.

Notice that the topologies induced on E by the two C^∞ structures are also different: Consider, for instance, the open subsets $\varphi^{-1}(U_\pi)$ and $\psi^{-1}(U_0)$, where U_π and U_0 denote small neighbourhoods of π and 0, respectively.

Problem 1.36 Consider the subset N of $\mathbb{R}^2$ (the *Noose*) defined (see Fig. 1.10) by

$$N = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(0, y) : 1 < y < 2\}.$$

(i) Prove that the function

$$\begin{array}{rll} \varphi\colon \quad N & \to \mathbb{R} & \\ (\sin 2\pi s, \cos 2\pi s) & \mapsto s & \text{if } 0 \leqslant s < 1, \\ (0, s) & \mapsto 1 - s & \text{if } 1 < s < 2, \end{array}$$

is a chart that defines a C^∞ structure on N.

(ii) Prove that the function

$$\begin{array}{rll} \psi\colon \quad N & \to \mathbb{R} & \\ (\sin 2\pi s, \cos 2\pi s) & \mapsto 1 - s & \text{if } 0 < s \leqslant 1, \\ (0, s) & \mapsto 1 - s & \text{if } 1 < s < 2, \end{array}$$

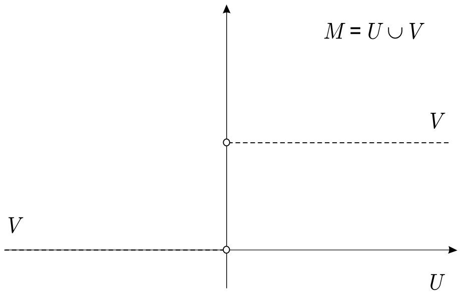


Fig. 1.11 An example of set with a C^∞ structure

also defines a C^∞ structure on N.

(iii) Prove that the two structures above are different.

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution

(i) Obviously $\varphi\colon N \to (-1, 1)$ is a one-to-one map. Endow N with the unique topology τ_a making φ a homeomorphism. Thus, the atlas $\{(N, \varphi)\}$, with the single chart φ, defines a C^∞ structure in N.

Notice that if N is endowed with the topology inherited from that of $\mathbb{R}^2$, then φ is not continuous at the point $(0, 1)$.

(ii) Proceed similarly to (i).

(iii) If (N, ψ) is assumed to belong to the structure defined from (N, φ), then $\psi \circ \varphi^{-1}\colon (-1, 1) \to (-1, 1)$ would be C^∞, but it is not even continuous.

Problem 1.37 Consider the sets

$$U = \left\{(s, 0) \in \mathbb{R}^2 : s \in \mathbb{R}\right\}, \qquad V = \left\{(s, 0) \in \mathbb{R}^2 : s < 0\right\} \cup \left\{(s, 1) \in \mathbb{R}^2 : s > 0\right\},$$

and the maps

$$\begin{aligned} &\varphi\colon U \to \mathbb{R}, \qquad && \varphi(s, 0) = s, && \\ &\psi\colon V \to \mathbb{R}, && \psi(s, 0) = s, \qquad && \psi(s, 1) = s, \\ &\gamma\colon V \to \mathbb{R}, && \gamma(s, 0) = s^3, && \gamma(s, 1) = s^3. \end{aligned}$$

(i) Prove that $\{(U, \varphi), (V, \psi)\}$ defines a C^∞ structure on the set $M = U \cup V$ (see Fig. 1.11).

(ii) Is (V, γ) a chart in the previous differentiable structure?

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution

(i) The maps φ and ψ are injective, and we have $\varphi(U) = \mathbb{R}$, $\psi(V) = \mathbb{R} \setminus \{0\}$, which are open subsets of $\mathbb{R}$. Moreover, both $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are the

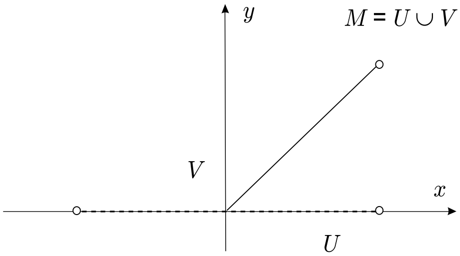


Fig. 1.12 Two charts which do not define an atlas

identity map on $\psi(U \cap V) = (-\infty, 0) = \varphi(U \cap V)$, and $\varphi(U \cap V)$, $\psi(U \cap V)$ are open subsets of $\mathbb{R}$. Hence $\mathscr{A} = \{(U, \varphi), (V, \psi)\}$ is a C^∞ atlas on M.

(ii) The map γ is injective, and $\gamma(V) = \mathbb{R} \setminus \{0\}$, $\gamma(U \cap V) = (-\infty, 0)$, are open subsets of $\mathbb{R}$. Moreover, the maps $\gamma \circ \varphi^{-1}$, $\varphi \circ \gamma^{-1}$, $\gamma \circ \psi^{-1}$, and $\psi \circ \gamma^{-1}$ are C^∞ maps. Thus, γ is, in fact, a chart of the above differentiable structure.

Problem 1.38 Let

$$S = \{(x, 0) \in \mathbb{R}^2 : x \in (-1, +1)\} \cup \{(x, x) \in \mathbb{R}^2 : x \in (0, 1)\}.$$

Let

$$\begin{aligned} &U = \{(x, 0) : x \in (-1, +1)\}, \qquad \varphi\colon U \to \mathbb{R}, \qquad \varphi(x, 0) = x, \\ &V = \{(x, 0) : x \in (-1, 0]\} \cup \{(x, x), x \in (0, 1)\}, \\ &\psi\colon V \to \mathbb{R}, \qquad \psi(x, 0) = x, \qquad \psi(x, x) = x \end{aligned}$$

(see Fig. 1.12). Is $\mathscr{A} = \{(U, \varphi), (V, \psi)\}$ an atlas on the set S?

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution We have $S = U \cup V$. Furthermore φ and ψ are injective maps onto the open subset $(-1, +1)$ of $\mathbb{R}$. Thus (U, φ) and (V, ψ) are charts on S. However, one has $\varphi(U \cap V) = \psi(U \cap V) = (-1, 0]$, which is not an open subset of $\mathbb{R}$. Thus $\mathscr{A}$ is not an atlas on S.

Problem 1.39 Consider on $\mathbb{R}^2$ the subsets

$$E_1 = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}, \qquad E_2 = \{(x, 1) \in \mathbb{R}^2 : x \in \mathbb{R}\}.$$

Define on $E = E_1 \cup E_2$ an equivalence relation $\sim$ by

$$\begin{aligned} (x_1, 0) \sim (x_2, 0) &\iff x_1 = x_2, \\ (x_1, 1) \sim (x_2, 1) &\iff x_1 = x_2, \\ (x_1, 0) \sim (x_2, 1) &\iff x_1 = x_2 < 0. \end{aligned}$$

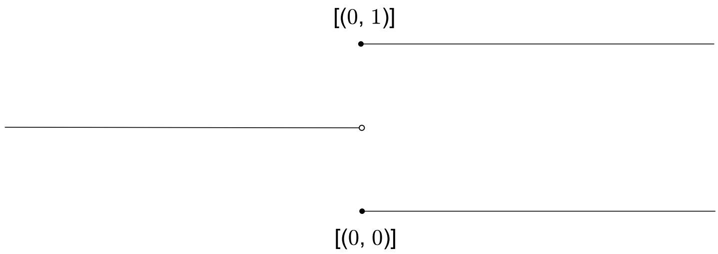


Fig. 1.13 A set with a C^∞ atlas, whose induced topology is not Hausdorff

The classes of the quotient set $S = E/\sim$ are represented by the elements $(x, 0)$ for $x < 0$, and the elements $(x, 0)$ and $(x, 1)$ for $x \geqslant 0$ (see Fig. 1.13). Prove that S admits a C^∞ atlas, but S is not Hausdorff with the induced topology.

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution Denote by $[(x, y)]$ the class of (x, y). We can endow S with a manifold structure by means of the charts (U_1, φ_1) and (U_2, φ_2), where

$$U_1 = \big\{\big[(x,0)\big] : x \in \mathbb{R}\big\}, \qquad U_2 = \big\{\big[(x,0)\big] : x < 0\big\} \cup \big\{\big[(x,1)\big] : x \geqslant 0\big\},$$
$$\varphi_1\big(\big[(x,0)\big]\big) = x, \qquad \varphi_2\big(\big[(x,0)\big]\big) = \varphi_2\big(\big[(x,1)\big]\big) = x.$$

One has $U_1 \cup U_2 = S$. Furthermore, $\varphi_1(U_1) = \mathbb{R}$, $\varphi_2(U_2) = \mathbb{R}$ are open sets and

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} \colon (-\infty, 0) &\to U_1 \cap U_2 \to (-\infty, 0) \\ x &\mapsto \big[(x,0)\big] \mapsto x \end{aligned}$$

is a diffeomorphism. Hence S admits a C^∞ atlas.

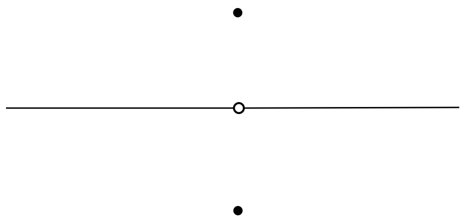
Nevertheless, the induced topology is not Hausdorff. The points $[(0, 1)]$ and $[(0, 0)]$ do not admit disjoint open neighbourhoods. In fact, if U is an open subset of S containing $[(0, 0)]$, then $\varphi_1(U \cap U_1)$ must be an open subset of $\mathbb{R}$. But $[(0, 0)] \in U \cap U_1$, hence $\varphi_1(U \cap U_1)$ is an open subset of $\mathbb{R}$ that contains 0, thus it contains an interval of the form $(-\alpha, \alpha)$, with $\alpha > 0$. Therefore, $\{[(x, 0)] : -\alpha < x < 0\} \subset U$. Similarly, an open subset V of S containing $[(0, 1)]$ can have a subset of the form $\{[(x, 0)] : -\beta < x < 0, \beta > 0\}$. Thus U and V cannot be disjoint.

Problem 1.40 Let S be the subset of $\mathbb{R}^2$ which consists of all the points of the set $U = \{(s, 0)\}$, $s \in \mathbb{R}$, and the point $(0, 1)$. Let U_1 be the set obtained from U replacing the point $(0, 0)$ by the point $(0, 1)$. We define the maps

$$\varphi \colon U \to \mathbb{R}, \qquad \varphi(s,0) = s, \qquad \varphi_1 \colon U_1 \to \mathbb{R}, \qquad \begin{cases} \varphi_1(s,0) = s, & s \neq 0, \\ \varphi_1(0,1) = 0. \end{cases}$$

Prove that $\{(U, \varphi), (U_1, \varphi_1)\}$ is a C^∞ atlas on S, but S is not Hausdorff with the induced topology.

Fig. 1.14 The straight line with a double point



Solution $U \cup V = S$, φ and φ_1 are injective maps in $\mathbb{R}$, and the changes of coordinates $\varphi \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi^{-1}$ are both the identity on the open subset $\mathbb{R} \setminus \{0\}$. So, these two charts define a C^∞ atlas on S.

Let V be a neighbourhood of $(0, 0)$ and W a neighbourhood of $(0, 1)$ in S. Then $\varphi(U \cap V)$ and $\varphi_1(U_1 \cap W)$ are open subsets of $\mathbb{R}$ containing 0, and so they will also contain some point $a \neq 0$. The point $(a, 0)$ belongs to $V \cap W$, hence the topology of S is not Hausdorff.

Problem 1.41 Consider the set S obtained identifying two copies L_1 and L_2 of the real line except at a point $p \in \mathbb{R}$ (see Fig. 1.14). Prove that S admits a C^∞ atlas but it is not Hausdorff with the induced topology.

Solution Take the usual charts on L_1 and L_2, i.e. the identity map on $\mathbb{R}$. Then $S = L_1 \cup L_2$, and the change of coordinates on the intersection $L_1 \cap L_2$ is C^∞ as it is the identity map. Nevertheless, the points $p_1 \in L_1$ and $p_2 \in L_2$, where p_i, $i = 1, 2$, stands for the representative of p in L_i, are obviously not separable.

Problem 1.42 Let $S = \mathbb{R} \times \mathbb{R}$, where in the first factor we consider the discrete topology, in the second factor the usual topology, and in S the product topology. Prove that S admits a C^∞ atlas and that S does not satisfy the second axiom of countability but it is paracompact.

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution For each $t \in \mathbb{R}$, let $L_t = \{(t, y) : y \in \mathbb{R}\} = \{t\} \times \mathbb{R}$, which is an open subset of S. The map $\varphi_t : L_t \to \mathbb{R}$, $\varphi_t(t, y) = y$, is a homeomorphism, hence $\{(L_t, \varphi_t)\}_{t \in \mathbb{R}}$ is a C^∞ atlas on S such that if $s \neq t$ then $L_t \cap L_s = \emptyset$. So S is a locally Euclidean space of dimension 1 which admits a differentiable structure. The topological space S has uncountable connected components; thus it is not second countable with the induced topology. The space S is paracompact. In fact, S is Hausdorff as a product of Hausdorff spaces and if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of S, then, for some fixed t, $\{U_\alpha \cap L_t\}_{\alpha \in A}$ is an open covering of L_t (which is paracompact since it is homeomorphic to $\mathbb{R}$ with the usual topology) which admits a locally finite refinement $\{V_\lambda^t\}_{\lambda \in \Lambda}$. Thus $\{V_\lambda^t\}_{\lambda \in \Lambda, t \in \mathbb{R}}$ is a locally finite refinement of $\{U_\alpha\}_{\alpha \in A}$.

One could alternatively argue that S is paracompact since each connected component of S is second countable.

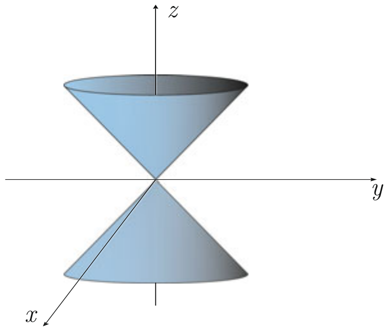


Fig. 1.15 The cone is not a locally Euclidean space because of the origin

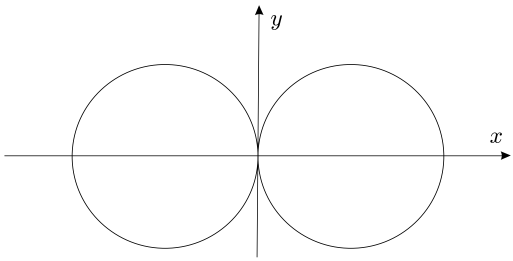


Fig. 1.16 Two tangent circles are not a locally Euclidean space

Problem 1.43 Consider the cone $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ (see Fig. 1.15) with the topology induced by the usual one of $\mathbb{R}^3$. Prove that the algebraic manifold S is not even a locally Euclidean space.

Solution The point $(0, 0, 0) \in S$ does not have a neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$. In fact, if such a homeomorphism $h \colon U \to V$ between an open neighbourhood U of $0 = (0, 0, 0)$ in S and an open subset V of $\mathbb{R}^2$ is assumed to exist, then, for small enough $\varepsilon > 0$, the open disk $B(h(0), \varepsilon)$ of centre $h(0)$ and radius ε would be contained in V. Now, if we drop the point $h(0)$ in $B(h(0), \varepsilon)$ the remaining set is connected. So it suffices to see that if we drop 0 in any of its neighbourhoods, the set $U \setminus \{0\}$ is not connected. In fact, $U \setminus \{0\} = U_+ \cup U_-$, where

$$U_+ = \big\{(x, y, z) \in U : z > 0\big\}, \qquad U_- = \big\{(x, y, z) \in U : z < 0\big\},$$

so $U_+ \cap U_- = \emptyset$, and U_+ and U_- are open subsets in the induced topology. Hence S is not even a locally Euclidean space.

Problem 1.44 Let S be the topological space defined by the union of the two circles in $\mathbb{R}^2$ with radius 1 and centres $(-1, 0)$ and $(1, 0)$, respectively (see Fig. 1.16), and the topology inherited from that of $\mathbb{R}^2$. Is S a locally Euclidean space?

Solution No, as none of the connected neighbourhoods in S of the point of tangency $(0, 0)$ is homeomorphic to an open subset of $\mathbb{R}$. In fact, let V be a neighbourhood of $(0, 0)$ in S inside the unit open ball centred at the origin. If such a neighbourhood V

were homeomorphic to $\mathbb{R}$, then $V \setminus \{(0,0)\}$ and $\mathbb{R} \setminus \{0\}$ would be homeomorphic; but this is not possible, as $V \setminus \{(0,0)\}$ has at least four connected components and $\mathbb{R} \setminus \{0\}$ has only two.

Problem 1.45 Let S be a set with a C^∞ atlas and consider the topological space S with the induced topology.

(i) Is S locally compact, locally connected and locally connected by arcs as a topological space? Does it satisfy the first axiom of countability? Does it satisfy the separation axiom T_1?
(ii) Does it satisfy the separation axiom T_2? And the second axiom of countability?
(iii) Does it satisfy the separation axiom T_3? Is S a regular topological space? Is S pseudometrizable? Does it satisfy all separations axioms T_i? Is S paracompact? Can it have continuous partitions of unity?
(iv) Does S satisfy the properties mentioned in (iii) if we constrain it to be T_2 and to satisfy the second axiom of countability?

Hint Consider:

1. Urysohn's Theorem: If S verifies the second axiom of countability, then it is equivalent for S to be pseudometrizable and to be regular.
2. Stone's Theorem: If S is pseudometrizable, then it is paracompact.

Solution

(i) S being locally Euclidean, it is locally compact, locally connected, locally connected by arcs, and satisfies the first axiom of countability.

   S satisfies the separation axiom T_1. In fact, let p and q be different points of S. If they belong to the domain of some chart (U, φ) of S, we can choose disjoint open subsets V_1, V_2 of $\mathbb{R}^n$ (assuming $\dim S = n$), contained in $\varphi(U)$, and such that $\varphi(p) \in V_1$, $\varphi(q) \in V_2$. Since φ is continuous, $\varphi^{-1}(V_1)$ and $\varphi^{-1}(V_2)$ are disjoint open subsets of S containing p and q, respectively. If p and q do not belong to the domain of a given chart of S, there must be a chart whose domain U_1 contains p but not q, and one chart whose domain U_2 contains q but not p.

   Notice that U_1 and U_2 are open subsets of S.
(ii) It does not necessarily satisfy the separation axiom T_2, as it can be seen in the counterexamples given in Problems 1.39, 1.40, 1.41. It does not necessarily satisfy the second axiom of countability, as the counterexample given in Problem 1.42 proves.
(iii) Not necessarily, since S is not necessarily T_2.
(iv) Yes, as we have:
   (a) S is locally compact, as it follows from (i). As S is also T_2, it is T_3 and hence regular.
   (b) By Urysohn's Theorem, S is pseudometrizable.
   (c) S being pseudometrizable and T_2, it satisfies all the separation axioms.
   (d) S being pseudometrizable, it is paracompact by Stone's Theorem.
   (e) S being paracompact, it admits continuous partitions of unity.

1.4 Differentiable Functions and Mappings

Problem 1.46 Consider the map

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto x^3 + x\,y + y^3 + 1.$$

(i) Compute the map $f_*\colon T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$.
(ii) Which of the points $(0,0)$, $(\frac{1}{3}, \frac{1}{3})$, $(-\frac{1}{3}, -\frac{1}{3})$, is f_* injective or surjective at?

Solution

(i)

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_p\right) = \frac{\partial f}{\partial x}(p)\left.\frac{\partial}{\partial t}\right|_{f(p)} = \left(3x^2 + y\right)(p)\left.\frac{\partial}{\partial t}\right|_{f(p)},$$
$$f_*\left(\left.\frac{\partial}{\partial y}\right|_p\right) = \frac{\partial f}{\partial y}(p)\left.\frac{\partial}{\partial t}\right|_{f(p)} = \left(x + 3y^2\right)(p)\left.\frac{\partial}{\partial t}\right|_{f(p)}.$$

(ii)

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(0,0)}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_1, \qquad f_*\left(\left.\frac{\partial}{\partial y}\right|_{(0,0)}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_1,$$

hence $f_{*(0,0)}$ is neither surjective nor injective.

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(\frac{1}{3},\frac{1}{3})}\right) = \frac{2}{3}\left.\frac{\partial}{\partial t}\right|_{\frac{32}{27}} = f_*\left(\left.\frac{\partial}{\partial y}\right|_{(\frac{1}{3},\frac{1}{3})}\right),$$

hence $f_{*(\frac{1}{3},\frac{1}{3})}$ is surjective, but not injective.

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(-\frac{1}{3},-\frac{1}{3})}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_{\frac{28}{27}} = f_*\left(\left.\frac{\partial}{\partial y}\right|_{(-\frac{1}{3},-\frac{1}{3})}\right),$$

hence $f_{*(-\frac{1}{3},-\frac{1}{3})}$ is neither surjective nor injective.

Problem 1.47 Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto \left(x^2 - 2y, 4x^3y^2\right),$$
$$g\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad (x, y) \mapsto \left(x^2y + y^2, x - 2y^3, y\,\mathrm{e}^x\right).$$

(i) Compute $f_{*(1,2)}$ and $g_{*(x,y)}$.
(ii) Find $g_*((4\frac{\partial}{\partial x} - \frac{\partial}{\partial y})_{(0,1)})$.
(iii) Calculate the conditions that the constants λ, μ, ν must satisfy for the vector

$$\left(\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}\right)_{g(0,0)}$$

to be the image of some vector by g_*.

Solution

(i)

$$f_{*(1,2)} \equiv \begin{pmatrix} 2 & -2 \\ 48 & 16 \end{pmatrix}, \qquad g_{*(x,y)} \equiv \begin{pmatrix} 2x\,y & x^2+2y \\ 1 & -6y^2 \\ y\mathrm{e}^x & \mathrm{e}^x \end{pmatrix}.$$

(ii)

$$g_*\left(\left(4\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)_{(0,1)}\right) \equiv \begin{pmatrix} 0 & 2 \\ 1 & -6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 3 \end{pmatrix}_{g(0,1)}$$
$$\equiv \left(-2\frac{\partial}{\partial x} + 10\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)_{(1,-2,1)}.$$

(iii) Since

$$g_{*(0,0)} \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the image by g_* of $T_{(0,0)}\mathbb{R}^2$ is the vector subspace of $T_{(0,0,0)}\mathbb{R}^3$ of vectors of type $(0, \mu, \nu)$.

Remark f_* cannot be injective at any point since $\dim \mathbb{R}^2 > \dim \mathbb{R}$.

Problem 1.48 The elements of $\mathbb{R}^4$ can be written as matrices of the form $A = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$. Let $A_0 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Let $T_\theta \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the differentiable transformation defined by $T_\theta(A) = A_0 A$.

(i) Calculate $T_{\theta *}$.
(ii) Compute $T_{\theta *}X$, where $X = \cos\theta \, \frac{\partial}{\partial x} - \sin\theta \, \frac{\partial}{\partial y} + \cos\theta \, \frac{\partial}{\partial z} - \sin\theta \, \frac{\partial}{\partial t}$.

Solution

(i)

$$T_{\theta *} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

(ii) It is immediate that $T_{\theta *}X = \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$. The result can also be obtained considering that if

$$X = \lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y} + \lambda_3 \frac{\partial}{\partial z} + \lambda_4 \frac{\partial}{\partial t}$$

is a vector field on $\mathbb{R}^4$, then $T_{\theta *}X = A_0 A$, where $A = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{pmatrix}$. We thus have

$$T_{\theta *}X \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \cos\theta \\ -\sin\theta & -\sin\theta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial z}.$$

Problem 1.49 Let E be the "Figure Eight" with its differentiable structure given by the global chart $(\sin 2s, \sin s) \mapsto s$, $s \in (0, 2\pi)$ (see Problem 1.35). Consider the vector $v = (\mathrm{d}/\mathrm{d}s)_0$ tangent at the origin $p = (0,0)$ to E and let $j\colon E \to \mathbb{R}^2$ be the canonical injection of E in $\mathbb{R}^2$.

(i) Compute $j_* v$.
(ii) Compute $j_* v$ if E is given by the chart $(\sin 2s, \sin s) \mapsto s$, $s \in (-\pi, \pi)$.

Solution

(i) The origin p corresponds to $s = \pi$, so

$$j_{*p} \equiv \begin{pmatrix} \frac{\partial \sin 2s}{\partial s} & 0 \\ 0 & \frac{\partial \sin s}{\partial s} \end{pmatrix}_{s=\pi} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

As $v = \frac{\mathrm{d}}{\mathrm{d}s}|_p$, we have

$$j_{*p} v \equiv \begin{pmatrix} 2 \\ -1 \end{pmatrix} (1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \equiv 2 \frac{\partial}{\partial x}\bigg|_p - \frac{\partial}{\partial y}\bigg|_p.$$

(ii) We now have

$$j_{*p} \equiv \begin{pmatrix} \frac{\partial \sin 2s}{\partial s} \\ \frac{\partial \sin s}{\partial s} \end{pmatrix}_{s=0} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

so $j_{*p} v = 2\frac{\partial}{\partial x}|_p + \frac{\partial}{\partial y}|_p$.

Problem 1.50 Consider the parametrisation (see Remark 1.4)

$$x = \sin\theta \cos\varphi, \qquad y = \sin\theta \sin\varphi, \qquad z = \cos\theta, \quad 0 < \theta < \pi,\ 0 < \varphi < 2\pi,$$

of S^2. Let $f\colon S^2 \to S^2$ be the map induced by the automorphism of $\mathbb{R}^3$ with matrix

$$\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}.$$

Consider the coordinate neighbourhood

$$U = \big\{(x, y, z) \in S^2 : x + z \neq 0\big\}.$$

Compute $f_*(\frac{\partial}{\partial\theta}|_p)$ and $f_*(\frac{\partial}{\partial\varphi}|_p)$ for $p \equiv (\theta_0, \varphi_0) \in U$ such that $f(p)$ also belongs to U.

Solution This parametrisation can be described by saying that we have a chart Φ from U to an open subset of $A = (0, \pi) \times (0, 2\pi)$ with

$$\Phi^{-1}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad u, v \in A,$$

and that we call

$$\theta = u \circ \Phi, \qquad \varphi = v \circ \Phi,$$

where u and v are the coordinate functions on A. Then we need to compute $f_*((\partial/\partial\theta)_p)$ and $f_*((\partial/\partial\varphi)_p)$, where $p \in U$. As $f(p) \in U$, we have

$$\begin{aligned}
f_*\left(\frac{\partial}{\partial\theta}\bigg|_p\right) &= \left(\frac{\partial(\theta\circ f)}{\partial\theta}\right)(p)\frac{\partial}{\partial\theta}\bigg|_{f(p)} + \left(\frac{\partial(\varphi\circ f)}{\partial\theta}\right)(p)\frac{\partial}{\partial\varphi}\bigg|_{f(p)} \\
&= \left(\frac{\partial(\theta\circ f\circ\Phi^{-1})}{\partial u}\right)\big(\Phi(p)\big)\frac{\partial}{\partial\theta}\bigg|_{f(p)} \\
&\quad + \left(\frac{\partial(\varphi\circ f\circ\Phi^{-1})}{\partial u}\right)\big(\Phi(p)\big)\frac{\partial}{\partial\varphi}\bigg|_{f(p)}, \\
f_*\left(\frac{\partial}{\partial\varphi}\bigg|_p\right) &= \left(\frac{\partial(\theta\circ f)}{\partial\varphi}\right)(p)\frac{\partial}{\partial\theta}\bigg|_{f(p)} + \left(\frac{\partial(\varphi\circ f)}{\partial\varphi}\right)(p)\frac{\partial}{\partial\varphi}\bigg|_{f(p)} \\
&= \left(\frac{\partial(\theta\circ f\circ\Phi^{-1})}{\partial v}\right)\big(\Phi(p)\big)\frac{\partial}{\partial\theta}\bigg|_{f(p)} \\
&\quad + \left(\frac{\partial(\varphi\circ f\circ\Phi^{-1})}{\partial v}\right)\big(\Phi(p)\big)\frac{\partial}{\partial\varphi}\bigg|_{f(p)}.
\end{aligned} \tag{$\star$}$$

Now,

$$\begin{aligned}
&\big(\theta\circ f\circ\Phi^{-1}\big)(u, v) \\
&\quad = (\theta\circ f)(\sin u\cos v, \sin u\sin v, \cos u) \\
&\quad = \theta\left(\frac{\sqrt{2}}{2}(\sin u\cos v + \cos u), \sin u \sin v, \frac{\sqrt{2}}{2}(-\sin u\cos v + \cos u)\right) \\
&\quad = \arccos\left(\frac{\sqrt{2}}{2}(-\sin u\cos v + \cos u)\right), \\
&\big(\varphi\circ f\circ\Phi^{-1}\big)(u, v) = \arctan\left(\sqrt{2}\,\frac{\sin u\sin v}{\sin u\cos v + \cos u}\right).
\end{aligned}$$

(Notice that, since $x + z \neq 0$ on U, the function arctan is well-defined on U.)

Then, we obtain by calculating and substituting the four partial derivatives in ($\star$) above:

$$f_*\left(\frac{\partial}{\partial\theta}\bigg|_p\right) = \frac{\sin\theta_0 + \cos\theta_0\cos\varphi_0}{\sqrt{1+\sin^2\theta_0\sin^2\varphi_0 + \sin 2\theta_0\cos\varphi_0}}\frac{\partial}{\partial\theta}\bigg|_{f(p)} + \frac{\sqrt{2}\sin\varphi_0}{1+\sin^2\theta_0\sin^2\varphi_0+\sin 2\theta_0\cos\varphi_0}\frac{\partial}{\partial\varphi}\bigg|_{f(p)},$$

$$f_*\left(\frac{\partial}{\partial\varphi}\bigg|_p\right) = \frac{-\sin\theta_0\sin\varphi_0}{\sqrt{1+\sin^2\theta_0\sin^2\varphi_0 + \sin 2\theta_0\cos\varphi_0}}\frac{\partial}{\partial\theta}\bigg|_{f(p)} + \frac{\sqrt{2}(\sin^2\theta_0 + \frac{1}{2}\sin 2\theta_0\cos\varphi_0)}{1+\sin^2\theta_0\sin^2\varphi_0+\sin 2\theta_0\cos\varphi_0}\frac{\partial}{\partial\varphi}\bigg|_{f(p)}.$$

1.5 Critical Points and Values

Problem 1.51 Consider the map

$$\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x^1, x^2, x^3) \mapsto (y^1, y^2) = (x^1x^2,\ x^3).$$

(i) Find the critical points of φ.
(ii) Let S^2 be the unit sphere of $\mathbb{R}^3$. Find the critical points of $\varphi|_{S^2}$.
(iii) Find the set C of critical values of $\varphi|_{S^2}$.
(iv) Does C have zero measure?

The relevant theory is developed, for instance, in Milnor [7].

Solution

(i) The Jacobian matrix $\varphi_* = \left(\begin{smallmatrix} x^2 & x^1 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right)$ has rank $\varphi_* < 2$ if and only if $x^1 = x^2 = 0$, that is, the set of critical points of φ is the x^3-axis.

(ii) Consider the charts defined by the parametrisation

$$x^1 = \sin u\cos v, \qquad x^2 = \sin u \sin v, \qquad x^3 = \cos u,$$

for $u \in (0,\pi)$, $v \in (0, 2\pi)$ and $u \in (0,\pi)$, $v \in (-\pi,\pi)$ (see Remark 1.4). We have

$$y^1 = \frac{1}{2}\sin^2 u \sin 2v, \qquad y^2 = \cos u.$$

So we can write

$$(\varphi|_{S^2})_* \equiv \begin{pmatrix} \frac{1}{2}\sin 2u\sin 2v & \sin^2 u\cos 2v \\ -\sin u & 0\end{pmatrix};$$

thus $\operatorname{rank}(\varphi|_{S^2})_* < 2$ if and only if either $\sin u = 0$ or $\cos 2v = 0$.

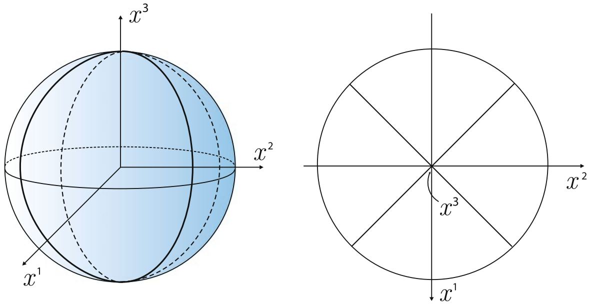


Fig. 1.17 The set of critical points of $\varphi|_{S^2}$

We have $\sin u \neq 0$ in both charts. In the first chart, we have $\cos 2v = 0$ for $v = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. In the second chart, one has $\cos 2v = 0$ for $v = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$. The sets of respective critical points coincide: They are the four half-circles in Fig. 1.17 excluding the poles, due to the parametrisation. Now, we must add the poles as they are critical points of φ by virtue of (i) above.

Hence, the set of critical points of $\varphi|_{S^2}$ is given by the meridians corresponding to $v = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

(iii) Since $\sin 2v = \pm 1$ for $v = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, the set of critical values of $\varphi|_{S^2}$ is

$$C = \left\{ \left(y^1, y^2\right) : y^1 = \pm\frac{1}{2}\sin^2 u,\, y^2 = \cos u \right\},$$

that is, the parabolas

$$2y^1 + \left(y^2\right)^2 = 1, \qquad 2y^1 - \left(y^2\right)^2 = -1.$$

Note that the images of the poles are included.

(iv) A subset S of an n-manifold M has measure zero if it is contained in a countable union of coordinate neighbourhoods U_i such that, φ_i being the corresponding coordinate map, $\varphi_i(U_i \cap S) \subset \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$. This is the case for $C \subset \mathbb{R}^2$, as it is a finite union of 1-submanifolds of $\mathbb{R}^2$.

Problem 1.52

(i) Let $N = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and $M = \mathbb{R}^2$. We define $f : M \to \mathbb{R}$ by $f(x, y) = y^2$. Prove that the set of critical points of $f|_N$ is the intersection with N of the set of critical points of f.

(ii) Let $N = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $M = \mathbb{R}^2$. We define $f : M \to \mathbb{R}$ by $f(x, y) = x^2 + y^2$. Is the set of critical points of $f|_N$ the same as the one of f?

Solution

(i) The set of critical points of f is N and $f|_N$ is the zero map. Thus all the points of N are critical for $f|_N$.

(ii) No. In this case, the set of critical points of f reduces to the origin, but $f|_N = 1$, so all the points of N are critical.

Problem 1.53 Find the critical points and the critical values of the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x + y^2, y + z^2)$.

Solution We have $f_* \equiv \left(\begin{smallmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \end{smallmatrix}\right)$. Since rank $f_* = 2$, f has no critical points, hence it has no critical values.

Problem 1.54 Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \mapsto x \sin y + y \sin z + z \sin x.$$

(i) Prove that $(0, 0, 0)$ is a non-degenerate critical point of f.
(ii) Calculate the index of f at $(0, 0, 0)$.

Solution

(i)

$$f_{*(0,0,0)} \equiv (\sin y + z \cos x,\ x \cos y + \sin z,\ y \cos z + \sin x)_{(0,0,0)} = (0, 0, 0).$$

Thus rank $f_{*(0,0,0)} = 0$, so $(0, 0, 0)$ is a critical point. The Hessian matrix of f at $(0, 0, 0)$ is

$$H^f_{(0,0,0)} = \begin{pmatrix} -z \sin x & \cos y & \cos x \\ \cos y & -x \sin y & \cos z \\ \cos x & \cos z & -y \sin z \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since $\det H^f_{(0,0,0)} = 2 \neq 0$, the point $(0, 0, 0)$ is non-degenerate.

(ii) The index of f at $(0, 0, 0)$ is the index of $H^f_{(0,0,0)}$, that is, the number of negative signs in a diagonal matrix representing the quadratic form $2(x\,y + x\,z + y\,z)$ associated to $H^f_{(0,0,0)}$. Applying the Gauss method of decomposition in squares, one has

$$\begin{aligned} 2x\,y + 2x\,z + 2y\,z &= 2\big((x + z)(y + z) - z^2\big) \\ &= 2\left(\frac{1}{4}(x + y + 2z)^2 - \frac{1}{4}(x - y)^2 - z^2\right) \\ &= \frac{1}{2}(x + y + 2z)^2 - \frac{1}{2}(x - y)^2 - 2z^2. \end{aligned}$$

As two negative signs appear, the index of f at $(0, 0, 0)$ is 2.

Problem 1.55 Consider the C^∞ manifold $\mathbb{R}^n$ and a submanifold L given by a vector subspace of $\mathbb{R}^n$ with $\dim L \leqslant n - 1$. Prove that L has zero measure.

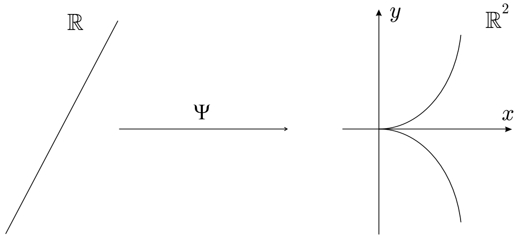


Fig. 1.18 The graph of the map $t \mapsto (t^2, t^3)$

Solution Let $\dim L = k \leqslant n - 1$. Consider the map

$$f : \mathbb{R}^k \to \mathbb{R}^n, \qquad f\big(x^1, x^2, \ldots, x^k\big) = x^i e_i,$$

where $\{e_i\}$ is a basis of L. By virtue of Sard's Theorem, $f(\mathbb{R}^k) = L$ has zero measure.

Problem 1.56 Let M_1 and M_2 be two C^∞ manifolds. Give an example of differentiable mapping $f : M_1 \to M_2$ such that all the points of M_1 are critical points and the set of critical values has zero measure.

Solution Let $f : M_1 \to M_2$ defined by $f(p) = q$, for every $p \in M_1$ and q a fixed point of M_2. Then the rank of f is zero, hence all the points of M_1 are critical. On the other hand, the set of critical values reduces to the point q, and the set $\{q\}$ has obviously zero measure.

1.6 Immersions, Submanifolds, Embeddings and Diffeomorphisms

Problem 1.57 Prove that the C^∞ map

$$\Psi : \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (x, y) = \big(t^2, t^3\big)$$

(see Fig. 1.18) is not an immersion.

Solution

$$\operatorname{rank}\Psi = \operatorname{rank}\left(\frac{\partial x}{\partial t} \;\; \frac{\partial y}{\partial t}\right) = \operatorname{rank}\big(2t \;\; 3t^2\big) = \begin{cases} 1 & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

For $t = 0$, we have $\operatorname{rank}\Psi = 0 < \dim\mathbb{R} = 1$, thus Ψ is not an immersion. Let us consider Ψ_{*t_0} in detail, as a map between tangent vector spaces. We have

$$\begin{aligned} \Psi_*: \qquad T_{t_0}\mathbb{R} &\to T_{\Psi(t_0)}\mathbb{R}^2 \\ \lambda \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0} &\mapsto \Psi_*\left(\lambda \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) \end{aligned}$$

and

$$\begin{aligned}
\Psi_*\left(\lambda \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}\right) &= \lambda\left(\frac{\partial(x\circ\Psi)}{\partial t}(t_0)\frac{\partial}{\partial x}\Big|_{\Psi(t_0)} + \frac{\partial(y\circ\Psi)}{\partial t}(t_0)\frac{\partial}{\partial y}\Big|_{\Psi(t_0)}\right)\\
&= \lambda\left(2t_0\frac{\partial}{\partial x}\Big|_{\Psi(t_0)} + 3t_0^2\frac{\partial}{\partial y}\Big|_{\Psi(t_0)}\right) \equiv \left(2\lambda t_0, 3\lambda t_0^2\right)\\
&= \begin{cases} (0,0)\in T_{(0,0)}\mathbb{R}^2 & \forall\lambda \text{ if } t_0 = 0,\\ (0,0)\in T_{\Psi(t_0)}\mathbb{R}^2 & \text{if } \lambda = 0,\\ \neq (0,0)\in T_{\Psi(t_0)}\mathbb{R}^2 & \text{if } t_0,\lambda \neq 0. \end{cases}
\end{aligned}$$

That is, $\Psi_*(T_0\mathbb{R}) = (0,0)\in T_{(0,0)}\mathbb{R}^2$. The whole tangent space $T_0\mathbb{R}$ is mapped by Ψ_{*0} onto only one point of the tangent space $T_{(0,0)}\mathbb{R}^2$.

Problem 1.58 Let $M = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. Define a C^∞ map by

$$f\colon M\to\mathbb{R}^2, \qquad (x,y)\mapsto\left(\frac{y}{1-x^2-y^2}, \mathrm{e}^{x^2}\right).$$

(i) Find the set S of points p of M at which f_{*p} is injective.
(ii) Prove that $f(S)$ is an open subset of $\mathbb{R}^2$.

Solution

(i) One has

$$\begin{aligned}
\operatorname{rank} f_* < 2 \quad &\Longleftrightarrow \quad 2x\,\mathrm{e}^{x^2}\frac{1-x^2+y^2}{(1-x^2-y^2)^2} = 0\\
&\Longleftrightarrow \quad x = 0 \quad \text{or} \quad 1 = x^2 - y^2.
\end{aligned}$$

Since $1 > x^2+y^2$, we have $1 > x^2-y^2$, so $S = M\setminus\{(0,y) : -1<y<1\}$.

(ii) Consider the subset $\{(0,y) : -1<y<1\}$ of M. We have

$$f\left(\{(0,y) : -1<y<1\}\right) = \left\{\left(\frac{y}{1-y^2}, 1\right)\right\} = (-\infty,\infty)\times\{1\}\subset\mathbb{R}^2.$$

Thus $f(M) = \{(x,y)\in\mathbb{R}^2 : 1\leqslant y<\mathrm{e}\}$, hence

$$f(S) = \left\{(x,y)\in\mathbb{R}^2 : 1<y<\mathrm{e}\right\},$$

which is an open subset of $\mathbb{R}^2$.

Problem 1.59 Let $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ be the C^∞ manifolds defined, respectively, by the differentiable structures obtained from the atlases $\{(\mathbb{R},\mathrm{id})\}$ and $\{(\mathbb{R},\varphi)\}$ on $\mathbb{R}$, where $\varphi\colon\mathbb{R}\to\mathbb{R}$, $\varphi(t) = t^3$. Prove that $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ are diffeomorphic (see Problem 1.22).

Solution To prove that $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ are diffeomorphic, we only have to give a map Φ such that its representative Ψ in the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\Phi} & \mathbb{R} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \mathbb{R} & \xrightarrow{\Psi} & \mathbb{R} \end{array}$$

be a diffeomorphism. Let $\Phi(t) = \sqrt[3]{t}$. One has $\Psi(t) = \varphi \circ \Phi \circ \mathrm{id}^{-1}(t) = t$.

Problem 1.60 Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $x = \mathrm{e}^u \cos v$, $y = \mathrm{e}^u \sin v$.

(i) Prove that the Jacobian matrix determinant of f does not vanish at any point of the plane.
(ii) Can f be taken as a local coordinate map on a neighbourhood of any point?
(iii) Is f a diffeomorphism?
(iv) Given a point $p_0 = (u_0, v_0)$, give an example of a maximal open neighbourhood of p_0 on which we can take f as a local coordinate map.

Solution

(i) Notice that $x^2 + y^2 = \mathrm{e}^{2u} > 0$; so that $f(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for all $(u,v) \in \mathbb{R}^2$. We have

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^u \cos v & -\mathrm{e}^u \sin v \\ \mathrm{e}^u \sin v & \mathrm{e}^u \cos v \end{pmatrix},$$

hence $\partial(x,y)/\partial(u,v) = \mathrm{e}^{2u} > 0$ for all $(u,v) \in \mathbb{R}^2$.

(ii) By (i), f is a local diffeomorphism at every point of $\mathbb{R}^2$. So f can be taken as a local coordinate map on a neighbourhood of every point.

(iii) The map f is not a diffeomorphism as it is not injective. We have $f(u,v) = f(u',v')$ if and only if $u = u'$ and $v - v' = 2k\pi$, $k \in \mathbb{Z}$. In fact, from the relations

$$\mathrm{e}^u \cos v = \mathrm{e}^{u'} \cos v', \qquad \mathrm{e}^u \sin v = \mathrm{e}^{u'} \sin v',$$

we obtain $\mathrm{e}^{2u} = \mathrm{e}^{2u'}$, and so $u = u'$. Then one has $\cos v = \cos v'$, $\sin v = \sin v'$, hence the difference between v and v' is an integer multiple of 2π.

(iv) The points having the same image as p_0 are the ones of the form $(u_0, v_0 + 2k\pi)$, $k \in \mathbb{Z}$. The nearest ones to p_0 are $(u_0, v_0 \pm 2\pi)$. Hence such a neighbourhood is $\mathbb{R} \times (v_0 - \pi, v_0 + \pi)$.

Problem 1.61 Let V be a finite-dimensional real vector space. Consider the open subset $\mathscr{E}$ of $\mathrm{End}_{\mathbb{R}} V$ defined by

$$\mathscr{E} = \{T \in \mathrm{End}_{\mathbb{R}} V : \det(I + T) \neq 0\},$$

where I denotes the identity endomorphism.

(i) Prove that the map

$$f\colon \mathscr{E} \to \operatorname{End}_{\mathbb{R}} V, \qquad T \mapsto (I-T)(I+T)^{-1},$$

is an involution of $\mathscr{E}$.

(ii) Consider on $\mathscr{E}$ the differentiable structure induced by $\operatorname{End}_{\mathbb{R}} V$. Prove that $f\colon \mathscr{E} \to \mathscr{E}$ is a diffeomorphism.

Solution

(i) If $T \in \mathscr{E}$, then $I + f(T) = 2I(I+T)^{-1} \in \mathscr{E}$. Hence

$$\det\bigl(I + f(T)\bigr) = \det\bigl(2I(I+T)^{-1}\bigr) = 2^{\dim V}/\det(I+T) \neq 0.$$

It is easily checked that $f(f(T)) = T$.

(ii) The map f is C^∞. In fact, the entries of $f(T)$ can be expressed as rational functions of the entries of T. As $f^{-1} = f$, we conclude.

Problem 1.62 Prove that the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) = \bigl(x\,\mathrm{e}^y + y,\; x\,\mathrm{e}^y - y\bigr),$$

is a C^∞ diffeomorphism.

Solution Solving the system

$$x\,\mathrm{e}^y + y = x', \qquad x\,\mathrm{e}^y - y = y',$$

in x and y, we conclude that the unique solution is

$$x = \frac{x'+y'}{2\mathrm{e}^{(x'-y')/2}}, \qquad y = \frac{x'-y'}{2};$$

hence the map is one-to-one. Let us see that both f and f^{-1} are C^∞. We have

$$f\colon (x,y) \mapsto \bigl(x\,\mathrm{e}^y + y, x\,\mathrm{e}^y - y\bigr), \qquad f^{-1}\colon (x,y) \mapsto \left(\frac{x+y}{2}\mathrm{e}^{(y-x)/2}, \frac{x-y}{2}\right).$$

Since the components of f and f^{-1} and their derivatives of any order are elementary functions, f and f^{-1} are C^∞. Thus f is a C^∞ diffeomorphism.

Problem 1.63 Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by

$$x' = \mathrm{e}^{2y} + \mathrm{e}^{2z}, \qquad y' = \mathrm{e}^{2x} - \mathrm{e}^{2z}, \qquad z' = x - y.$$

Find the image set $\varphi(\mathbb{R}^3)$ and prove that φ is a diffeomorphism from $\mathbb{R}^3$ to $\varphi(\mathbb{R}^3)$.

Solution Solving, one has

$$x = z' + y, \qquad x' = e^{2y} + e^{2z}, \qquad y' = e^{2z'} e^{2y} - e^{2z},$$

and so

$$e^{2y} = \frac{x' + y'}{1 + e^{2z'}}, \qquad e^{2z} = \frac{x' e^{2z'} - y'}{1 + e^{2z'}}.$$

Hence it must be $x' > 0$, $x' + y' > 0$, $x' e^{2z'} > y'$.

Thus,

$$\varphi(\mathbb{R}^3) = \{(x, y, z) \in \mathbb{R}^3 : x > 0,\ x + y > 0,\ x\, e^{2z} > y\}.$$

The map φ is injective, since the above formulae give the unique point (x, y, z) having (x', y', z') as its image by φ.

In order to see that φ is a diffeomorphism from $\mathbb{R}^3$ to $\varphi(\mathbb{R}^3)$, it suffices to prove that the determinant of its Jacobian matrix J_φ never vanishes. We have

$$\det J_\varphi = \det \begin{pmatrix} 0 & 2e^{2y} & 2e^{2z} \\ 2e^{2x} & 0 & -2e^{2z} \\ 1 & -1 & 0 \end{pmatrix} = -4\left(e^{2y+2z} + e^{2x+2z}\right) \neq 0.$$

Problem 1.64 Consider the C^∞ function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (x \cos z - y \sin z,\ x \sin z + y \cos z,\ z).$$

Prove that $f|_{S^2}$ is a diffeomorphism from the unit sphere S^2 onto itself.

Solution For each $(x, y, z) \in S^2$, one has $f(x, y, z) \in S^2$, so that $(f|_{S^2})(S^2) \subset S^2$. Furthermore, given $(u, v, w) \in S^2$, we have to prove that there exists $(x, y, z) \in S^2$ such that $f(x, y, z) = (u, v, w)$, that is,

$$x \cos z - y \sin z = u, \qquad x \sin z + y \cos z = v, \qquad z = w.$$

Solving this system in x, y, z, we have

$$x = u \cos w + v \sin w, \qquad y = -u \sin w + v \cos w, \qquad z = w.$$

These equations are the ones of the components of the inverse function of $f|_{S^2}$, which is clearly C^∞, hence $f|_{S^2}$ is a diffeomorphism.

Problem 1.65 Let $\{(E, \varphi)\}$ and $\{(E, \psi)\}$ be the atlases on the "Figure Eight" built in Problem 1.35. Exhibit a diffeomorphism between the differentiable manifolds E_φ and E_ψ defined by the differentiable structures obtained from the atlases $\{(E, \varphi)\}$ and $\{(E, \psi)\}$, respectively.

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution Let

$$f\colon E_\varphi \to E_\psi, \qquad f(\sin 2s, \sin s) = \bigl(\sin 2(s-\pi), \sin(s-\pi)\bigr).$$

Since $(\psi \circ f \circ \varphi^{-1})(s) = s - \pi$, it follows that f is a diffeomorphism.

Problem 1.66 Let (N, φ), (N, ψ) be the atlases on the "Noose" built in Problem 1.36. Exhibit a diffeomorphism between the differentiable manifolds N_φ and N_ψ defined, respectively, by the differentiable structures obtained from the atlases $\{(N, \varphi)\}$ and $\{(N, \psi)\}$.

The relevant theory is developed, for instance, in Brickell and Clark [1].

Solution The map $f\colon (N, \varphi) \to (N, \psi)$, $(x, y) \mapsto (-x, y)$, mapping a point to its symmetric with respect to the y-axis, is a diffeomorphism. One has

$$\begin{aligned} (-1,1) \xrightarrow{\varphi^{-1}} (N, \varphi) &\xrightarrow{f} \quad (N, \psi) \quad \xrightarrow{\psi} (-1,1) \\ s \longmapsto \varphi^{-1}(s) &\longmapsto f\bigl(\varphi^{-1}(s)\bigr) \longmapsto s. \end{aligned}$$

In fact,

$$\varphi^{-1}(s) = \begin{cases} (0, 1-s) & \text{if } -1 < s < 0, \\ (\sin 2\pi s, \cos 2\pi s) & \text{if } 0 \leqslant s < 1, \end{cases}$$

$$f\bigl(\varphi^{-1}(s)\bigr) = \begin{cases} (0, 1-s) & \text{if } -1 < s < 0, \\ (\sin 2\pi(1-s), \cos 2\pi(1-s)) & \text{if } 0 \leqslant s < 1, \end{cases}$$

and

$$\bigl(\psi \circ f \circ \varphi^{-1}\bigr)(s) = s, \qquad s \in (-1, 1).$$

Problem 1.67 The aim of the present problem is to prove that the manifold of affine straight lines of the plane, the 2-dimensional real projective space minus a point, and the infinite Möbius strip are diffeomorphic. Explicitly:

(a) Let M denote the *set of affine straight lines of the plane*, that is,

$$M = \bigl\{r(a, v) : a, v \in \mathbb{R}^2, v \neq 0\bigr\},$$

where

$$r(a, v) = \{a + tv : t \in \mathbb{R}\} \subset \mathbb{R}^2$$

is the (affine) straight line of $\mathbb{R}^2$ determined by a and v.

Consider, for each $p \in \mathbb{R}^2$, the set

$$U_p = \{L \in M : p \notin L\}, \qquad A_p = \mathbb{R}^2 \backslash \{p\}, \qquad \varphi_p\colon U_p \to A_p,$$

where $\varphi_p(L)$, for $L \in U_p$, is the foot of the perpendicular from p to L (see Fig. 1.19, left).

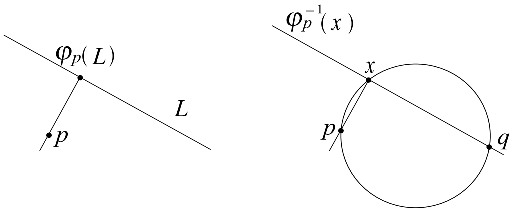


Fig. 1.19 Charts for the affine straight lines

(b) Let P_0 denote the *punctured projective space*, that is, $\mathbb{R}P^2 \backslash \{\pi(0,0,1)\}$, where

$$\pi : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}P^2$$

stands for the natural map sending a point to its equivalence class (see Problem 1.81).

Let $\pi(w) \in P_0$. Then w is a non-zero vector of $\mathbb{R}^3$, which is not parallel to the axis z; hence the plane orthogonal to w through the point $(0,0,1)$ intersects the plane xy (which we shall identify with $\mathbb{R}^2$) in a straight line, that is in a point of M, which we shall denote by $f(\pi(w))$. We thus have a map from P_0 to the set of affine straight lines of the plane,

$$f : P_0 \to M,$$

which is well-defined since $f(\pi(w))$ only depends on the direction determined by w, that is, on $\pi(w)$.

Moreover, let

$$V_1 = \{\pi(w_1, w_2, w_3) : w_1 \neq 0\}, \qquad \psi_1\big(\pi(w_1, w_2, w_3)\big) = \left(\frac{w_2}{w_1}, \frac{w_3}{w_1}\right),$$

$$V_2 = \{\pi(w_1, w_2, w_3) : w_2 \neq 0\}, \qquad \psi_2\big(\pi(w_1, w_2, w_3)\big) = \left(\frac{w_1}{w_2}, \frac{w_3}{w_2}\right),$$

if $\pi(w_1, w_2, w_3)$ belongs to either V_1 or V_2, respectively. Then (cf. Problem 1.81), the set $\{(V_i, \psi_1)\}$, $i = 1, 2$, is an atlas for P_0.

(c) Let $\widetilde{M}$ denote the infinite Möbius strip, defined, as in Problem 1.31 above, as the infinite strip $[0, \pi] \times \mathbb{R}$ under the identification $(0, t) \equiv (\pi, -t)$ for all $t \in \mathbb{R}$.

Consider the one-to-one map h between $\widetilde{M}$ and P_0 defined as follows. Let $\mu : [0, \pi] \times \mathbb{R} \to \widetilde{M}$ be the natural projection, so that $\mu(\alpha, t) = \{(\alpha, t)\}$ for $\alpha \in (0, \pi)$, and $\mu(0, t) = \mu(\pi, -t) = \{(0, t), (\pi, -t)\}$, and define

$$\begin{aligned} h: \qquad \widetilde{M} &\longrightarrow P_0 \\ \mu(\alpha, t) &\longmapsto h\big(\mu(\alpha, t)\big) = \pi(\cos\alpha, \sin\alpha, t). \end{aligned}$$

This map is well-defined. This is obvious if $\alpha \in (0, \pi)$, and

$$h\big(\mu(0, t)\big) = \pi(1, 0, t), \qquad h\big(\mu(\pi, -t)\big) = \pi(-1, 0, -t) = \pi(1, 0, t).$$

Then:

1. Prove that

$$\{(U_p, \varphi_p)\}_{p\in\mathbb{R}^2}$$

with U_p and φ_p as given in (a), is an atlas on M.
2. Prove, by using the atlas given in part 1, that the map f in (b) is a diffeomorphism from M to P_0.
3. Prove that one can endow $\widetilde{M}$ with a structure of differentiable manifold by means of the bijective map h given in (c), which becomes so a diffeomorphism.

Hint (to (a)) According to [5, p. 17], if a collection of couples (U_i, φ_i) (i describing a certain set of indices) on a set X is given such that:

(i) Any U_i is a subset of X and $X = \bigcup U_i$.
(ii) Any φ_i is a bijection of U_i on an open subset $\varphi_i(U_i)$ of $\mathbb{R}^n$, and for all i, j, the set $\varphi_i(U_i \cap U_j)$ is an open subset of $\mathbb{R}^n$,
(iii) For any pair i, j, the map $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a diffeomorphism,

then X admits a unique structure of topological space such that any U_i is an open subset and the maps φ_i are homeomorphisms.

Solution

1. Given a point $x \in A_p$, that is, different from p, there exists only one straight line perpendicular to the segment px passing through x. This straight line is precisely $\varphi_p^{-1}(x)$. This proves that φ_p is bijective. Since obviously A_p is an open subset of $\mathbb{R}^2$ and $\bigcup_{p\in\mathbb{R}^2} U_p = M$, it only rests to check the differentiability of the changes of charts to verify that

$$\{(U_p, \varphi_p)\}_{p\in\mathbb{R}^2}$$

is an atlas.

To check it, let $p, q \in \mathbb{R}^2$, $p \neq q$ (see Fig. 1.19, right). We have

$$U_p \cap U_q = \{L \in M : p \notin L,\ q \notin L\}.$$

To see that $\varphi_p(U_p \cap U_q)$, which is the domain of the map $\varphi_q \circ \varphi_p^{-1}$, is an open subset of A_p, we shall prove that its complementary is a closed subset. Let x be a point in its complementary subset, that is, such that x does not belong to the domain of $\varphi_q \circ \varphi_p^{-1}$. Then, either x does not belong to the domain of φ_p^{-1}, which is A_p, that is, $x = p$; or contrarily $x \neq p$, but $\varphi_p^{-1}(x)$ does not belong to the domain of φ_q, that is, to U_q. But this happens if and only if $q \in \varphi_p^{-1}(x)$, that is, q belongs to the straight line through x perpendicular to the segment xp, that is, if and only if x sees the segment pq under a $\pi/2$ angle. So x belongs to the

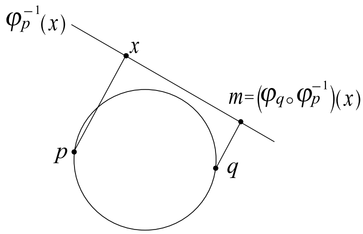


Fig. 1.20 Change of charts for the affine straight lines

circle C_{pq} with diameter being the segment pq. Summarising, either $x = p$ (so also $x \in C_{pq}$), or $x \neq p$ and $x \in C_{pq}$; in short, $x \in C_{pq}$. Since C_{pq} is a closed subset of $\mathbb{R}^2$, we have proved that $\varphi_p(U_p \cap U_q)$ is an open subset of A_p.

It only remains to prove the differentiability of the map $\varphi_q \circ \varphi_p^{-1}$. To this end, let $J : \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation of angle $\pi/2$, expressed by

$$J(a, b) = (-b, a),$$

and suppose that $x \notin C_{pq}$. Then

$$\varphi_p^{-1}(x) = r\big(x, J(x - p)\big),$$

so

$$\big(\varphi_q \circ \varphi_p^{-1}\big)(x) = \varphi_q\big(r\big(x, J(x - p)\big)\big).$$

Denote this point simply by m (see Fig. 1.20). Since m belongs to the straight line $r(x, J(x - p))$, we can put

$$m = x + tJ(x - p),$$

with t such that $\langle q - m, J(x - p)\rangle = 0$. Hence, since J is an isometry, we get

$$t = \frac{\langle q - x, J(x - p)\rangle}{\langle x - p, x - p\rangle}, \qquad \big(\varphi_q \circ \varphi_p^{-1}\big)(x) = x + \frac{\langle q - x, J(x - p)\rangle}{\langle x - p, x - p\rangle}\, J(x - p),$$

which is C^∞, for the scalar product is a polynomial in the components of its factors, so the components of $(\varphi_q \circ \varphi_p^{-1})(x)$ are rational functions of the components of x.

Consequently, we have proved that $\{(U_p, \varphi_p)\}_{p\in\mathbb{R}^2}$ is an atlas on M, which is thus a 2-dimensional C^∞ manifold when endowed with the differentiable structure corresponding to the given atlas.

2. We have seen that f is well-defined. Conversely, each point $L \in M$, that is, each straight line in the plane xy, determines with the point $(0, 0, 1) \in \mathbb{R}^3$ a plane which cannot be parallel to the plane xy, so its normal straight line $\pi(w)$ is not parallel to $(0, 0, 1)$. In other words, $\pi(w) = f^{-1}(L)$, so proving that f is a bijective map.

Let us compute f and f^{-1}. Let $L = r(a, v) \in M$. The plane of $\mathbb{R}^3$ containing L and passing through the point $(0, 0, 1)$ must be parallel to the director vector of L, which is $(v, 0)$, and also to the vector $(a, 0) - (0, 0, 1) = (a, -1)$, (where we use the notation $(x, b) \in \mathbb{R}^3$, for $x = (x_1, x_2) \in \mathbb{R}^2$, to denote the point (x_1, x_2, b)). Since these two vectors are linearly independent, we have

$$\begin{aligned} f^{-1}(L) = f^{-1}\bigl(r(a, v)\bigr) &= \pi\bigl((v, 0) \wedge (a, -1)\bigr) = \pi(-v_2, v_1, -v_2 a_1 + v_1 a_2) \\ &= \pi\bigl(Jv, \langle Jv, a\rangle\bigr). \end{aligned}$$

Conversely, let

$$\pi(w) = \pi(w_1, w_2, w_3) \in P_0$$

and, to be short, write $\bar{w} = (w_1, w_2) \in \mathbb{R}^2$, so that $w = (\bar{w}, w_3)$. If $f(\pi(w)) = r(a, v)$, we should have

$$\pi(\bar{w}, w_3) = \pi\bigl(Jv, \langle Jv, a\rangle\bigr).$$

We put $Jv = \bar{w}$, that is $v = -J\bar{w}$, so one should have

$$w_3 = \langle Jv, a\rangle = \langle \bar{w}, a\rangle.$$

As $\langle \bar{w}, \bar{w}\rangle \neq 0$, for in the opposite case $\pi(w) \notin P_0$, we can get that condition letting simply

$$a = \frac{w_3 \bar{w}}{\langle \bar{w}, \bar{w}\rangle},$$

that is,

$$f\bigl(\pi(w)\bigr) = f\bigl(\pi(\bar{w}, w_3)\bigr) = r\left(\frac{w_3 \bar{w}}{\langle \bar{w}, \bar{w}\rangle}, -J\bar{w}\right),$$

which is the expression we looked for.

Now consider one of the charts of M,

$$\varphi_p \colon U_p \to A_p,$$

and compute

$$\bigl(\varphi_p \circ f \circ \psi_1^{-1}\bigr)(x), \quad x = (x_1, x_2) \in \mathbb{R}^2 \text{ in its domain.}$$

We have $\psi_1^{-1}(x) = \pi(1, x_1, x_2)$. Hence

$$f\bigl(\psi_1^{-1}(x)\bigr) = r\left(\frac{x_2(1, x_1)}{1 + x_1^2}, (x_1, -1)\right).$$

Now, $f(\psi_1^{-1}(x))$ belongs to the domain U_p of φ_p if and only if the straight line

$$r\left(\frac{x_2(1,x_1)}{1+x_1^2},(x_1,-1)\right)$$

does not contain p. That is, x does not belong to the domain of the map $\varphi_p\circ\psi_1^{-1}$ if and only if p belongs to that straight line, i.e. if

$$p-\frac{x_2(1,x_1)}{1+x_1^2}$$

is parallel to $(x_1,-1)$, or equivalently, if and only if it is orthogonal to $J(x_1,-1)$, or even if and only if

$$\left\langle p-\frac{x_2(1,x_1)}{1+x_1^2},(1,x_1)\right\rangle=0.$$

This equation is polynomial in the components of x, so that the set of points satisfying the condition is a closed subset of $\mathbb{R}^2$, as we wanted. Suppose then that x belongs to the domain of $\varphi_p\circ f\circ\psi_1^{-1}$, which, as we have just shown, is an open subset of $\mathbb{R}^2$. Then

$$\left(\varphi_p\circ f\circ\psi_1^{-1}\right)(x)$$

can be written as $a+tv$, where

$$a=\frac{x_2(1,x_1)}{1+x_1^2},\qquad v=(x_1,-1),$$

and t such that $\langle p-(a+tv),v\rangle=0$, that is,

$$\left(\varphi_p\circ f\circ\psi_1^{-1}\right)(x)=a+\frac{\langle p-a,v\rangle}{\langle v,v\rangle}v.$$

All the expressions involved in this formula through a and v are polynomial in the components of x, so the map

$$\varphi_p\circ f\circ\psi_1^{-1}$$

is of class C^∞. The proof for $\varphi_p\circ f\circ\psi_2^{-1}$ is similar, hence f is C^∞.

Differentiability of f^{-1} is easier to prove. Let $x=(x_1,x_2)\in A_p$, such that $x\neq p$. We have

$$\begin{aligned}\left(f^{-1}\circ\varphi_p^{-1}\right)(x)&=f^{-1}\left(r\left(x,J(x-p)\right)\right)=\pi\left(p-x,\langle x,p-x\rangle\right)\\&=\pi\left(p_1-x_1,\ p_2-x_2,\ \langle x,p-x\rangle\right),\end{aligned}$$

which belongs to the domain of, say, ψ_1, if and only if $p_1 \neq x_1$, that is, the domain of $\psi_1 \circ f^{-1} \circ \varphi_p^{-1}$ is the open subset $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq p_1\}$. For this open subset we have

$$\left(\psi_1 \circ f^{-1} \circ \varphi_p^{-1}\right)(x) = \left(\frac{p_2 - x_2}{p_1 - x_1}, \frac{\langle x, p - x\rangle}{p_1 - x_1}\right),$$

clearly showing the differentiability of that map; the proof for ψ_2 is similar, so we have proved that f^{-1} also is a differentiable map, that is, that

$$f \colon P_0 \to M$$

is a diffeomorphism.

3. The map h is surjective, for if $\pi(w) = \pi(\bar{w}, w_3) \in P_0$, one should have $\bar{w} \neq 0$. We can thus normalise the representative element, letting

$$\pi(w) = \pi\left(\frac{\bar{w}}{|\bar{w}|}, \frac{w_3}{|\bar{w}|}\right).$$

As $\bar{w}/|\bar{w}|$ is a unitary vector, there exists only one $\alpha \in [0, 2\pi)$ such that

$$\frac{\bar{w}}{|\bar{w}|} = (\cos\alpha, \sin\alpha),$$

so that

$$\pi(w) = \pi\left(\cos\alpha, \sin\alpha, \frac{w_3}{|\bar{w}|}\right).$$

If $\alpha \in (0, \pi)$, we have

$$\pi(w) = h\left(\mu\left(\alpha, \frac{w_3}{|\bar{w}|}\right)\right);$$

if $\alpha \in (\pi, 2\pi)$, we have

$$\begin{aligned} h\left(\mu\left(\alpha - \pi, -\frac{w_3}{|\bar{w}|}\right)\right) &= \pi\left(-\cos\alpha, -\sin\alpha, -\frac{w_3}{|\bar{w}|}\right) \\ &= \pi\left(\cos\alpha, \sin\alpha, \frac{w_3}{|\bar{w}|}\right) = \pi(w); \end{aligned}$$

if $\alpha = 0$, then

$$\pi(w) = \pi\left(1, 0, \frac{w_3}{|\bar{w}|}\right) = h\left(\mu\left(0, \frac{w_3}{|\bar{w}|}\right)\right);$$

finally, if $\alpha = \pi$, then

$$\pi(w) = \pi\left(-1, 0, \frac{w_3}{|\bar{w}|}\right) = h\left(\mu\left(\pi, \frac{w_3}{|\bar{w}|}\right)\right).$$

In other words, h is surjective. It is also injective, for if

$$h\big(\mu(\alpha, t)\big) = h\big(\mu(\beta, s)\big),$$

we have

$$\pi(\cos\alpha, \sin\alpha, t) = \pi(\cos\beta, \sin\beta, s).$$

As the two first coordinates of these representative elements constitute unitary vectors, and these have to be a multiple of each other, we necessarily have

$$(\cos\alpha, \sin\alpha, t) = \pm(\cos\beta, \sin\beta, s);$$

hence, as $\alpha, \beta \in [0, \pi]$, either $\alpha = \beta$ and $t = s$; or $\alpha = 0$, $\beta = \pi$, $t = -s$, and so $\mu(\alpha, t) = \mu(\beta, s)$; or $\alpha = \pi$, $\beta = 0$, $t = -s$, and so $\mu(\alpha, t) = \mu(\beta, s)$. The map h is injective hence bijective.

So, one can indeed endow $\widetilde{M}$ with a structure of differentiable manifold by means of the bijective map h, which thus becomes a diffeomorphism.

Problem 1.68 Prove that the map

$$p\colon \mathbb{R} \to S^1, \qquad t \mapsto (\cos 2\pi t, \sin 2\pi t),$$

is a covering map.

Solution We must prove:

(i) p is C^∞ and surjective.
(ii) For each $x \in S^1$, there exists a neighbourhood U of x in S^1 such that $p^{-1}(U) = \bigcup U_i$, $i \in I$, where the U_i are disjoint open subsets of $\mathbb{R}$ such that, for each $i \in I$, $p\colon U_i \to U$ is a diffeomorphism.

Now, (i) is immediate. Moreover p is a local diffeomorphism.

As for (ii), let $y \in \mathbb{R}$; then

$$p\colon (y - \pi, y + \pi) \to S^1 \setminus \{p(y + \pi)\}$$

is a diffeomorphism and

$$p^{-1}\big(S^1 \setminus \{p(y + \pi)\}\big) = \bigcup_{k \in \mathbb{Z}} \big(y + (2k - 1)\pi, y + (2k + 1)\pi\big).$$

Of course, one can take smaller intervals as domains of the diffeomorphisms.

Problem 1.69 Consider the curves:

(a)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto \big(t, |t|\big),$$

(b)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto \big(t^3 - 4t, t^2 - 4\big),$$

(c)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (\cos 2\pi t, \sin 2\pi t, t),$$

(d)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto (\cos 2\pi t, \sin 2\pi t),$$

(e)
$$\sigma\colon (1, \infty) \to \mathbb{R}^2$$
$$t \mapsto \left(\frac{1}{t}\cos 2\pi t, \frac{1}{t}\sin 2\pi t\right),$$

(f)
$$\sigma\colon (1, \infty) \to \mathbb{R}^2$$
$$t \mapsto \left(\frac{1+t}{2t}\cos 2\pi t, \frac{1+t}{2t}\sin 2\pi t\right),$$

(g)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \to \left(2\cos\left(t - \frac{\pi}{2}\right), \sin 2\left(t - \frac{\pi}{2}\right)\right),$$

(h)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto \left(2\cos\left(f(t) - \frac{\pi}{2}\right), \sin 2\left(f(t) - \frac{\pi}{2}\right)\right),$$

where $f(t)$ denotes a monotonically increasing C^∞ function on $-\infty < t < \infty$ such that $f(0) = \pi$, $\lim_{t\to-\infty} f(t) = 0$ and $\lim_{t\to\infty} f(t) = 2\pi$ (for instance, $f(t) = \pi + 2\arctan t$).

(i)
$$\sigma\colon \mathbb{R} \to \mathbb{R}^2$$
$$t \;\mapsto \begin{cases} (1/t, \sin\pi\, t) & \text{if } 1 \leqslant t < \infty, \\ (0, t+2) & \text{if } -\infty < t \leqslant -1, \end{cases}$$

where in addition one smoothly connects, for $-1 \leqslant t \leqslant 1$, the two curves $\sigma|_{(-\infty,-1]}$ and $\sigma|_{[1,\infty)}$ with a C^∞ curve.

1. Is σ an immersion in (a)? (resp., in (b), (d), (g))?
2. Is σ an injective immersion in (b) (resp., in (d), (g), (h), (i))?
3. Is σ an embedding in (c)? (resp., in (e), (f), (h), (i))?

The relevant theory is developed, for instance, in Warner [8].

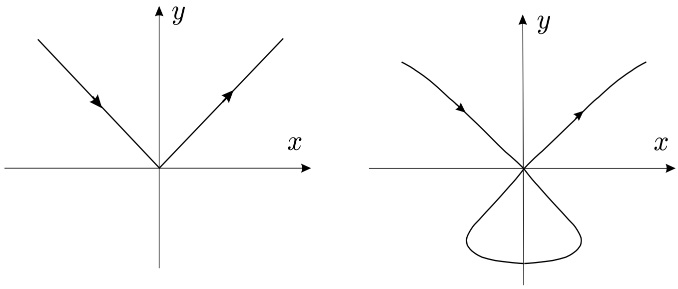


Fig. 1.21 (a) σ is not an immersion. (b) σ is a non-injective immersion

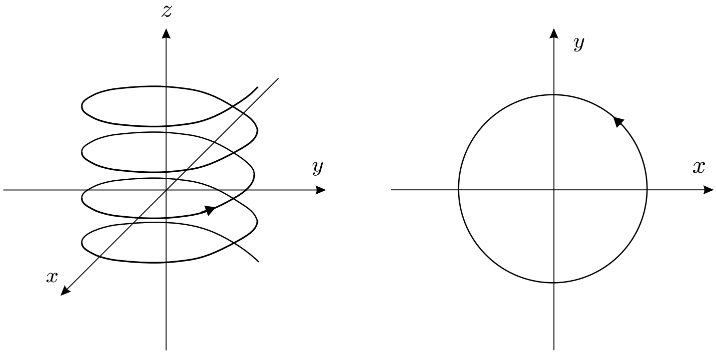


Fig. 1.22 (c) σ is an embedding. (d) σ is a non-injective immersion

Solution

(a) σ is not an immersion, as it is not a differentiable map at the origin (see Fig. 1.21).

We recall that

$$\sigma'(t_0) = \sigma_{*t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \right),$$

that is, $\sigma'(t_0)$ is the image of the canonical vector at $t_0 \in \mathbb{R}$.

(b) σ is a differentiable map, and since $\sigma'(t) = (3t^2 - 4, 2t) \neq (0, 0)$ for all t, the map σ is an immersion. But for $t = \pm 2$, it has a self-intersection, so it is not an injective immersion.

(c) σ is an immersion, as

$$\sigma'(t) = (-2\pi \sin 2\pi\, t, 2\pi \cos 2\pi\, t, 1) \neq (0, 0, 0), \quad t \in \mathbb{R}.$$

It is trivially injective and since the map $\sigma : \mathbb{R} \to \sigma(\mathbb{R})$ is open, σ is an embedding (see Fig. 1.22).

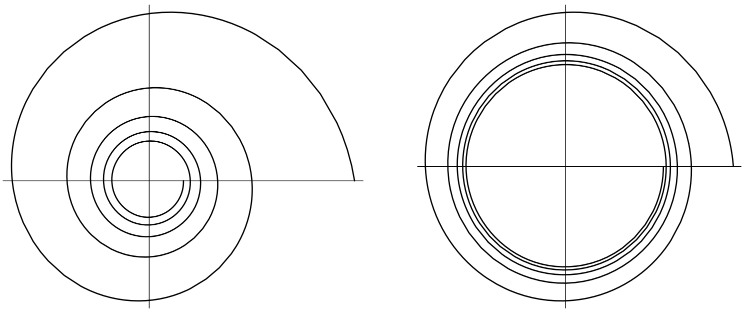


Fig. 1.23 (e) σ is an embedding. (f) σ is an immersion

(d) σ is an immersion since

$$\sigma'(t) = (-2\pi \sin 2\pi\, t, 2\pi \cos 2\pi\, t) \neq (0,0)$$

for all t, but σ is obviously not injective. Nevertheless, $\sigma(\mathbb{R})$ is an embedded submanifold. (See Problems 1.25 and 1.71.)

(e) σ is an immersion, as

$$\sigma'(t) = \left(-\frac{1}{t^2}\cos 2\pi\, t - \frac{2\pi}{t}\sin 2\pi\, t, -\frac{1}{t^2}\sin 2\pi\, t + \frac{2\pi}{t}\cos 2\pi\, t\right) = (0,0)$$

if and only if each component vanishes or, equivalently, the square of each component, or even the sum of those squares vanishes, that is, $(1/t^4) + 4\pi^2/t^2 = 0$, or $1 + 4t^2\pi^2 = 0$, which leads us to a contradiction. Since $\sigma\colon (1,\infty) \to \sigma(1,\infty)$ is an injective and open map, it follows that σ is an embedding (see Fig. 1.23).

(f) σ is an immersion, as

$$\sigma'(t) = \left(-\frac{\cos 2\pi\, t}{2t^2} - \frac{t+1}{t}\pi \sin 2\pi\, t, -\frac{\sin 2\pi\, t}{2t^2} + \frac{t+1}{t}\pi \cos 2\pi\, t\right) = (0,0)$$

if and only if the sum of the squares of the components vanishes, that is, if $(1/4t^4) + ((t+1)\pi/t)^2 = 0$, or $1 + 4t^2(t+1)^2\pi^2 = 0$, which leads us to a contradiction. Finally, σ is an embedding, as $\sigma\colon (1,\infty) \to \sigma(1,\infty)$ is an open injective map.

(g) The image is a "Figure Eight", whose image makes a complete circuit starting at the origin as t goes from 0 to 2π, in the sense shown in Fig. 1.24(g). The curve is an immersion, as

$$\sigma'(t) = \left(-2\sin\left(t - \frac{\pi}{2}\right), 2\cos 2\left(t - \frac{\pi}{2}\right)\right) \neq (0,0)$$

for all t; but it is not an injective immersion since $\sigma(\{0, \pm 2\pi, \pm 4\pi, \ldots\}) = \{(0,0)\}$.

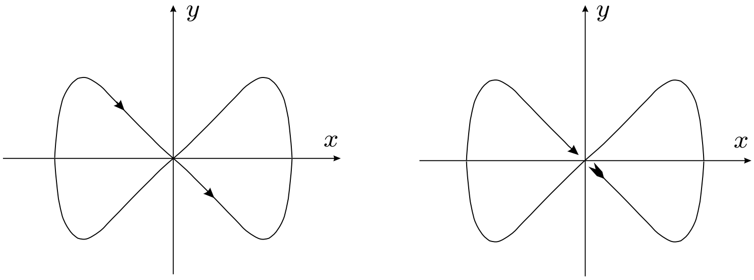
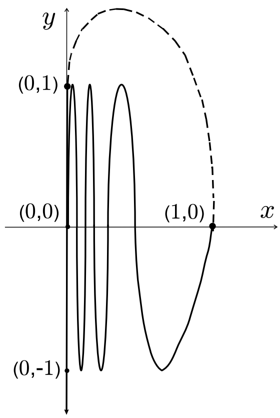


Fig. 1.24 (g) σ is an immersion. (h) σ is an injective immersion but not an embedding

Fig. 1.25 (i) σ is not an embedding



(h) We have a "Figure Eight" as in (g), but the curve now passes through $(0, 0)$ only once. Though it is an injective immersion, it is not an embedding, as the "Figure Eight" is compact and $\mathbb{R}$ is not (see Fig. 1.24(h)).

(i) σ is an injective immersion. It is not an embedding: In fact, take a point p on the vertical segment $\{0\} \times (-1, +1)$ of the graph of the curve. Then an open neighbourhood of p in that vertical interval is never the intersection of an open neighbourhood of p in $\mathbb{R}^2$ with the graph of the curve (see Fig. 1.25, where the C^∞ curve connecting $\sigma|_{(-\infty,-1]}$ and $\sigma|_{[1,\infty)}$ is dotted).

Problem 1.70 Prove that the map

$$\begin{aligned} \varphi\colon \qquad \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto \big(x^3 - 3x\,z^2 + yz,\ y - 3x\,z,\ z\big) \end{aligned}$$

is a harmonic map and a homeomorphism of $\mathbb{R}^3$ whose Jacobian vanishes on the plane $x = 0$.

Solution Since the Laplacian on the Euclidean space $\mathbb{R}^3$ is

$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

we have

$$\begin{aligned}\Delta\varphi &= \Delta\big(x^3 - 3x\,z^2 + yz,\ y - 3x\,z,\ z\big) = \big(\Delta\big(x^3 - 3x\,z^2 + yz\big),\ \Delta(y - 3x\,z),\ \Delta z\big)\\ &= (0,0,0),\end{aligned}$$

so φ is indeed a harmonic map.

Its inverse map φ^{-1} is easily seen to be

$$(x,y,z) \longmapsto \big(\sqrt[3]{x - yz},\ y + 3z\sqrt[3]{x - yz},\ z\big),$$

from which it follows that φ is a homeomorphism.

Finally, one has for its Jacobian,

$$\det\begin{pmatrix} 3x^2 - 3z^2 & z & -6x\,z + y \\ -3z & 1 & -3x \\ 0 & 0 & 1 \end{pmatrix} = 3x^2,$$

which obviously vanishes on $x = 0$.

Remark H. Lewy proved in [6] that a one-to-one harmonic map $\mathbb{R}^2 \to \mathbb{R}^2$ has non-vanishing Jacobian. The previous example, due to J.C. Wood [9], and which can be modified for $n > 3$, proves that Lewy's Theorem [6] fails for $n \geqslant 3$.

1.7 Constructing Manifolds by Inverse Image. Implicit Map Theorem

Problem 1.71 Prove that the sphere S^n is a closed embedded submanifold of $\mathbb{R}^{n+1}$.

Solution The map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x^1,\dots,x^{n+1}) = \sum_{i=1}^{n+1}(x^i)^2$, is trivially C^∞ and has rank constant and equal to 1 on $\mathbb{R}^{n+1} \setminus \{0\}$. Since $S^n = f^{-1}(1)$, S^n is a closed embedded submanifold of $\mathbb{R}^{n+1}$.

Problem 1.72 Prove that each of the functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by:

(a) $f(x,y,z) = x^2 + y^2 - z^2 - 1$, (b) $f(x,y,z) = x^2 - y^2 - z^2 - 1$,

defines a structure of differentiable manifold on $f^{-1}(0)$. The corresponding manifolds are called *one-sheet* and *two-sheet hyperboloids*, respectively. Find in each case a finite atlas defining its C^∞ structure.

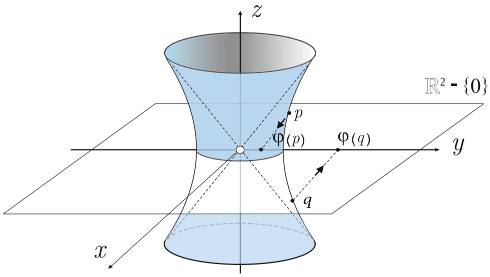


Fig. 1.26 An atlas with one chart for the one-sheet hyperboloid

Solution In the case (a), the rank of the Jacobian matrix of f is zero if and only if $x = y = z = 0$, but $(0,0,0) \notin f^{-1}(0)$. Thus the one-sheet hyperboloid is an embedded submanifold of $\mathbb{R}^3$.

In the case (b), one proceeds as in (a), now with the Jacobian matrix $J = \operatorname{diag}(2x, -2y, -2z)$. Thus the two-sheet hyperboloid is a C^∞ submanifold of $\mathbb{R}^3$.

As for the atlas, we prove below that the one-sheet hyperboloid is diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$ and hence it suffices to consider only one chart. This fact can be visualized by the map φ onto the plane $z = 0$ mapping each point p of the hyperboloid to the intersection $\varphi(p)$ with that plane of the straight line parallel to the asymptotic line by the meridian passing through the point (see Fig. 1.26).

Notice that there is another choice, mapping the points of the hyperboloid with $z < 0$ to the interior of the disk $x^2 + y^2 < 1$ minus the origin, and the points with $z > 0$ to the points with $x^2 + y^2 > 1$.

The equations of φ are given by

$$x' = x\left(1 - \frac{z}{\sqrt{1+z^2}}\right), \qquad y' = y\left(1 - \frac{z}{\sqrt{1+z^2}}\right),$$

and, as a computation shows, the inverse map φ^{-1} is given by

$$x = \frac{x'^2 + y'^2 + 1}{2(x'^2 + y'^2)}x', \qquad y = \frac{x'^2 + y'^2 + 1}{2(x'^2 + y'^2)}y', \qquad z = \frac{1 - x'^2 - y'^2}{2\sqrt{x'^2 + y'^2}}.$$

To have an atlas in the case (b), one needs at least two charts, as after finding x, y or z in the equation $x^2 - y^2 - z^2 - 1 = 0$, none of them is uniquely defined. Let $H = f^{-1}(0)$. Then the charts (U_1, φ_1), (U_2, φ_2), given by

$$\begin{aligned} U_1 &= \{(x,y,z) \in H : x > 0\}, \qquad &\varphi_1 \colon U_1 \to \mathbb{R}^2, \qquad &\varphi_1(x,y,z) = (y,z), \\ U_2 &= \{(x,y,z) \in H : x < 0\}, \qquad &\varphi_2 \colon U_2 \to \mathbb{R}^2, \qquad &\varphi_2(x,y,z) = (y,z), \end{aligned}$$

obviously define an atlas for the manifold.

Problem 1.73 Let H be the two-sheet hyperboloid defined as in Problem 1.72. By using the charts defined there and proceeding directly, prove that the natural injection $j \colon H \to \mathbb{R}^3$ has rank 2 at every point.

Solution Take the atlas in Problem 1.72(b). We have $U_1 = \varphi_1^{-1}(\mathbb{R}^2)$, $U_2 = \varphi_2^{-1}(\mathbb{R}^2)$, and the corresponding coordinate functions in $\mathbb{R}^3$ are given by the inclusion $j\colon H = U_1 \cup U_2 \to \mathbb{R}^3$, so that

$$\begin{aligned} j \circ \varphi_1^{-1}\colon \quad \mathbb{R}^2 &\to \mathbb{R}^3 \\ (y,z) &\mapsto \big(\sqrt{1+y^2+z^2}, y, z\big), \\ j \circ \varphi_2^{-1}\colon \quad \mathbb{R}^2 &\to \mathbb{R}^3 \\ (y,z) &\mapsto \big(-\sqrt{1+y^2+z^2}, y, z\big). \end{aligned}$$

We have

$$\operatorname{rank} j_{*p} = \begin{cases} \operatorname{rank}(j \circ \varphi_1^{-1})_{*(y,z)} & \text{if } p \in U_1,\ (y,z) = \varphi_1(p), \\ \operatorname{rank}(j \circ \varphi_2^{-1})_{*(y,z)} & \text{if } p \in U_2,\ (y,z) = \varphi_2(p), \end{cases}$$

that is,

$$\operatorname{rank} j_{*p} = \operatorname{rank} \begin{pmatrix} \frac{y}{\sqrt{1+y^2+z^2}} & \frac{z}{\sqrt{1+y^2+z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2,$$

$$\operatorname{rank} j_{*p} = \operatorname{rank} \begin{pmatrix} \frac{-y}{\sqrt{1+y^2+z^2}} & \frac{-z}{\sqrt{1+y^2+z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2,$$

if $p \in U_1$, $(y,z) = \varphi_1(p)$, and $p \in U_2$, $(y,z) = \varphi_2(p)$, respectively.

Problem 1.74 Prove that the subset H of the Euclidean space $\mathbb{R}^3$ of all the points (x,y,z) of $\mathbb{R}^3$ satisfying $x^3 + y^3 + z^3 - 2x\,y\,z = 1$ admits a C^∞ 2-manifold structure.

Solution The map

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad f(x,y,z) = x^3 + y^3 + z^3 - 2x\,y\,z - 1,$$

is C^∞ and its Jacobian matrix is

$$J = \begin{pmatrix} 3x^2 - 2y\,z & 3y^2 - 2x\,z & 3z^2 - 2x\,y \end{pmatrix},$$

which vanishes only if $(x,y,z) = (0,0,0)$. In fact, multiplying the identities

$$3x^2 = 2y\,z, \qquad 3y^2 = 2x\,z, \qquad 3z^2 = 2x\,y,$$

we get $27x^2y^2z^2 = 8x^2y^2z^2$, from which $x\,y\,z = 0$. If $x \neq 0$ then by the first of the three equations above we would have the absurd $y \neq 0$, $z \neq 0$. Thus $x = 0$. By the same reason, one has $y = z = 0$; but $(0,0,0) \notin H$.

Problem 1.75 Prove that the subset M of the Euclidean space $\mathbb{R}^3$ which consists of all the points (x, y, z) of $\mathbb{R}^3$ satisfying

$$x^2 - y^2 + 2xz - 2yz = 1, \qquad 2x - y + z = 0,$$

admits a structure of C^∞ 1-manifold.

Solution The functions

$$f_1(x, y, z) = x^2 - y^2 + 2xz - 2yz - 1, \qquad f_2(x, y, z) = 2x - y + z,$$

are C^∞ functions. The rank of the Jacobian matrix of f_1, f_2 with respect to x, y, z, is less than 2 if and only if $x - 2y - z = 0$, but the points satisfying this equation do not belong to $M = f^{-1}(0)$.

Problem 1.76 Prove that, if $F\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is any differentiable function on $\mathbb{R}^{n-1}$, then the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined on $\mathbb{R}^n$ by

$$f\left(x^1, \dots, x^n\right) = F\left(x^1, \dots, x^{n-1}\right) - x^n,$$

defines a structure of C^∞ manifold on $f^{-1}(0)$. Prove that this manifold is diffeomorphic to $\mathbb{R}^{n-1}$. Illustrate the result considering the C^∞ manifolds on $\mathbb{R}^3$ thus determined by the functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\text{(a)}\quad f(x, y, z) = x^2 + y^2 - z, \qquad \text{(b)}\quad f(x, y, z) = x^2 - y^2 - z,$$

which are examples of *paraboloids*: *Elliptic* (*of revolution*) in the case (a), and *hyperbolic* in the case (b).

Solution The rank of the Jacobian matrix of f is 1 everywhere, thus $f^{-1}(0)$ admits a structure of C^∞ manifold. Furthermore, it suffices to consider the chart $(f^{-1}(0), \varphi)$, where

$$\varphi\colon f^{-1}(0) \to \mathbb{R}^{n-1}, \qquad \varphi\left(x^1, \dots, x^n\right) = \left(x^1, \dots, x^{n-1}\right).$$

In the particular case of the paraboloids, taking into account the previous considerations, it is clear that:

Case (a): It is only necessary to consider the chart (U, φ) with

$$U = f^{-1}(0), \qquad \varphi\colon f^{-1}(0) \to \mathbb{R}^2, \qquad \varphi(x, y, z) = (x, y).$$

Case (b): Proceed as in (a).

Problem 1.77 Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be any homogeneous polynomial function (with degree no less than one) with at least one positive value. Prove that the function $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = F(x) - 1$, defines on $f^{-1}(0)$ a structure of C^∞ manifold.

Solution The Jacobian matrix of f is

$$J_f = \begin{pmatrix} \frac{\partial F}{\partial x^1} & \cdots & \frac{\partial F}{\partial x^n} \end{pmatrix}.$$

If $\deg F = 1$, then at least one of the elements $(\partial F/\partial x^i)(p)$ does not vanish.

If $\deg F = r > 1$ and the matrix $((\partial F/\partial x^i)(p))$ is zero at a point $p = (x^1, \dots, x^n)$, then $F(p)$ is also zero at that point. In fact, since F is homogeneous of degree r one has

$$rF(p) = x^1 \frac{\partial F}{\partial x^1}(p) + \cdots + x^n \frac{\partial F}{\partial x^n}(p).$$

Thus $f(p) = F(p) - 1 = -1$, hence on $f^{-1}(0)$ the Jacobian matrix J_f does not vanish. That is, $\operatorname{rank} J_f = 1$ on $f^{-1}(0)$, so that $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$. Notice that $f^{-1}(0)$ is not empty as if F has a positive value, then it also takes all the positive values, since $F(tp) = t^r F(p)$.

1.8 Submersions. Quotient Manifolds

Problem 1.78 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 - 1$.

(i) Prove that $C = f^{-1}(0)$ is an embedded 2-submanifold of $\mathbb{R}^3$.

(ii) Prove that a vector

$$v = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \right)_{(0,1,1)}$$

is tangent to C if and only if $b = 0$.

(iii) If $j : S^1 \to \mathbb{R}^2$ is the inclusion map, prove that $j \times \mathrm{id}_{\mathbb{R}} : S^1 \times \mathbb{R} \to \mathbb{R}^3$ induces a diffeomorphism from $S^1 \times \mathbb{R}$ to C.

Solution

(i) f is a differentiable map and $\operatorname{rank} f_* = \operatorname{rank}(2x \;\; 2y \;\; 0)$. Hence the rank of f is 1 at every point except at $\{(0, 0, z) : z \in \mathbb{R}\}$, but these points do not belong to C. Thus, by virtue of the Implicit Map Theorem for Submersions 1.14 (f is a submersion in some neighbourhood of C), C is a closed embedded submanifold of $\mathbb{R}^3$ and $\dim C = \dim \mathbb{R}^3 - \dim \mathbb{R} = 2$.

(ii) Given $v \in T_p\mathbb{R}^3$, $p \in C$, one has $v \in T_pC$ if and only if $v(f) = 0$, but $v(f) = (2ax + 2by)_{(0,1,1)} = 2b$, thus $v \in T_pC$ if and only if $b = 0$.

(iii) $\operatorname{im}(j \times \mathrm{id}_{\mathbb{R}}) = C$, as $(x, y) \in S^1$ if and only if $x^2 + y^2 = 1$, or similarly $(x, y, z) \in C$, for all $z \in \mathbb{R}$. Hence $F = j \times \mathrm{id}_{\mathbb{R}} : S^1 \times \mathbb{R} \to \mathbb{R}^3$ is a differentiable map (as it is a product of differentiable maps) that can be factorized by C, which is an embedded submanifold of $\mathbb{R}^3$. That is, there exists a differentiable map f_0 that makes commutative the diagram

$$\begin{tikzcd} S^1 \times \mathbb{R} \arrow[rr, "j \times \mathrm{id}_{\mathbb{R}}"] \arrow[dr, "f_0"'] & & \mathbb{R}^3 \\ & C \arrow[ur, "i"'] & \end{tikzcd}$$

where i denotes the embedding of C in $\mathbb{R}^3$. On the other hand, $j \times \mathrm{id}_{\mathbb{R}}$ is also an embedding, since j is. Thus the map f_0^{-1} that makes commutative the diagram

$$\begin{tikzcd} C \arrow[rr, "i"] \arrow[dr, "f_0^{-1}"'] & & \mathbb{R}^3 \\ & S^1 \times \mathbb{R} \arrow[ur, "j \times \mathrm{id}_{\mathbb{R}}"'] & \end{tikzcd}$$

is C^∞. Thus f_0 is a diffeomorphism.

Problem 1.79 Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by

$$u = x^2 + y^2 + z^2 - 1, \qquad v = ax + by + cz, \quad a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1.$$

(i) Find the points at which φ is a submersion.
(ii) Find $\varphi^{-1}(0)$.
(iii) Find the points where φ is not a submersion, and its image.

Solution

(i)

$$\operatorname{rank} \varphi_* = \operatorname{rank} \begin{pmatrix} 2x & 2y & 2z \\ a & b & c \end{pmatrix} = 2$$

at the points (x, y, z) in which the vector (x, y, z) is not a multiple of (a, b, c). Hence φ is a submersion on $\mathbb{R}^3 \setminus \langle (a, b, c) \rangle$, where $\langle (a, b, c) \rangle$ denotes the straight line generated by (a, b, c).

(ii) Let $(a, b, c)^\perp$ denote the plane through the origin orthogonal to the vector (a, b, c). Then:

$$\begin{aligned} \varphi^{-1}(0) &= \big\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ ax + by + cz = 0 \big\} \\ &= S^2 \cap (a, b, c)^\perp. \end{aligned}$$

(iii) The map φ is not a submersion at the points of $\langle (a, b, c) \rangle$, whose image is

$$\varphi\big(\langle a, b, c \rangle\big) = \big\{ \big(\lambda^2 - 1, \lambda\big) \big\} \subset \mathbb{R}^2 = \big\{ (u, v) \in \mathbb{R}^2 : u = v^2 + 1 \big\}.$$

Problem 1.80 Consider the differentiable map $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$u = x^2 + y^2 + z^2 + t^2 - 1, \qquad v = x^2 + y^2 + z^2 + t^2 - 2y - 2z + 5.$$

(i) Find the set of points of $\mathbb{R}^4$ where φ is not a submersion, and its image.
(ii) Calculate a basis of $\ker \varphi_{*(0,1,2,0)}$.
(iii) Calculate the image by φ_* of $(1,0,2,1) \in T_{(1,2,0,1)}\mathbb{R}^2$ and the image by φ^* of $(\mathrm{d}u + 2\mathrm{d}v)_{(-1,5)} \in T^*_{(-1,5)}\mathbb{R}^2$, choosing the point $(0,0,0,0)$ in $\varphi^{-1}((-1,5))$.

Solution

(i) φ is not a submersion at the points of $\mathbb{R}^4$ where

$$\operatorname{rank} \varphi_* = \operatorname{rank} \begin{pmatrix} 2x & 2y & 2z & 2t \\ 2x & 2y-2 & 2z-2 & 2t \end{pmatrix} < 2.$$

Hence, the set is

$$A = \big\{(x,y,z,t) \in \mathbb{R}^4 : x = 0,\ y = z,\ t = 0\big\}.$$

Therefore,

$$\varphi(A) = \big\{(u,v) \in \mathbb{R}^2 : u = 2\lambda^2 - 1,\ v = 2\lambda^2 - 4\lambda + 5,\ \lambda \in \mathbb{R}\big\}.$$

(ii) We have $\varphi_* \colon T_{(0,1,2,0)}\mathbb{R}^4 \to T_{(4,4)}\mathbb{R}^2$. Every vector $X \in T_{(0,1,2,0)}$ is of the type

$$X = \lambda_1 \left.\frac{\partial}{\partial x}\right|_p + \lambda_2 \left.\frac{\partial}{\partial y}\right|_p + \lambda_3 \left.\frac{\partial}{\partial z}\right|_p + \lambda_4 \left.\frac{\partial}{\partial t}\right|_p,$$

where $p = (0,1,2,0)$. Since $\varphi_{*(0,1,2,0)} \equiv \left(\begin{smallmatrix} 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{smallmatrix}\right)$, we have

$$\varphi_{*(0,1,2,0)} X \equiv (2\lambda_2 + 4\lambda_3) \left.\frac{\partial}{\partial u}\right|_{(4,4)} + 2\lambda_3 \left.\frac{\partial}{\partial v}\right|_{(4,4)}.$$

If $X \in \ker \varphi_{*p}$, we deduce $\lambda_2 = \lambda_3 = 0$. Thus

$$\ker \varphi_{*p} = \left\{ \lambda \left.\frac{\partial}{\partial x}\right|_p + \mu \left.\frac{\partial}{\partial t}\right|_p : \lambda, \mu \in \mathbb{R} \right\}$$

and $\{\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial t}|_p\}$ is a basis of $\ker \varphi_{*p}$.

(iii) $\varphi_{*(1,2,0,1)}(1,0,2,1) = 4\frac{\partial}{\partial u}|_{(5,7)}$. Let $p = (0,0,0,0)$, so $\varphi(p) = (-1,5)$ and $\varphi^*_{(-1,5)}(\mathrm{d}u + 2\,\mathrm{d}v) = -4(\mathrm{d}y + \mathrm{d}z)_{(0,0,0,0)}$.

Problem 1.81 We define an equivalence relation $\sim$ in the open subset $\mathbb{R}^{n+1} \setminus \{0\}$ by the condition that two vectors of $\mathbb{R}^{n+1} \setminus \{0\}$ are equivalent if they are proportional. The quotient space $\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ is the *real projective space* of dimension n.

(i) Prove that, giving $\mathbb{R}\mathrm{P}^n$ the quotient topology induced by the previous equivalence relation, it is Hausdorff.

(ii) Let $[x^1, \dots, x^{n+1}]$ be the equivalence class in $\mathbb{R}\mathrm{P}^n$ of $(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$. For each $i = 1, 2, \dots, n+1$, let U_i be the subset of points $[x^1, \dots, x^{n+1}]$ of $\mathbb{R}\mathrm{P}^n$ such that $x^i \neq 0$. Prove that the functions $\varphi_i : U_i \to \mathbb{R}^n$ defined by

$$\varphi_i\left(\left[x^1, \dots, x^{n+1}\right]\right) = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right)$$

are homeomorphisms and that the changes of coordinates $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are differentiable. Hence the systems (U_i, φ_i), $i = 1, 2, \dots, n+1$, define an atlas on the space $\mathbb{R}\mathrm{P}^n$.

(iii) Prove that the projection map

$$\begin{aligned} \pi\colon \qquad \mathbb{R}^{n+1} \setminus \{0\} &\to \mathbb{R}\mathrm{P}^n \\ \left(x^1, \dots, x^{n+1}\right) &\mapsto \left[x^1, \dots, x^{n+1}\right] \end{aligned}$$

is a submersion. Hence $\mathbb{R}\mathrm{P}^n$ is a quotient manifold of $\mathbb{R}^{n+1} \setminus \{0\}$.

Solution

(i) The relation $\sim$ is open, i.e. given the open subset $U \subset \mathbb{R}^{n+1} \setminus \{0\}$, then $[U] = \bigcup_{x \in U} [x]$ is an open subset of $(\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\})$. In fact, since U is open, so is $U_\lambda = \{\lambda x : x \in U\}$, $\lambda \neq 0$ being fixed, and $[U] = \bigcup_{\lambda \neq 0} U_\lambda$. Moreover, the graph of $\sim$ is the subset

$$\Gamma = \left\{(x, \lambda x) : \lambda \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{R}^{n+1} \setminus \{0\}\right\}$$

of $(\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\})$. The subset Γ is closed, as if $(x_n, \lambda_n x_n) \mapsto (x, y)$ then (λ_n) is bounded. Thus it has a convergent subsequence (λ_{n_k}). Let $\lambda = \lim_{k \to \infty} \lambda_{n_k}$. Then

$$y = \lim_{n \to \infty} \lambda_n x_n = \lim_{k \to \infty} \lambda_{n_k} x_{n_k} = \lambda x.$$

So $(x, y) \in \Gamma$. We conclude that the quotient space is Hausdorff.

(ii) It is obvious that the functions φ_i are homeomorphisms. As for the changes of coordinates

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),$$

we clearly have for $i < j$:

$$\varphi_j(U_i \cap U_j) = \left\{\left(t^1, \dots, t^n\right) \in \mathbb{R}^n : t^i \neq 0\right\}.$$

Furthermore,

$$\varphi_j^{-1}\left(t^1, \dots, t^n\right) = \left[t^1, \dots, t^{j-1}, 1, t^j, \dots, t^n\right].$$

So, for $(t^1, \ldots, t^n) \in \varphi_j(U_i \cap U_j)$ we have $\varphi_j^{-1}(t^1, \ldots, t^n)$ as above and moreover

$$\begin{aligned}\varphi_{ij}(t^1, \ldots, t^n) &= \left(\frac{t^1}{t^i}, \ldots, \frac{t^{i-1}}{t^i}, \frac{t^{i+1}}{t^i}, \ldots, \frac{t^{j-1}}{t^i}, \frac{1}{t^i}, \frac{t^j}{t^i}, \ldots, \frac{t^n}{t^i}\right)\\ &= (x^1, \ldots, x^n).\end{aligned}$$

The equations

$$x^1 = \frac{t^1}{t^i}, \quad \ldots, \quad x^{i-1} = \frac{t^{i-1}}{t^i}, \quad x^i = \frac{t^{i+1}}{t^i}, \quad \ldots, \quad x^{j-2} = \frac{t^{j-1}}{t^i},$$
$$x^{j-1} = \frac{1}{t^i}, \quad x^j = \frac{t^j}{t^i}, \quad \ldots, \quad x^n = \frac{t^n}{t^i},$$

correspond to differentiable functions on U_{ij}.

(Note that we have supposed $i < j$, which is not restrictive.)

(iii) $\mathbb{R}^{n+1} \setminus \{0\}$ is an open submanifold of $\mathbb{R}^{n+1}$. Using the identity chart on $\mathbb{R}^{n+1} \setminus \{0\}$ and an arbitrarily fixed chart φ_i as in (ii) above on $U_i \subset \mathbb{R}\mathrm{P}^n$, the projection map π has on $\pi^{-1}(U_i)$ (where $x^i \neq 0$) the representative map

$$\begin{aligned}\varphi_i \circ \pi \circ \mathrm{id}^{-1}: \qquad & \pi^{-1}(U_i) \to \mathbb{R}^n\\ & (x^1, \ldots, x^{n+1}) \mapsto \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right),\end{aligned}$$

which is easily seen to have rank n. Since i is arbitrary, π is a submersion, thus concluding.

Problem 1.82 Construct an atlas on the real projective space $\mathbb{R}\mathrm{P}^n$ considered as the quotient space of the sphere S^n by identification of antipodal points. Prove that the projection map $\pi: S^n \to \mathbb{R}\mathrm{P}^n$ is a submersion. Hence $\mathbb{R}\mathrm{P}^n$ is a quotient manifold of S^n.

Hint Use the atlas given by the $2n + 2$ open hemispheres defined by the coordinate axes, and the canonical projections.

Solution As we know, $\mathbb{R}\mathrm{P}^n$ is the quotient space of the subspace $\mathbb{R}^{n+1} \setminus \{0\}$ of $\mathbb{R}^{n+1}$, by the relation $\sim$ given by $x \sim y$ if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. The projection $\pi: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n$ is an open mapping. On S^n the relation above reduces to $x \sim \pm x$, that is, $[x] = \{x, -x\}$ for every $x \in S^n$. Hence on S^n the above relation corresponds to the antipodal identification.

Consider the restriction to S^n of the projection π, that we continue denoting by $\pi: S^n \to \mathbb{R}\mathrm{P}^n$, $\pi(x) = [x]$, and which is still open and surjective. In fact, given $[x] \in \mathbb{R}\mathrm{P}^n$, then $x/|x| \in S^n$ and $\pi(x/|x|) = \pi(x) = [x]$. Hence, $\mathbb{R}\mathrm{P}^n$ can be considered (as a topological space) as the quotient space of S^n obtained by identifying antipodal points. From which it follows, since $\pi\colon S^n \to \mathbb{RP}^n$ is continuous, that $\mathbb{RP}^n$ is compact and connected.

Notice that if $U \subset S^n$ is contained in an open hemisphere and $x \in U$, then $-x \notin U$, hence $\pi|_U\colon U \to \pi(U)$ is injective; that is, $\pi|_U$ is a homeomorphism. This property allows us to construct an atlas on $\mathbb{RP}^n$ from an atlas on S^n whose coordinate domains are contained in open hemispheres of S^n. For instance, the atlas consisting in the $2n+2$ open hemispheres defined by the coordinate hyperplanes and the canonical projections. Let, for instance, $V_i^+ = \{x \in S^n : x^i > 0\}$, and

$$\begin{aligned} h_i^+\colon \quad & V_i^+ \to \mathbb{R}^n \\ & \left(x^1, \dots, x^{n+1}\right) \mapsto \left(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1}\right). \end{aligned}$$

We then define in $\mathbb{RP}^n$, $V_i = \pi(V_i^+)$ and $\varphi_i^+ = h_i^+ \circ (\pi|_{V_i^+})^{-1}\colon V_i \to \mathbb{R}^n$. The map φ_i^+ is a homeomorphism, since it is a composition of homeomorphisms. Considering $V_i^- = \{x \in S^n : x^i < 0\}$, it follows that $\pi(V_i^+) = \pi(V_i^-) = V_i$ and the similar homeomorphism is $\varphi_i^- = h_i^- \circ (\pi|_{V_i^-})^{-1}\colon V_i \to \mathbb{R}^n$. Notice that $\varphi_i^-(V_i) = \varphi_i^+(V_i)$, but $\varphi_i^- \neq \varphi_i^+$; in fact, we have $\varphi_i^-([x]) = -\varphi_i^+([x])$. Since $\varphi_i^- = -\varphi_i^+$ (they differ by the diffeomorphism $t \to -t$ of $\mathbb{R}^n$), we shall forget the charts (V_i, φ_i^-), and we shall consider only the charts (V_i, φ_i^+), $i = 1, \dots, n+1$. If $i \neq j$, then $V_i \cap V_j \neq \emptyset$, and moreover,

$$\varphi_i^+ \circ \left(\varphi_j^+\right)^{-1}\colon \varphi_j^+(V_i \cap V_j) \to \varphi_i^+(V_i \cap V_j)$$

is given by

$$\begin{aligned} \varphi_i^+ \circ \left(\varphi_j^+\right)^{-1} &= h_i^+ \circ (\pi|_{V_i^+})^{-1} \circ (\pi|_{V_j^+}) \circ \left(h_j^+\right)^{-1} \\ &= h_i^+ \circ \left(\pi^{-1} \circ \pi\right)\big|_{V_i^+ \cap V_j^+} \circ h_j^{+-1} \\ &= h_i^+ \circ \left(h_j^+\right)^{-1}, \end{aligned}$$

which is differentiable since it is a change of coordinates in S^n, known to be differentiable. By the constructions above, for a given i, the projection map π has locally the representative map

$$\varphi_i^+ \circ (\pi|_{V_i^+}) \circ \left(h_i^+\right)^{-1}\colon h_i^+\left(V_i^+\right) \to \varphi_i^+(V_i),$$

which is the identity map, so having rank n. Since i is arbitrary, π is a submersion, and we have finished.

Problem 1.83 (The Real Grassmannian as a Quotient Manifold) Let

$$M \subset \mathbb{R}^n \times \overset{(k)}{\cdots} \times \mathbb{R}^n$$

be the subset of k-tuples $(v_1, \dots, v_k)$ of linearly independent vectors of $\mathbb{R}^n$. Let $\mathrm{GL}(k, \mathbb{R})$ act on M on the right by $(v_1, \dots, v_k) \cdot A = (v'_1, \dots, v'_k)$, where

$$v'_j = \sum_{i=1}^{k} a^i_j v_i, \qquad A = \left(a^i_j\right) \in \mathrm{GL}(k, \mathbb{R}), \quad i, j = 1, \dots, k.$$

Prove:

(i) M is an open subset of $\mathbb{R}^n \times \overset{(k)}{\cdots} \times \mathbb{R}^n$.

(ii) If $\sim$ is the equivalence relation induced by this action, then the quotient manifold $M/\!\sim$ exists and can be identified to the Grassmannian $G_k(\mathbb{R}^n)$ of all k-planes in $\mathbb{R}^n$.

Solution

(i) Let us denote by x^i_j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$, the natural coordinates on $\mathbb{R}^n \times \overset{(k)}{\cdots} \times \mathbb{R}^n$. Given $(v_1, \dots, v_k) \in M$, we write

$$X = \begin{pmatrix} x^1_1(v_1, \dots, v_k) & \cdots & x^1_k(v_1, \dots, v_k) \\ \vdots & & \vdots \\ x^n_1(v_1, \dots, v_k) & \cdots & x^n_k(v_1, \dots, v_k) \end{pmatrix}, \qquad (\star)$$

that is, $x^i_j(v_1, \dots, v_k)$ is the ith component of the column vector v_j.

Let $\Delta_{i_1 \dots i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, denote the determinant of the $k \times k$ submatrix of ($\star$) defined by the rows $i_1, \dots, i_k$. The subset M is open, as it is defined by the inequality

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \Delta^2_{i_1 \dots i_k} > 0.$$

(ii) Let $((x^i_j), (y^r_s))$, $i, r = 1, \dots, n$; $j, s = 1, \dots, k$, be the natural coordinates on the product manifold $M \times M$, and let z^a_b, $a, b = 1, \dots, k$, be the entries of a matrix in $\mathrm{GL}(k, \mathbb{R})$.

The graph $\mathscr{G}$ of $\sim$ is the image of the differentiable map

$$\varphi \colon M \times \mathrm{GL}(k, \mathbb{R}) \to M \times M, \qquad \varphi(X, Z) = (X, XZ).$$

The graph $\mathscr{G}$ is closed in $M \times M$, as it follows by taking into account that a pair $((v_1, \dots, v_k), (w_1, \dots, w_k)) \in M \times M$ belongs to $\mathscr{G}$ if and only if $w_i \in \langle v_1, \dots, v_k \rangle$, $1 \leqslant i \leqslant k$, and that every vector subspace of $\mathbb{R}^n$ is a closed subset. Hence, by applying the Theorem of the Closed Graph 1.16, we only need to prove that $\mathscr{G}$ is an embedded submanifold.

Certainly, φ is injective as $\varphi(X, Z) = \varphi(X', Z')$ means $X = X'$, $XZ = X'Z'$, and since $\operatorname{rank} X = k$, the latter equation implies $Z = Z'$.

Next we prove that $\varphi\colon M \times \mathrm{GL}(k,\mathbb{R}) \to \mathscr{G}$ is a homeomorphism. Assume

$$\lim_{h\to\infty} \varphi(X_h, Z_h) = \lim_{h\to\infty} (X_h, X_h Z_h) = (X, Y).$$

Hence $\lim_{h\to\infty} X_h = X$. As $\mathscr{G}$ is closed in $M \times M$, there exists $Z \in \mathrm{GL}(k,\mathbb{R})$ such that $Y = XZ$. We only need to prove that $\lim_{h\to\infty} Z_h = Z$. Set $X_h = (v_{1,h}, \dots, v_{k,h})$, $X = (v_1, \dots, v_k)$. As the vectors $v_1, \dots, v_k$ are linearly independent, we can complete them up to a basis $(v_1, \dots, v_k, v_{k+1}, \dots, v_n)$ in $\mathbb{R}^n$. Let

$$v_{j,h} = \sum_{i=1}^{k} a^i_{j,h} v_i + \sum_{i=k+1}^{n} b^i_{j,h} v_i, \quad 1 \leqslant j \leqslant k, \tag{$\star\star$}$$

be the expression of $v_{j,h}$ in this basis. As $\lim_{h\to\infty} v_{j,h} = v_j$, we obtain $\lim_{h\to\infty} a^i_{j,h} = \delta_{ij}$, for $i, j = 1, \dots, k$, and $\lim_{h\to\infty} b^i_{j,h} = 0$, for $k+1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$. Set $\widehat{X} = (v_{k+1}, \dots, v_n)$, and let A_h, B_h be the matrices of sizes $k \times k$, $(n-k) \times k$, respectively, given by

$$A_h = \left(a^i_{j,h}\right)^{i=1,\dots,k}_{j=1,\dots,k}, \qquad B_h = \left(b^i_{j,h}\right)^{k+1\leqslant i\leqslant n}_{1\leqslant j\leqslant k}.$$

Then, ($\star\star$) can be rewritten as $X_h = XA_h + \widehat{X}B_h$; hence

$$X_h Z_h = XA_h Z_h + \widehat{X}B_h Z_h,$$

and passing to the limit, we obtain

$$XZ = X \lim_{h\to\infty} (A_h Z_h) + \widehat{X} \lim_{h\to\infty} (B_h Z_h).$$

Taking components we have $Z = \lim_{h\to\infty}(A_h Z_h)$ and $\lim_{h\to\infty}(B_h Z_h) = 0$. Since A_h goes to the $k \times k$ identity matrix $I_k = (\delta_{ij})$ as $h \to \infty$, we can conclude.

Let us compute φ_*. We have

$$\xi^i_j = \varphi_* \left(\frac{\partial}{\partial x^i_j}\bigg|_{(X,Z)} \right) = \left(\frac{\partial}{\partial x^i_j} + \sum_{s=1}^{k} z^s_j \frac{\partial}{\partial y^i_s} \right)_{(X,XZ)},$$

$$\zeta^a_b = \varphi_* \left(\frac{\partial}{\partial z^a_b}\bigg|_{(X,Z)} \right) = \sum_{r=1}^{n} x^r_a \frac{\partial}{\partial y^r_b}\bigg|_{(X,XZ)},$$

where $1 \leqslant i \leqslant n$ and $j, a, b = 1, \dots, k$.

We claim that the tangent vectors ξ^i_j, ζ^a_b, are linearly independent for every $(X, Z) \in M \times \mathrm{GL}(k,\mathbb{R})$. In fact, if

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \lambda^i_j \xi^i_j + \sum_{a,b=1}^{k} \mu^a_b \zeta^a_b = 0, \tag{$\star\star\star$}$$

for some scalars λ^i_j, μ^a_b, then by applying the equation ($\star\star\star$) to the function x^i_j, we obtain $\lambda^i_j = 0$. Hence, this equation reduces to

$$\sum_{a,b=1}^{k} \mu^a_b \sum_{r=1}^{n} x^r_a \frac{\partial}{\partial y^r_b}\bigg|_{(X,XZ)} = 0,$$

or else,

$$\sum_{r=1}^{n}\sum_{b=1}^{k}\left(\sum_{a=1}^{k} x^r_a \mu^a_b\right)\frac{\partial}{\partial y^r_b}\bigg|_{(X,XZ)} = 0.$$

Hence

$$\begin{pmatrix} x^1_1 & \cdots & x^1_k \\ \vdots & & \vdots \\ x^n_1 & \cdots & x^n_k \end{pmatrix}\begin{pmatrix} \mu^1_1 & \cdots & \mu^1_k \\ \vdots & & \vdots \\ \mu^k_1 & \cdots & \mu^k_k \end{pmatrix} = 0.$$

As $\operatorname{rank}(x^i_j) = k$, the previous equality implies $(\mu^a_b) = 0$.

Finally, let us show that $M/\sim$ can be identified to the Grassmannian. We have a natural surjective map

$$\Psi : M \to G_k(\mathbb{R}^n), \qquad \Psi(v_1, \dots, v_k) = \langle v_1, \dots, v_k\rangle.$$

We have $\dim\langle v_1, \dots, v_k\rangle = k$ as $v_1, \dots, v_k$ are linearly independent. Moreover,

$$\Psi(v_1, \dots, v_k) = \Psi(v'_1, \dots, v'_k) = V$$

if and only if $\{v_1, \dots, v_k\}$ and $\{v'_1, \dots, v'_k\}$ are two bases of V. Hence there exists $A \in \mathrm{GL}(k, \mathbb{R})$ such that $(v'_1, \dots, v'_k) = (v_1, \dots, v_k) \cdot A$, thus proving that the fibres of Ψ are exactly the orbits of $\mathrm{GL}(k, \mathbb{R})$.

Problem 1.84 Let $\pi : M \to N$ be a differentiable map. Prove that π is a submersion if and only if it admits local sections through each point, i.e. for every $q_0 = \pi(p_0)$, $p_0 \in M$, there exist an open neighbourhood V of q_0 in N, and a differentiable map $\sigma : V \to M$ such that (see Fig. 1.27):

(i) $\sigma(q_0) = p_0$;
(ii) $\pi \circ \sigma = \mathrm{id}_V$.

Solution From (ii) we have $\pi_{*p_0} \circ \sigma_{*q_0} = \mathrm{id}_{T_{q_0}N}$. Since the identity map is surjective, $\pi_* : T_{p_0}M \to T_{q_0}N$ is surjective. Conversely, if π is a submersion at p_0, by the Theorem of the Rank 1.11, there exist local coordinates $(x^1, \dots, x^m)$, $(y^1, \dots, y^n)$, centred at p_0, q_0 in M, N, respectively, such that $y^i \circ \pi = x^i$, $1 \leqslant i \leqslant n$. Notice that $m \geqslant n$, as π is a submersion. Hence we can define a map σ on the domain of

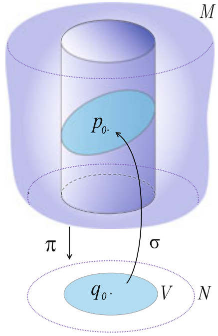


Fig. 1.27 A local section σ of a submersion $\pi: M \to N$

$(y^1, \dots, y^n)$ by setting

$$x^i \circ \sigma = \begin{cases} y^i & \text{if } 1 \leqslant i \leqslant n, \\ 0 & \text{if } n+1 \leqslant i \leqslant m. \end{cases}$$

Then, for every $i = 1, \dots, n$, we have

$$y^i \circ (\pi \circ \sigma) = \left(y^i \circ \pi\right) \circ \sigma = x^i \circ \sigma = y^i,$$

thus proving that σ is a local section of π.

Problem 1.85 Let M, N be smooth manifolds and let $S \hookrightarrow N$ be an embedded submanifold. A smooth map $f: M \to N$ is said to intersect S transversally at a point $x \in S$ if either:

(i) $f(x) \notin S$, or
(ii) $f(x) \in S$ and $T_{f(x)}N = f_*(T_x M) + T_{f(x)}S$.

The map f is said to *intersect* S *transversally* if f intersects S transversally at every point.

Prove: If f intersects S transversally, then

1. $f^{-1}(S)$ is an embedded submanifold of M.
2. The codimension of $f^{-1}(S)$ in M equals that of S in N.
3. For every $x \in f^{-1}(S)$, there is an exact sequence of vector spaces,

$$0 \to T_x\left(f^{-1}(S)\right) \to T_x M \to T_{f(x)}N / T_{f(x)}S \to 0.$$

Solution Let us fix a point $x \in M$ such that $f(x) \in S$. As the questions are local, we can assume $N = \mathbb{R}^m$, and also that S is defined by the equations $y^{q+1} = \cdots = y^m = 0$, where $(y^1, \dots, y^q, y^{q+1}, \dots, y^m)$ is a coordinate system on N defined around $f(x)$. Hence $\dim S = q$, or equivalently, $\operatorname{codim} S = m - q$. Let $\pi: N = \mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^{m-q} \to \mathbb{R}^{m-q}$ be the projection onto the last $m - q$ components. The composite mapping $\pi \circ f$ is submersive at each point in $f^{-1}(S)$ as, by applying π_*

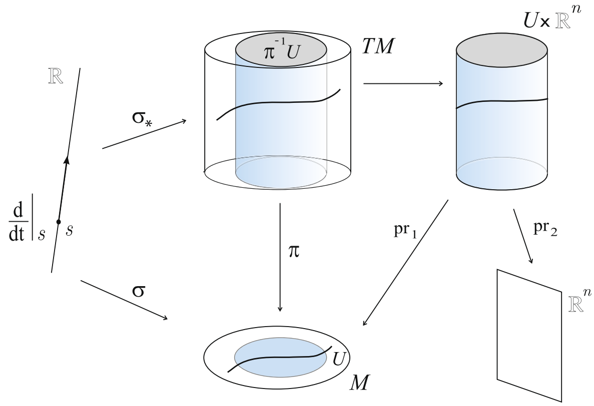


Fig. 1.28 The tangent vector field σ' to a curve σ as a curve in TM

to both sides of the formula

$$T_{f(x)}N = f_*(T_x M) + T_{f(x)}S,$$

one obtains that

$$T_{\pi(f(x))}\mathbb{R}^{m-q} = \pi_*(T_{f(x)}N) = (\pi \circ f)_*(T_x M),$$

because the tangent vectors in $T_{f(x)}S$ go to zero according to the equations for the submanifold S.

Consequently, the differentials $(\mathrm{d}_x(y^{q+1} \circ f), \dots, \mathrm{d}_x(y^m \circ f))$ are linearly independent and hence, $f^{-1}(S)$ is locally defined by the equations

$$y^{q+1} \circ f = 0 = \cdots = y^m \circ f = 0,$$

thus proving parts 1 and 2 (as $\operatorname{codim} f^{-1}(S) = m - q$), and part 3 also follows by simply taking dimensions in the short sequence in the statement.

1.9 The Tangent Bundle

Problem 1.86 Prove that if σ is a C^∞ curve in the C^∞ manifold M, then the tangent vector field σ' is a C^∞ curve in the tangent bundle TM.

Solution Given a C^∞ curve $\sigma : \mathbb{R} \to M$, the tangent vector field σ' is, by definition, a map that we can write as

$$\sigma' = \sigma_* \circ \frac{\mathrm{d}}{\mathrm{d}t} : \mathbb{R} \to TM,$$

where $\mathrm{d}/\mathrm{d}t$ denotes the canonical vector field on $\mathbb{R}$ and hence it can be considered as a curve in TM (see Fig. 1.28), so that for a coordinate neighbourhood $U \subset M$

with coordinate functions $x^1, \dots, x^n$, one has

$$\sigma_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_s\right) = \sum_{i=1}^{n} \frac{\mathrm{d}(x^i \circ \sigma)}{\mathrm{d}t}(s) \frac{\partial}{\partial x^i}\bigg|_{\sigma(s)}.$$

Thus,

$$\sigma'(s) = \left(\left(x^1 \circ \sigma\right)(s), \dots, \left(x^1 \circ \sigma\right)(s), \frac{\mathrm{d}(x^1 \circ \sigma)}{\mathrm{d}t}(s), \dots, \frac{\mathrm{d}(x^n \circ \sigma)}{\mathrm{d}t}(s)\right).$$

The coordinate functions $\{x^i\}$ and σ are C^∞. Hence the composition $x^i \circ \sigma$ is C^∞ for each $i = 1, \dots, n$; and the functions $\mathrm{d}(x^i \circ \sigma)/\mathrm{d}t$, which are coordinate functions on the open subset $\pi^{-1}(U)$ of TM, are also C^∞.

Problem 1.87 Assume that the manifold M admits a basis $\{X_1, \dots, X_n\}$ for the $(C^\infty M)$-module $\mathfrak{X}(M)$ of C^∞ vector fields on M.

Prove that the map

$$M \times \mathbb{R}^n \xrightarrow{F} TM = \bigcup_{p \in M} T_pM$$

$$\left(p, a^1, \dots, a^n\right) \mapsto F\left(p, a^1, \dots, a^n\right) = \sum_i a^i X_i\big|_p \in T_pM$$

is a diffeomorphism, that is, that TM is then trivial.

Remark Compare with Problem 2.18.

Solution To begin with, we prove that for every $p \in M$, the tangent vectors $X_1|_p, \dots, X_n|_p$ are linearly independent and hence they are a basis of T_pM. Let $(U, x^1, \dots, x^n)$ be a coordinate system defined on an open neighbourhood U of p, and let $f \in C^\infty M$ be a function such that:

(a) $f = 1$ on an open neighbourhood $V \subset U$.
(b) $\operatorname{supp} f \subset U$.

Then $f\partial/\partial x^i$ defines a global vector field. Hence there exists an $n \times n$ matrix with entries $f_i^h \in C^\infty M$ such that $f\partial/\partial x^i = \sum_h f_i^h X_h$. Evaluating at p, we obtain that

$$\left(\partial/\partial x^i\right)_p = \sum_h f_i^h(p) X_h\big|_p. \qquad (\star)$$

Moreover, as $(\partial/\partial x^1)_p, \dots, (\partial/\partial x^n)_p$ is a basis of T_pM, there exist scalars λ_h^i such that $X_h|_p = \sum_i \lambda_h^i (\partial/\partial x^i)_p$, and substituting this expression into ($\star$), we obtain $(\partial/\partial x^j)_p = \sum_h f_j^h(p)\lambda_h^i(\partial/\partial x^i)_p$. As $\{(\partial/\partial x^i)_p\}$ is a basis, we conclude $f_j^h(p)\lambda_h^i = \delta_j^i$, thus proving that the matrix $(f_j^h(p))$ is invertible.

Moreover,

(i) F is injective, as if

$$F\big(p, a^1, \dots, a^n\big) = F\big(p', \bar{a}^1, \dots, \bar{a}^n\big),$$

it follows that $p = p'$. Furthermore, $\sum_i a^i X_i|_p = \sum_i \bar{a}^i X_i|_p$, from which, since the $X_i|_p$ are a basis of T_pM, we have $a^i = \bar{a}^i$ for every $i = 1, \dots, n$.

(ii) F is surjective, since each $v \in T_pM$ is of the form $v = \lambda^i X_i|_p$, that is, $v = F(p, \lambda^1, \dots, \lambda^n)$.

(iii) F is differentiable. In fact, let (U, φ) be a chart around $p \in M$, with $\varphi = (x^1, \dots, x^n)$, and consider the associated chart $(\pi^{-1}(U), \Phi)$ in TM, where

$$\pi : TM \to M, \qquad v \mapsto \pi(v) = p, \quad v \in T_pM,$$

that is,

$$\pi^{-1}(U) \xrightarrow{\Phi} \varphi(U) \times \mathbb{R}^n$$
$$v = \sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial x^i}\bigg|_p \longmapsto \Phi(v) = \big(\varphi(p), \lambda^1, \dots, \lambda^n\big).$$

Now, as $X_1, \dots, X_n$ are C^∞ vector fields, we have $X_i|_U = \sum_{j=1}^n f_i^j \frac{\partial}{\partial x^j}$, where $f_i^j : U \to \mathbb{R}$ are C^∞ functions. Hence, given $t = (t^1, \dots, t^n) \in \varphi(U)$ such that $\varphi(p) = (t^1, \dots, t^n)$, one has

$$\begin{aligned}
&\big(\Phi \circ F \circ (\varphi \times \mathrm{id}_{\mathbb{R}^n})^{-1}\big)\big(t^1, \dots, t^n, a^1, \dots, a^n\big) \\
&\quad = (\Phi \circ F)\big(p, a^1, \dots, a^n\big) = \Phi\big(a^i X_i\big|_p\big) \\
&\quad = \Phi\left(\sum_{j=1}^{n} a^i f_i^j\big(\varphi^{-1}(t)\big) \frac{\partial}{\partial x^j}\bigg|_{\varphi^{-1}(t)}\right) \\
&\quad = \left(t^1, \dots, t^n, \sum_i a^i f_i^1\big(\varphi^{-1}(t)\big), \dots, \sum_i a^i f_i^n\big(\varphi^{-1}(t)\big)\right).
\end{aligned}$$

Thus $\Phi \circ F \circ (\varphi \times \mathrm{id}_{\mathbb{R}^n})^{-1}$ is C^∞, hence F is C^∞.

Moreover F^{-1} is C^∞. In fact,

$$\begin{aligned}
&\big((\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ F^{-1} \circ \Phi^{-1}\big)\big(t^1, \dots, t^n, \lambda^1, \dots, \lambda^n\big) \\
&\quad = \big((\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ F^{-1}\big)\left(\sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(t)}\right) \\
&\quad = \big((\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ F^{-1}\big)\left(\sum_{i,j} \lambda^i \tilde{f}_i^j\big(\varphi^{-1}(t)\big) X_j\big|_{\varphi^{-1}(t)}\right)
\end{aligned}$$

$$= (\varphi \times \mathrm{id}_{\mathbb{R}^n})\Big(\varphi^{-1}(t), \sum_i \lambda^i \tilde{f}_i^1\big(\varphi^{-1}(t)\big), \ldots, \sum_i \lambda^i \tilde{f}_i^n\big(\varphi^{-1}(t)\big)\Big)$$

$$= \Big(t^1, \ldots, t^n, \sum_i \lambda^i \tilde{f}_i^1\big(\varphi^{-1}(t)\big), \ldots, \sum_i \lambda^i \tilde{f}_i^n\big(\varphi^{-1}(t)\big)\Big),$$

where $(\tilde{f}_j^i) = (f_j^i)^{-1}$. Hence $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ F^{-1} \circ \Phi^{-1}$ is C^∞ and thus F^{-1} is C^∞.

Problem 1.88 Let $j\colon S^2 \to \mathbb{R}^3$ be the natural inclusion map. Prove that the map $j_*\colon T\,S^2 \to T\,\mathbb{R}^3$ is an embedding.

Solution Let $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$. As $j(S^2) \subset U$, we have $j_* T S^2 \subset TU$, and since U is open in $\mathbb{R}^3$, it suffices to prove that $j_*\colon TS^2 \to TU$ is an embedding. Consider the map

$$\varphi\colon U \to S^2 \times \mathbb{R}^+, \qquad \varphi(x) = \left(\frac{x}{|x|}, |x|\right).$$

Then, φ is a diffeomorphism whose inverse map is $\varphi^{-1}(y,\lambda) = \lambda y$, $\lambda \in \mathbb{R}^+$, $y \in S^2$. One has $(\varphi \circ j)(y) = (y, 1)$, for all $y \in S^2$. Hence $\varphi_* \circ j_* = (\varphi \circ j)_*$ establishes a diffeomorphism between TS^2 and $T(S^2 \times \{1\}) \subset TS^2 \times T\mathbb{R}^+$. As φ_* is a diffeomorphism we conclude that j_* is a diffeomorphism between TS^2 and the closed submanifold $\varphi^{-1}(T(S^2 \times \{1\})) \subset TU$.

1.10 Vector Fields

Problem 1.89 Consider the vector fields

$$X = x\,y\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}, \qquad Y = y\frac{\partial}{\partial y},$$

on $\mathbb{R}^3$ and the map $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 y$. Compute:

(i) $[X,Y]_{(1,1,0)}$; (ii) $(fX)_{(1,1,0)}$;

(iii) $(Xf)(1,1,0)$; (iv) $f_*(X_{(1,1,0)})$.

Solution

(i)

$$[X,Y]_{(1,1,0)} = \left(-y\,x\frac{\partial}{\partial x}\right)_{(1,1,0)} = -\frac{\partial}{\partial x}\bigg|_{(1,1,0)}.$$

(ii)

$$(fX)_{(1,1,0)} = f(1,1,0)X_{(1,1,0)} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)_{(1,1,0)}.$$

(iii)

$$(Xf)(1,1,0) = X_{(1,1,0)}f = \left(\frac{\partial f}{\partial x}\right)(1,1,0) = 2.$$

(iv)

$$f_*(X_{(1,1,0)}) \equiv \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}\right)_{(1,1,0)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_1,$$

where t denotes the canonical coordinate on $\mathbb{R}$.

Problem 1.90 Write in cylindrical coordinates the vector field on $\mathbb{R}^3$ defined by

$$X = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$

Solution The change from cylindrical coordinates (ρ, θ, z) to Cartesian coordinates is $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$. The Jacobian matrix of this transformation is

$$A = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The field X is written in cylindrical coordinates as

$$X = f_1(\rho,\theta,z)\frac{\partial}{\partial\rho} + f_2(\rho,\theta,z)\frac{\partial}{\partial\theta} + f_3(\rho,\theta,z)\frac{\partial}{\partial z}.$$

Therefore,

$$\begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Hence

$$X = (2\cos\theta - \sin\theta)\frac{\partial}{\partial\rho} - \frac{2\sin\theta + \cos\theta}{\rho}\frac{\partial}{\partial\theta} + 3\frac{\partial}{\partial z}.$$

Problem 1.91 Find the tangent space at the point $p = (1,1,1)$ to the surface S in $\mathbb{R}^3$ defined by the equation $f \equiv x^3 - y^3 + xyz - xy = 0$.

Solution One has

$$\mathrm{d}f = \left(3x^2 + yz - y\right)\mathrm{d}x + \left(-3y^2 + xz - x\right)\mathrm{d}y + xy\,\mathrm{d}z.$$

So, $(\mathrm{d}f)_p = (3\,\mathrm{d}x - 3\,\mathrm{d}y + \mathrm{d}z)_p$.

If

$$X = \left(\lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y} + \lambda_3 \frac{\partial}{\partial z}\right)_p$$

is a vector tangent to S, then $\mathrm{d}f(X) = 0$, and conversely. So, at $(1, 1, 1)$ we must have $\lambda_3 = -3\lambda_1 + 3\lambda_2$. Hence,

$$\begin{aligned} X_p &= \lambda_1 \frac{\partial}{\partial x}\bigg|_p + \lambda_2 \frac{\partial}{\partial y}\bigg|_p + (-3\lambda_1 + 3\lambda_2)\frac{\partial}{\partial z}\bigg|_p \\ &= \lambda_1 \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial z}\right)_p + \lambda_2 \left(\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)_p, \end{aligned}$$

so the vectors $(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial z})_p$ and $(\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z})_p$ are a basis of the tangent space to S at $(1, 1, 1)$.

Problem 1.92 Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the C^∞ function defined by $f(x, y, z) = x^2 + y^2 - 1$, which defines a differentiable structure on $S = f^{-1}(0)$. Consider the vector fields on $\mathbb{R}^3$:

(a) $X = (x^2 - 1)\dfrac{\partial}{\partial x} + x\,y\dfrac{\partial}{\partial y} + x\,z\dfrac{\partial}{\partial z}$; (b) $Y = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + 2x\,z^2\dfrac{\partial}{\partial z}$.

Are they tangent to S?

Hint If $p \in S$ and $X \in T_p\mathbb{R}^3$, X is tangent to the submanifold S if and only if $Xf = 0$.

Solution

(a)

$$Xf = (x^2 - 1)\frac{\partial f}{\partial x} + x\,y\frac{\partial f}{\partial y} + x\,z\frac{\partial f}{\partial z} = 2x(x^2 + y^2 - 1).$$

Thus if $p = (x, y, z) \in S$, $X_p f = 0$. Hence X is tangent to S.

(b) $Yf = 2x^2 + 2y^2$. If $p = (x, y, z) \in S$, then $Y_p f = 2$, so Y is not tangent to S.

Problem 1.93 Find the tangent plane to the one-sheet hyperboloid $H \equiv x^2 + y^2 - z^2 = 1$ at a generic point of itself.

Solution Consider the parametrisation (see Remark 1.4) given by

$$x = \cosh u \sin v, \qquad y = \cosh u \cos v, \qquad z = \sinh u, \quad u \in \mathbb{R},\ v \in (0, 2\pi).$$

We have on the hyperboloid:

$$\frac{\partial}{\partial u} = \sinh u \sin v \frac{\partial}{\partial x} + \sinh u \cos v \frac{\partial}{\partial y} + \cosh u \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial v} = \cosh u \cos v \frac{\partial}{\partial x} - \cosh u \sin v \frac{\partial}{\partial y},$$

that is, $\partial/\partial u$ and $\partial/\partial v$ are respectively the restrictions to the hyperboloid of the vector fields on $\mathbb{R}^3 \setminus \{0\}$ given by

$$X = \frac{x\,z}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y\,z}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} + \sqrt{x^2+y^2}\frac{\partial}{\partial z}, \qquad Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Hence, for $p = (x_0, y_0, z_0) \in H$,

$$\begin{aligned} T_pH &= \left\{\lambda \frac{\partial}{\partial u}\Big|_p + \mu \frac{\partial}{\partial v}\Big|_p,\ \lambda, \mu \in \mathbb{R}\right\} = \{aX_p + bY_p : a, b \in \mathbb{R}\} \\ &= \left\{\left(a\frac{x_0 z_0}{\sqrt{x_0^2+y_0^2}} + by_0\right)\frac{\partial}{\partial x}\Big|_p + \left(\frac{y_0 z_0}{\sqrt{x_0^2+y_0^2}} - bx_0\right)\frac{\partial}{\partial y}\Big|_p \right. \\ &\qquad \left. + a\sqrt{x_0^2+y_0^2}\frac{\partial}{\partial z}\Big|_p,\ a, b \in \mathbb{R}\right\}. \end{aligned}$$

Problem 1.94 Show that the vector fields X, Y, Z given by

$$\begin{aligned} X_p &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}\right)_p, \\ Y_p &= \left(-z\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial t}\right)_p, \\ Z_p &= \left(-t\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}\right)_p, \end{aligned}$$

where $p \in S^3 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}$, define a global parallelisation of S^3.

Solution The vector fields are tangent to S^3, as $\langle X_p, N_p\rangle = \langle Y_p, N_p\rangle = \langle Z_p, N_p\rangle = 0$, where N_p denotes the outward-pointing unit normal vector to S^3 at p,

$$N_p = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)_p,$$

and $\langle\cdot,\cdot\rangle$ denotes the Euclidean product of $T_p\mathbb{R}^4 \equiv \mathbb{R}^4$.

Furthermore, the fields are linearly independent, as

$$\operatorname{rank}\begin{pmatrix} -y & x & -t & z \\ -z & t & x & -y \\ -t & -z & y & x \end{pmatrix} < 3$$

if and only if $p = (0, 0, 0, 0) \notin S^3$.

The fields X, Y, Z are the restriction to S^3 of the fields written similarly on $\mathbb{R}^4$, which are C^∞ on $\mathbb{R}^4$. Since S^3 is an embedded submanifold in $\mathbb{R}^4$, the vector fields given on S^3 are C^∞ on S^3.

Problem 1.95 Give a C^∞ non-vanishing vector field on the sphere S^{2n+1}.

Solution We have $S^{2n+1} = \{p = (x^1, \dots, x^{2n+2}) \in \mathbb{R}^{2n+2} : \sum_{i=1}^{2n+2} (x^i)^2 = 1\}$.

The vector field X defined by

$$X_p = -x^2 \frac{\partial}{\partial x^1}\bigg|_p + x^1 \frac{\partial}{\partial x^2}\bigg|_p + \cdots - x^{2n+2} \frac{\partial}{\partial x^{2n+1}}\bigg|_p + x^{2n+1} \frac{\partial}{\partial x^{2n+2}}\bigg|_p,$$

where $p \in S^{2n+1}$, is tangent to S^{2n+1}. In fact, it is clearly orthogonal to the normal vector

$$N_p = \sum_{i=1}^{2n+2} x^i \frac{\partial}{\partial x^i}\bigg|_p$$

at p with respect to the Euclidean product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^{2n+2}$. Moreover, X is C^∞ since the functions x^i, $i = 1, \dots, 2n+2$, are C^∞. Hence $X \in \mathfrak{X}(S^{2n+1})$.

Problem 1.96 Find the general expression for $X \in \mathfrak{X}(\mathbb{R}^2)$ in the following cases:

(i) $[\frac{\partial}{\partial x}, X] = X$ and $[\frac{\partial}{\partial y}, X] = X$;

(ii) $[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, X] = X$.

Hint (to (ii)) Take new coordinates $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$.

Solution

(i) Let $X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$. Then,

$$\left[\frac{\partial}{\partial x}, X\right] = \frac{\partial a(x,y)}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b(x,y)}{\partial x}\frac{\partial}{\partial y} = X,$$

$$\left[\frac{\partial}{\partial y}, X\right] = \frac{\partial a(x,y)}{\partial y}\frac{\partial}{\partial x} + \frac{\partial b(x,y)}{\partial y}\frac{\partial}{\partial x} = X,$$

from which

$$\frac{\partial a(x,y)}{\partial x} = a(x,y), \qquad \frac{\partial b(x,y)}{\partial x} = b(x,y), \tag{$\star$}$$

$$\frac{\partial a(x,y)}{\partial y} = a(x,y), \qquad \frac{\partial b(x,y)}{\partial y} = b(x,y). \tag{$\star\star$}$$

Solving, from ($\star$) we have

$$a(x,y) = Af(y)\mathrm{e}^x, \qquad b(x,y) = Bg(y)\mathrm{e}^x.$$

Substituting these expressions in (⋆⋆), one has

$$f'(y) = f(y), \qquad g'(y) = g(y),$$

from which $f(y) = C\mathrm{e}^y$, $g(y) = D\mathrm{e}^y$. Hence

$$a(x, y) = E\mathrm{e}^{x+y}, \qquad b(x, y) = F\mathrm{e}^{x+y},$$

and

$$X = \mathrm{e}^{x+y}\left(E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y}\right).$$

(ii) Taking u and v as in the hint, we have $\frac{\partial}{\partial u} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, and one can write $X = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}$. We have

$$\left[\frac{\partial}{\partial u}, X\right] = \frac{\partial a(u, v)}{\partial u}\frac{\partial}{\partial u} + \frac{\partial b(u, v)}{\partial u}\frac{\partial}{\partial v} = X,$$

from which

$$\frac{\partial a(u, v)}{\partial u} = a(u, v), \qquad \frac{\partial b(u, v)}{\partial u} = b(u, v).$$

Hence, as in (i) above, we have $a(u, v) = f(v)\mathrm{e}^u$, $b(u, v) = g(v)\mathrm{e}^u$. So

$$X = f\left(\frac{1}{2}(x - y)\right)\mathrm{e}^{(x+y)/2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + g\left(\frac{1}{2}(x - y)\right)\mathrm{e}^{(x+y)/2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right),$$

that is,

$$X = \mathrm{e}^{(x+y)/2}\left\{h\left(\frac{1}{2}(x - y)\right)\frac{\partial}{\partial x} + k\left(\frac{1}{2}(x - y)\right)\frac{\partial}{\partial y}\right\},$$

for arbitrary C^∞ functions h, k.

Problem 1.97 Consider the two vector fields on $\mathbb{R}^{n+1}$ defined by

$$e_0 = \partial_0, \qquad e_1 = \sum_\alpha f_\alpha\left(x^i\right)\partial_\alpha,$$

where $\partial_0 = \partial/\partial x^0$ and

$$\partial_\alpha = \partial/\partial x^\alpha, \quad 1 \leqslant \alpha \leqslant n, \qquad f_\alpha\left(x^i\right) = f_\alpha\left(x^0, \dots, x^n\right), \quad 0 \leqslant i \leqslant n.$$

We define recursively $e_r = [e_0, e_{r-1}]$, $2 \leqslant r \leqslant n$. Then:

(i) Compute e_r in terms of the vector fields ∂_α.
(ii) Find functions $f_\alpha(x^i)$, such that $e_0, \dots, e_n$ are linearly independent.

Solution

(i) $e_2 = [\partial_0, \sum_\alpha f_\alpha(x^i)\partial_\alpha] = \sum_\alpha \partial_0(f_\alpha(x^i))\partial_\alpha$.

We proceed by induction: Suppose $e_r = \sum_\alpha \partial_0^{r-1}(f_\alpha(x^i))\partial_\alpha$; then,

$$\begin{aligned}
e_{r+1} &= [\partial_0, e_r] = \sum_\alpha \bigl[\partial_0, \partial_0^{r-1}\bigl(f_\alpha\bigl(x^i\bigr)\bigr)\partial_\alpha\bigr] = \sum_\alpha \partial_0\bigl(\partial_0^{r-1}\bigl(f_\alpha\bigl(x^i\bigr)\bigr)\bigr)\partial_\alpha \\
&= \sum_\alpha \partial_0^r\bigl(f_\alpha\bigl(x^i\bigr)\bigr)\partial_\alpha.
\end{aligned}$$

(ii) Take $f_\alpha(x^i) = (x^0)^{\alpha-1}$. Then

$$\begin{aligned}
e_0 &= \partial_0, \\
e_1 &= \sum_\alpha f_\alpha\bigl(x^i\bigr)\partial_\alpha = \sum_\alpha \bigl(x^0\bigr)^{\alpha-1}\partial_\alpha = \partial_1 + x^0\partial_2 + \cdots + \bigl(x^0\bigr)^{n-1}\partial_n, \\
e_2 &= \sum_\alpha \partial_0\bigl(f_\alpha\bigl(x^i\bigr)\bigr)\partial_\alpha = \sum_\alpha \partial_0\bigl(\bigl(x^0\bigr)^{\alpha-1}\bigr)\partial_\alpha = \sum_\alpha (\alpha-1)\bigl(\bigl(x^0\bigr)^{\alpha-2}\bigr)\partial_\alpha \\
&= \partial_2 + 2x^0\partial_3 + 3\bigl(x^0\bigr)^2\partial_4 + \cdots + (n-1)\bigl(x^0\bigr)^{n-2}\partial_n, \\
e_3 &= \sum_\alpha \partial_0^2\bigl(f_\alpha\bigl(x^i\bigr)\bigr)\partial_\alpha = \sum_\alpha (\alpha-1)(\alpha-2)\bigl(x^0\bigr)^{\alpha-3}\partial_\alpha, \\
&\ \ \dots \\
e_n &= \sum_\alpha (\alpha-1)(\alpha-2)\dots(\alpha-n+1)\bigl(x^0\bigr)^{\alpha-n}\partial_\alpha = (n-1)!\,\partial_n.
\end{aligned}$$

1.10.1 Integral Curves

Problem 1.98 Is every vector field on the real line $\mathbb{R}$ complete?

Solution Let

$$X = x^2\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}).$$

The integral curves are the solutions of the equation $x'(t) = x^2(t)$, i.e. $x'(t)/x^2(t) = 1$, whose solution is $x(t) = -1/(t + A)$. The integral curve through x_0 verifies $x(0) = x_0$, hence $x_0 = -1/A$, thus it is the curve

$$x(t) = \frac{x_0}{1 - tx_0},$$

which is not defined for $t = 1/x_0$, so X is not complete.

Problem 1.99 Compute the integral curves of the vector field on $\mathbb{R}^3$ given by

$$X = y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}.$$

Solution The tangent vector at a point p of an integral curve γ of the vector field X coincides with the value of X at p.

Let $\gamma(t) = (x(t), y(t), z(t))$. Hence, $\gamma'(t) = (x'(t), y'(t), z'(t))$, where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 2,$$

from which the integral curves are of the type $\gamma(t) = (A\mathrm{e}^t + B, A\mathrm{e}^t, 2t + C)$ and the curve passing through (x_0, y_0, z_0) for $t = 0$ is

$$\gamma(t) = \big(x_0 + y_0\big(\mathrm{e}^t - 1\big),\ y_0\mathrm{e}^t,\ 2t + z_0\big).$$

Problem 1.100 For each of the following vector fields find its integral curves and study whether it is complete or not:

(i) $X = \dfrac{\partial}{\partial x} \in \mathfrak{X}\big(\mathbb{R}^2 \setminus \{0\}\big)$; (ii) $X = \dfrac{\partial}{\partial y} + \mathrm{e}^x \dfrac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$;

(iii) $X = \mathrm{e}^{-x} \dfrac{\partial}{\partial x}$; (iv) $X = y\dfrac{\partial}{\partial x}$, $Y = \dfrac{x^2}{2}\dfrac{\partial}{\partial y}$, $[X, Y]$;

(v) $X = x\dfrac{\partial}{\partial x}$; (vi) $X = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$.

The last four vector fields belong to $\mathfrak{X}(\mathbb{R}^2)$.

Solution

(i) The integral curves are the solutions of the system

$$x'(t) = 1, \qquad y'(t) = 0;$$

thus,

$$x(t) = t + A, \qquad y(t) = B,$$

and the integral curve of X through a given point (x_0, y_0) is

$$x(t) = t + x_0, \qquad y(t) = y_0.$$

If $x_0 > 0$, the maximal integral curve through $(x_0, 0)$ is defined only for the interval $(-x_0, +\infty)$. Hence X is not complete.

(ii) The integral curves are the solutions of the system

$$x'(t) = 0, \qquad y'(t) = 1, \qquad z'(t) = \mathrm{e}^{x(t)};$$

thus,

$$x(t) = A, \qquad y(t) = t + B, \qquad z(t) = \mathrm{e}^A t + C.$$

The integral curve of X through (x_0, y_0, z_0) is

$$x(t) = x_0, \qquad y(t) = t + y_0, \qquad z(t) = \mathrm{e}^{x_0} t + z_0,$$

which is defined for $t \in \mathbb{R}$, so X is complete.

(iii) The integral curves are the solutions of the system

$$\mathrm{e}^{x(t)} x'(t) = 1, \qquad y'(t) = 0;$$

thus,

$$\mathrm{e}^{x(t)} = t + A, \qquad y(t) = B;$$

that is,

$$x(t) = \log(t + A), \qquad y(t) = B.$$

The integral curve of X through (x_0, y_0) is

$$x(t) = \log\big(t + \mathrm{e}^{x_0}\big), \qquad y(t) = y_0.$$

X is not complete as this curve is only defined for $t \in (-\mathrm{e}^{x_0}, +\infty)$.

(iv) The integral curves of X are the solutions of the system

$$x'(t) = y(t), \qquad y'(t) = 0.$$

The integral curve through (x_0, y_0) is $x(t) = y_0 t + x_0$, $y(t) = y_0$. Hence, X is complete.

Similarly, for Y we have

$$x'(t) = 0, \qquad y'(t) = \frac{x^2(t)}{2}.$$

Hence $x(t) = x_0$. So $y'(t) = \frac{1}{2} x_0^2$ and thus $y(t) = \frac{1}{2} x_0^2 t + y_0$. Hence, Y is complete.

As for

$$[X, Y] = xy \frac{\partial}{\partial y} - \frac{x^2}{2} \frac{\partial}{\partial x},$$

we have the system

$$x'(t) = -\frac{x^2(t)}{2}, \qquad y'(t) = y(t)x(t).$$

As in Problem 1.98, we obtain

$$x(t) = \frac{2x_0}{x_0 t + 2}.$$

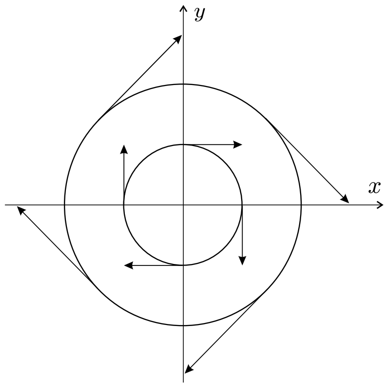


Fig. 1.29 Integral curves of the vector field $X = y\,\partial/\partial x - x\,\partial/\partial y$

So we have $y'(t)/y(t) = 2x_0/(x_0 t + 2)$, thus $\log y(t) = 2\log(x_0 t + 2) + \log B$. Since $y(0) = y_0$ it follows that $y_0 = 4B$. Therefore,

$$y(t) = \frac{y_0}{4}(x_0 t + 2)^2.$$

Hence $[X, Y]$ is not complete as its integral curve is not defined for $t = -2/x_0$.

(v) The integral curves are the solutions of the system

$$x'(t) = x(t), \qquad y'(t) = 0.$$

Hence the integral curve through (x_0, y_0) is

$$x(t) = x_0 \mathrm{e}^t, \qquad y(t) = y_0.$$

The graph is a horizontal half-line on $\mathbb{R}^2$ of exponential speed, with $x \in (-\infty, 0)$ or $(0, +\infty)$ depending on either $x_0 < 0$ or $x_0 > 0$, respectively. The graph is the point $(0, y_0)$ if $x_0 = 0$. X is complete.

(vi) The integral curves are the solutions of the system

$$x'(t) = y(t), \qquad y'(t) = -x(t).$$

That is,

$$x(t) = A\sin t + B\cos t, \qquad y(t) = -B\sin t + A\cos t.$$

As $x(0) = x_0 = B$, $y(0) = y_0 = A$, the integral curve through (x_0, y_0) is

$$x(t) = y_0\sin t + x_0\cos t, \qquad y(t) = -x_0\sin t + y_0\cos t.$$

Since $x^2(t) + y^2(t) = x_0^2 + y_0^2$, the integral curves are the circles with centre at the origin (see Fig. 1.29). The vector field is complete.

1.10.2 Flows

Problem 1.101 For each $t \in \mathbb{R}$, consider the map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x, y) \mapsto \varphi_t(x, y) = (x\cos t + y\sin t, -x\sin t + y\cos t).$$

(i) Prove that φ_t is a 1-parameter group of transformations of $\mathbb{R}^2$.
(ii) Calculate the associated vector field X.
(iii) Describe the orbits.
(iv) Prove that X is invariant by φ_t, that is, that $\varphi_{t*}X_p = X_{\varphi_t(p)}$.

Solution

(i) Each φ_t is trivially C^∞. Furthermore,
(a) $\varphi_0(x, y) = (x, y)$, thus $\varphi_0 = \mathrm{id}_{\mathbb{R}^2}$.
(b)

$$\begin{aligned}(\varphi_t \circ \varphi_s)(x, y) &= \varphi_t(x\cos s + y\sin s, -x\sin s + y\cos s)\\ &= \big(x\cos(s+t) + y\sin(s+t), -x\sin(s+t) + y\cos(s+t)\big)\\ &= \varphi_{t+s}(x, y).\end{aligned}$$

(ii) We have $X = \lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y}$, with

$$\lambda_1(x, y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(x\cos t + y\sin t) = y,$$

$$\lambda_2(x, y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(-x\sin t + y\cos t) = -x,$$

that is, $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.
(iii) The orbit through $p = (x_0, y_0)$ is the image of the map $\mathbb{R} \to \mathbb{R}^2$ given by

$$t \mapsto (x_0\cos t + y_0\sin t, -x_0\sin t + y_0\cos t),$$

that is, a circle centred at the origin and passing through $p = (x_0, y_0)$. If $p = (0, 0)$, the orbit reduces to the point p.
(iv) If $p = (x_0, y_0)$, then

$$X_{\varphi_t(p)} = (-x_0\sin t + y_0\cos t)\left.\frac{\partial}{\partial x}\right|_{\varphi_t(p)} - (x_0\cos t + y_0\sin t)\left.\frac{\partial}{\partial y}\right|_{\varphi_t(p)}.$$

Hence

$$\varphi_{t*}X_p \equiv \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}y_0\\ -x_0\end{pmatrix} \equiv X_{\varphi_t(p)}.$$

Problem 1.102 Let TM be the tangent bundle over a differentiable manifold M. Let $\varphi\colon \mathbb{R}\times TM \to TM$ defined by $\varphi(t,X)=\mathrm{e}^t X$.

(i) Prove that φ is a 1-parameter group of transformations of TM.
(ii) Calculate the vector field Y on TM associated to φ.
(iii) Prove that Y is invariant under φ.

Solution

(i) Let $\varphi_t\colon TM \to TM$, $X \mapsto \mathrm{e}^t X$. Obviously $\varphi_0 = \mathrm{id}_{TM}$. Furthermore,

$$(\varphi_t \circ \varphi_s)X = \varphi_t\big(\mathrm{e}^s X\big) = \mathrm{e}^{s+t} X = \varphi_{t+s} X,$$

so $\varphi_t \circ \varphi_s = \varphi_{t+s}$.

Let us see that φ is differentiable. Pick $(t_0, X_0) \in \mathbb{R}\times TM$. Let π denote the canonical projection from TM to M. Let $p=\pi(X_0)\in M$ and $(U, \psi = (x^1,\dots,x^n))$ be a coordinate system on a neighbourhood of p. Let $(\pi^{-1}(U), \Psi)$ be the chart in TM built from (U,ψ), that is,

$$\Psi = (\psi \times \mathrm{id}_{\mathbb{R}^n})\circ\tau\colon \pi^{-1}(U) \to \psi(U)\times\mathbb{R}^n,$$

with $\tau\colon \pi^{-1}(U)\to U\times\mathbb{R}^n$, where

$$\tau\left(\left(\lambda^1\frac{\partial}{\partial x^1}+\cdots+\lambda^n\frac{\partial}{\partial x^n}\right)_x\right) = \big(x,\lambda^1,\dots,\lambda^n\big).$$

Let us denote $\Psi = (x^1,\dots,x^n,y^1,\dots,y^n)$. Then, given $Z_0\in T_qM$, $q\in U$, such that

$$\begin{aligned}\Psi(Z_0) &= \big(\big(x^1\circ\pi\big)(Z_0),\dots,\big(x^n\circ\pi\big)(Z_0), y^1(Z_0),\dots,y^n(Z_0)\big)\\ &= \big(x^1(q),\dots,x^n(q),y^1(Z_0),\dots,y^n(Z_0)\big) = \big(a^1,\dots,a^n,b^1,\dots,b^n\big),\end{aligned}$$

we have, taking on $\mathbb{R}$ the chart $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$:

$$\begin{aligned}&\big(\Psi\circ\varphi\circ(\mathrm{id}_{\mathbb{R}}\times\Psi)^{-1}\big)\big(t,a^1,\dots,a^n,b^1,\dots,b^n\big)\\ &\quad= (\Psi\circ\varphi)(t,Z_0) = \Psi\big(\mathrm{e}^t Z_0\big)\\ &\quad= \big(x^1(q),\dots,x^n(q),\mathrm{e}^t y^1(Z_0),\dots,\mathrm{e}^t y^n(Z_0)\big)\\ &\quad= \big(a^1,\dots,a^n,\mathrm{e}^t b^1,\dots,\mathrm{e}^t b^n\big).\end{aligned}$$

Hence φ is differentiable.

(ii) Let Y be the vector field generated by φ. Let $X_0\in TM$ and $p=\pi(X_0)$ and consider as before the charts (U,ψ) in p, with $\psi=(x^1,\dots,x^n)$ and $(\pi^{-1}(U),\Psi)$ in X_0, with $\Psi=(x^1,\dots,x^n,y^1,\dots,y^n)$.

Then $Y\colon TM \to TTM$ has at X_0 the expression

$$Y_{X_0} = \sum_{i=1}^{n}\left(Y_{X_0}\big(x^i\big)\frac{\partial}{\partial x^i}\bigg|_{X_0} + Y_{X_0}\big(y^i\big)\frac{\partial}{\partial y^i}\bigg|_{X_0}\right).$$

As Y is generated by φ, one has

$$Y_{X_0}\big(x^i\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(x^i\big(\varphi(t, X_0)\big)\big), \qquad Y_{X_0}\big(y^i\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(y^i\big(\varphi(t, X_0)\big)\big).$$

Thus

$$Y_{X_0}\big(x^i\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(x^i\big(\mathrm{e}^t X_0\big)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(x^i(p)\big) = 0,$$

$$Y_{X_0}\big(y^i\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(\mathrm{e}^t y^i(X_0)\big) = y^i(X_0).$$

So $Y_{X_0} = \sum_{i=1}^{n} y^i(X_0)\frac{\partial}{\partial y^i}|_{X_0}$, hence $Y = \sum_{i=1}^{n} y^i\frac{\partial}{\partial y^i}$.

(iii) It suffices to prove that $(\varphi_{t*})_{X_0} Y_{X_0} = Y_{\mathrm{e}^t X_0}$. We know that

$$Y_{\mathrm{e}^t X_0} = \mathrm{e}^t \sum_{i=1}^{n} y^i(X_0)\frac{\partial}{\partial y^i}\bigg|_{\mathrm{e}^t X_0}.$$

On the other hand,

$$\varphi_t\big(a^1, \dots, a^n, b^1, \dots, b^n\big) = \big(a^1, \dots, a^n, \mathrm{e}^t b^1, \dots, \mathrm{e}^t b^n\big),$$

so that the matrix associated to φ_{t*} is $\left(\begin{smallmatrix} I_n & 0 \\ 0 & \mathrm{e}^t I_n \end{smallmatrix}\right)$.

Since $Y_{X_0} = (0, \dots, 0, y^1(X_0), \dots, y^n(X_0))$, we have

$$\begin{aligned}(\varphi_{t*})_{X_0}(Y_{X_0}) &= \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{e}^t I_n \end{pmatrix} {}^t\big(0, \dots, 0,\ y^1(X_0), \dots,\ y^n(X_0)\big) \\ &= \big(0, \dots, 0,\ \mathrm{e}^t y^1(X_0), \dots,\ \mathrm{e}^t y^n(X_0)\big) = Y_{\mathrm{e}^t X_0},\end{aligned}$$

as expected.

1.10.3 Transforming Vector Fields

Problem 1.103 Consider the projection $p\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Find the condition that a vector field on $\mathbb{R}^2$ must verify to be p-related to some vector field on $\mathbb{R}$.

Solution Let

$$X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} \in \mathfrak{X}\big(\mathbb{R}^2\big).$$

In order for X to be p-related to some vector field on $\mathbb{R}$, it must happen that for each couple of points $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ such that $p(x_0, y_0) = p(x_1, y_1)$ one has

$$p_*\left(\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right)_{(x_0,y_0)}\right) = p_*\left(\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right)_{(x_1,y_1)}\right). \tag{$\star$}$$

Since for the given pair of points we have $x_0 = x_1$, we can write such a couple of points in the form $(x, y_0), (x, y_1)$, and we have

$$p_*\left(\left.\frac{\partial}{\partial x}\right|_{(x,y_0)}\right) = p_*\left(\left.\frac{\partial}{\partial x}\right|_{(x,y_1)}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_x,$$

$$p_*\left(\left.\frac{\partial}{\partial y}\right|_{(x,y_0)}\right) = p_*\left(\left.\frac{\partial}{\partial y}\right|_{(x,y_1)}\right) = 0,$$

where t is the canonical coordinate on $\mathbb{R}$. Substituting in ($\star$), we obtain the condition we are looking for: $a(x, y_0) = a(x, y_1)$, for all x, y_0, y_1.

Problem 1.104 Let $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ be endowed with the natural differentiable structure as an open subset of $\mathbb{R}^2$, and let $f\colon M \to \mathbb{R}$, $(x, y) \mapsto x$.

(i) Prove that $X_{(x,y)} \equiv (x/r^3, y/r^3)$, where $r = \sqrt{x^2 + y^2}$, is a C^∞ vector field on M.
(ii) Is X f-related to a vector field on $\mathbb{R}$?

Solution

(i) The functions $M \to \mathbb{R}$, given by $(x, y) \mapsto x/\sqrt{(x^2 + y^2)^3}$, and $(x, y) \mapsto y/\sqrt{(x^2 + y^2)^3}$, are C^∞ on M.
(ii) No, as if X were f-related to a vector field on $\mathbb{R}$, then (as in Problem 1.103) it would be $f_* X_p = f_* X_{p'}$ if $p = (x_0, y_0)$, $p' = (x_0, y'_0)$, $y_0 \neq y'_0$, and this is not the case, as it is proved below.

The matrix associated to f_* with respect to the bases $\{\partial/\partial x, \partial/\partial y\}$ and $\{\mathrm{d}/\mathrm{d}t\}$, that is, the Jacobian matrix of $\mathrm{id}_{\mathbb{R}} \circ f \circ \mathrm{id}_M^{-1}$, is $(1\ 0)$. Consequently, if $p = (x_0, y_0) \in M$, we have

$$f_* X_p = \frac{x_0}{\sqrt{(x_0^2 + y_0^2)^3}} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{x_0}.$$

Hence $f_* X_p \neq f_* X_{p'}$.

Problem 1.105 Consider the 2-torus $T^2 = S^1 \times S^1$. Consider the submersion

$$f\colon \mathbb{R}^2 \to T^2, \qquad f(\theta, \theta') = \left(\mathrm{e}^{\theta\mathrm{i}}, \mathrm{e}^{\theta'\mathrm{i}}\right),$$

and a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. Under which condition is X f-projectable onto a vector field Y on T^2?

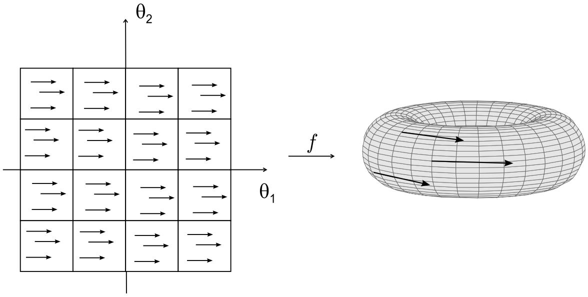


Fig. 1.30 A vector field on $\mathbb{R}^2$ inducing a vector field on T^2

Solution It is immediate that the condition is $f_*(X_{(\theta+2k\pi,\theta'+2k'\pi)}) = f_*(X_{(\theta,\theta')})$. Equivalently, X must be invariant under the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ defined by $(k,k') \cdot (\theta,\theta') = (\theta + 2k\pi, \theta' + 2k'\pi)$ (see Fig. 1.30).

Problem 1.106 Let $f\colon M \to N$ be a C^∞ map and X and Y be f-related C^∞ vector fields. Prove that f maps integral curves of X into integral curves of Y.

Solution For the integral curve of X through $p \in M$, $\sigma\colon (-\varepsilon,\varepsilon) \to M$, one has

$$\text{(a)}\quad \sigma \text{ is } C^\infty; \qquad \text{(b)}\quad \sigma(0) = p;$$

$$\text{(c)}\quad \sigma_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) = X_{\sigma(t_0)}, \qquad \text{for all } t_0 \in (-\varepsilon,\varepsilon).$$

Then the map $f \circ \sigma\colon (-\varepsilon,\varepsilon) \to N$ satisfies:

(i) $f \circ \sigma$ is differentiable as a composition of differentiable maps.
(ii) $(f \circ \sigma)(0) = f(p)$.
(iii)

$$\begin{aligned}
(f \circ \sigma)_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) &= (f_{*\sigma(t_0)} \circ \sigma_{*t_0})\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) \\
&= f_{*\sigma(t_0)}\left(\sigma_{*t_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right)\right) = f_*(X_{\sigma(t_0)}) \\
&= Y_{(f\circ\sigma)(t_0)} \quad (X \text{ and } Y \text{ are } f\text{-related}).
\end{aligned}$$

That is, $f \circ \alpha$ is the integral curve of Y passing through $f(p)$.

Problem 1.107 Let $\varphi\colon M \to N$ be a diffeomorphism between the C^∞ manifolds M and N. Given $X \in \mathfrak{X}(M)$, the *vector field image* $\varphi \cdot X$ of X is defined by

$$(\varphi \cdot X)_x = \varphi_*(X_{\varphi^{-1}(x)}).$$

Prove:

(i) In fact, $\varphi \cdot X \in \mathfrak{X}(N)$.
(ii) $\varphi \cdot [X, Y] = [\varphi \cdot X, \varphi \cdot Y]$, $X, Y \in \mathfrak{X}(M)$.

Solution

(i) From the definition of φ_*, it is immediate that the image of a vector is a vector. Moreover, $\varphi \cdot X$ is C^∞, which follows from

$$\varphi \cdot X = \varphi_* \circ X \circ \varphi^{-1}.$$

Further, we have, denoting by π_{TM} (resp., π_{TN}) the projection map of the tangent bundle over M (resp., N), that $\pi_{TN} \circ (\varphi \cdot X) = \mathrm{id}$. In fact,

$$\pi_{TN} \circ \varphi_* \circ X \circ \varphi^{-1} = \varphi \circ \pi_{TM} \circ X \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \mathrm{id}.$$

(ii) From the definition of $\varphi \cdot X$ it follows that $(\varphi \cdot X) f = X(f \circ \varphi) \circ \varphi^{-1}$. Hence, for any $p \in N$, one has

$$\begin{aligned}
\big(\varphi \cdot [X, Y]\big)_p f &= [X, Y]_{\varphi^{-1}(p)}(f \circ \varphi) \\
&= X_{\varphi^{-1}(p)}\big(Y(f \circ \varphi)\big) - Y_{\varphi^{-1}(p)}\big(X(f \circ \varphi)\big) \\
&= X_{\varphi^{-1}(p)}\big((\varphi \cdot Y)(f) \circ \varphi\big) - Y_{\varphi^{-1}(p)}\big((\varphi \cdot X)(f) \circ \varphi\big) \\
&= (\varphi \cdot X)_p\big((\varphi \cdot Y) f\big) - (\varphi \cdot Y)_p\big((\varphi \cdot X) f\big) = [\varphi \cdot X, \varphi \cdot Y]_p f.
\end{aligned}$$

Problem 1.108 Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \mathrm{e}^x$. Find the vector field image $f \cdot \partial/\partial x$.

Solution The Jacobian of f is e^x. We have, for any fixed x_0, that

$$\left(f \cdot \frac{\partial}{\partial x}\right)_{x_0} = f_*\left(\left.\frac{\partial}{\partial x}\right|_{f^{-1}(x_0)}\right) = f_*\left(\left.\frac{\partial}{\partial x}\right|_{\log x_0}\right) = \left(x \frac{\partial}{\partial x}\right)_{x_0}.$$

Hence

$$f \cdot \frac{\partial}{\partial x} = x \frac{\partial}{\partial x}.$$

Problem 1.109 Let $\pi : M \to N$ be a surjective submersion of manifolds such that the set $\pi^{-1}(q)$ is a compact connected set for each point $q \in N$. Suppose that the vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are π-related, i.e. $\pi_{*p} X_p = Y_{\pi(p)}$ for all $p \in M$.

Prove that the vector fields X and Y are complete or incomplete simultaneously.

Solution Since the vector fields X and Y are π-related, for any integral curve $\Gamma(t)$ of X its image $\gamma(t) = \pi(\Gamma(t))$ is an integral curve of Y, i.e. Y is complete if X is complete.

Suppose now that the vector field Y is complete. Choose a point $\tilde{p} \in M$. Let $\gamma(t)$, $t \in [0, T]$, $0 < T < \infty$, be any integral curve of Y such that $\gamma(0) = \pi(\tilde{p})$. It is evident that the set $\gamma([0, T]) \subset N$ is compact (as a continuous image of a compact set). To solve the problem we will prove first that its preimage $\pi^{-1}(\gamma([0, t]))$ is a compact subset of M.

Indeed, using the well-known Whitney Embedding Theorem, we can suppose that M is an embedded submanifold of $\mathbb{R}^l$ for some $l \in \mathbb{N}$; in particular, the topology on M is induced by the standard topology on $\mathbb{R}^l$. Suppose that the set $\pi^{-1}(\gamma([0, T]))$ is not compact. Then the set either $\pi^{-1}(\gamma([0, T/2]))$ or $\pi^{-1}(\gamma([T/2, T]))$ is not compact. So that there exists a sequence of compact sets

$$[0, T] = [a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

such that each set $A_n = \pi^{-1}(\gamma([a_n, b_n]))$ is not compact. There exists a unique common point $t_0 = \bigcap_{n\in\mathbb{N}}[a_n, b_n]$. Put $q_0 = \gamma(t_0)$. Since the set $\pi^{-1}(q_0)$ is compact (in M and, consequently, in $\mathbb{R}^l$), there exists an open ball B in $\mathbb{R}^l$ containing this set and such that the intersection of this set with the sphere $S = \partial B$ is empty. Now we prove that $A_n \cap S \neq \emptyset$. To this end assume that $A_n \cap S = \emptyset$, i.e. $A_n \subset (\mathbb{R}^l \setminus \{\bar{B}\}) \cup B$. But each (closed) set A_n is not compact, i.e. is unbounded in $\mathbb{R}^l$ and thus $A_n \cap (\mathbb{R}^l \setminus \{\bar{B}\}) \neq \emptyset$. Also $A_n \cap B \neq \emptyset$ as $\pi^{-1}(q_0) \subset A_n \cap B$. Taking into account that the set $\gamma([a_n, b_n])$ is connected and the map π is open (an image of each open subset is open), we obtain that the set

$$\pi(A_n \cap B) \cap \pi\big(A_n \cap \big(\mathbb{R}^l \setminus \{\bar{B}\}\big)\big)$$

is not empty, i.e. it contains some point q' for which

$$\pi^{-1}\big(q'\big) \cap B \neq \emptyset \quad \text{and} \quad \pi^{-1}\big(q'\big) \cap \big(\mathbb{R}^l \setminus \{\bar{B}\}\big) \neq \emptyset.$$

Also by assumption,

$$\pi^{-1}\big(q'\big) \subset A_n \subset \big(\mathbb{R}^l \setminus \{\bar{B}\}\big) \cup B.$$

But the set $\pi^{-1}(q')$ is connected, thus we are led to a contradiction. Thus there exists a point $p_n \in S \cap A_n$. By compactness of the (closed) set $S \cap M \subset M$, some subsequence $\{p_{n_k}\} \subset \{p_n\}$ converges to a point $\bar{p} \in S \cap M$. By continuity of π and γ, the sequence

$$\big\{\pi(p_{n_k}) = \gamma(t_{n_k})\big\}, \quad t_{n_k} \in [a_{n_k}, b_{n_k}],$$

converges to the point $\pi(\bar{p}) = \gamma(t_0) = q_0$. But $\bar{p} \in S$ and $\pi^{-1}(q_0) \cap S = \emptyset$, which is a contradiction, i.e. the set $\pi^{-1}(\gamma([0, T]))$ is compact.

Since the vector field X is smooth, for each point $p \in M$ there exists a neighbourhood U of p, a positive number $\varepsilon \in \mathbb{R}$ and a local 1-parameter group $\phi_t : U \to M$ of X defined for all $t \in (-\varepsilon, \varepsilon)$. Since the set $\pi^{-1}(\gamma([0, T]))$ is compact, there exists a finite covering $\{U^k\}$, $k = 1, \ldots, n$, of it, with corresponding groups ϕ_t^k, $t \in (-\varepsilon_k, \varepsilon_k)$. In other words, for each point $p \in \pi^{-1}(\gamma([0, T]))$ there exists the integral curve $\Gamma(t)$, $\Gamma(0) = p$ of X defined for all $t \in (-\varepsilon_0, \varepsilon_0)$, where $\varepsilon_0 = \min\{\varepsilon_1, \dots, \varepsilon_n\}$.

Consider now the point $\tilde{p}$ (recall that $\pi(\tilde{p}) = \gamma(0)$) and the integral curve $\Gamma_b \colon [0, b] \to M$ of X such that $\Gamma(0) = \tilde{p}$ and $0 < b$. Since the vector fields X and Y are π-related, we have that

$$\pi\big(\Gamma(t)\big) = \gamma(t) \quad \text{if } 0 \leqslant t \leqslant \min\{b, T\}.$$

If $b < T$, then $\Gamma(b) \in \pi^{-1}(\gamma([0, T]))$ and, consequently, as we remarked above, the extension $\Gamma_{b+(\varepsilon_0/2)}$ exists. Thus there exists the integral curve Γ_T which is an extension of Γ_b. Since T is arbitrary, the vector field X is complete.

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Chapter 2
Tensor Fields and Differential Forms

Abstract After providing some definitions and results on tensor fields and differential forms, this chapter deals with some aspects of general vector bundles, including the 'cocycle approach'; other topics are: Tensors and tensor fields, exterior forms, Lie derivative and the interior product; calculus of differential forms and distributions. Some examples related to manifolds studied in the previous chapter are also present, such as the infinite Möbius strip, considered as a vector bundle, and the tautological bundle over the real Grassmannian. Certain problems intend to make the reader familiar with computations of vector fields, differential forms, Lie derivative, the interior product, the exterior differential, and their relationships. Other group of problems tries to develop practical abilities in computing integral distributions and differential ideals.

> L'algorithme du Calcul différentiel absolu, c'est à dire l'instrument matériel des méthodes (...) se trouve tout entier dans une remarque due a M. Christoffel (...) Mais les méthodes mêmes et les avantages, qu'ils présentent, ont leur raison d'être et leur source dans les rapports intimes, que les lient à la notion de variété à n dimensions, qui nous devons aux génies de Gauss et de Riemann. D'après cette notion une variété V_n est définie intrinséquement dans ses propriétés métriques par n variables indépendants et par toute une classe de formes quadratiques des différentielles de ces variables, dont deux quelconques son transformables l'une en l'autre par une transformation ponctuelle. Par conséquence une V_n reste invariée vis-à-vis de toute transformation de ses coordonnées. La Calcul differentiel absolu, en agissant sur des formes covariantes ou contrevariants au ds^2 de V_n pour en dériver d'autres de même nature, est lui aussi dans ses formules et dans ses résultats indépendent du choix des variables indépendantes. Étant de la sorte essentiellement attaché à V_n, il est l'instrument naturel de toutes les recherches, qui ont pour object une telle variété, ou dans lesquelles on rencontre comme élément caractéristique une forme quadratique positive des différentielles de n variables ou de leurs dérivées.[1]

[1]The algorithm of absolute differential Calculus, that is, the material instrument of the methods (...) is fully included in a remark by Mr. Christoffel (...) But the methods themselves and their

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*, Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_2,

GREGORIO RICCI-CURBASTRO AND TULLIO LEVI-CIVITA, *Méthodes de calcul differentiel absolu et leurs applications*, Math. Annalen *54* (1900), no. 1–2, 127–128. (With kind permission from Springer.)

However Einstein realised his problems: '*If all accelerated systems are equivalent, then Euclidean geometry cannot hold in all of them.*' Einstein then remembered that he had studied Gauss' theory of surfaces as a student and suddenly realised that the foundations of geometry have physical significance. He consulted his friend [and mathematician] Grossmann who was able to tell Einstein of the important developments of Riemann, Ricci (Ricci-Curbastro) and Levi-Civita. Einstein wrote: '... *in all my life I have not laboured nearly so hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now.*' In 1913 Einstein and Grossmann published a joint paper where the tensor calculus of Ricci and Levi-Civita is employed to make further advances. Grossmann gave Einstein the Riemann–Christoffel tensor which, together with the Ricci tensor which can be derived from it, were to become the major tools in the future theory. Progress was being made in that gravitation was described for the first time by the metric tensor but still the theory was not right. (...) It was the second half of 1915 that saw Einstein finally put the theory in place. Before that however he had written a paper in October 1914 nearly half of which is a treatise on tensor analysis and differential geometry. This paper led to a correspondence between Einstein and Levi-Civita in which Levi-Civita pointed out technical errors in Einstein's work on tensors. Einstein was delighted to be able to exchange ideas with Levi-Civita whom he found much more sympathetic to his ideas on relativity than his other colleagues.

JOHN O'CONNOR AND EDMUND F. ROBERTSON, Article *General Relativity,* in 'The MacTutor History of Mathematics archive,' School of Mathematics and Statistics, University of St. Andrews, Scotland. (With kind permission from the authors.)

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advantages have their foundation and their source in the intimate links they have with the notion of n-dimensional manifold, which we owe to the geniuses of Gauss and Riemann. According to this notion, a manifold V_n is intrinsically defined with respect to its metric properties by n independent variables and by a full class of quadratic forms of the differentials of these variables, such that any two may be mutually transformed by a pointwise transformation. Consequently, a V_n remains invariant under any transformation of its coordinates. The absolute differential Calculus, dealing with covariant or contravariant forms of the ds^2 of V_n, in order to obtain other ones of the same nature, is itself independent of the choice of independent variables inside its formulas and its results. Being so essentially linked to V_n, it is a natural tool of all the researches on such a manifold (...) or one meets positive quadratic differential forms and their derivatives."

2.1 Some Definitions and Theorems on Tensor Fields and Differential Forms

Definitions 2.1 Let $\xi = (E, \pi, M)$ be a locally trivial bundle with fibre F over M. A *chart* on ξ is a pair (U, Ψ) consisting of an open subset $U \subset M$ and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \Psi = \pi$, where $\mathrm{pr}_1 : U \times F \to U$ is the first projection map. Ψ is called a *trivialisation of* ξ *over* U.

Let V be real vector space of finite dimension n, and let $\xi = (E, \pi, M)$ be a locally trivial bundle of fibre V. A structure of *vector bundle on* ξ is given by a family $\mathscr{A} = \{(U_\alpha, \Psi_\alpha)\}$ of charts on ξ satisfying:

(i) U_α is an open covering of the base space M.
(ii) For each pair (α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$, one has

$$\left(\Psi_\beta \circ \Psi_\alpha^{-1}\right)(p, v) = \left(p, g_{\beta\alpha}(p)v\right), \quad (p, v) \in (U_\alpha \cap U_\beta) \times V,$$

where $g_{\alpha\beta}$ is a C^∞ map from $U_\alpha \cap U_\beta$ to the group $\mathrm{GL}(V)$ of automorphisms of V.
(iii) If $\mathscr{A}' \supset \mathscr{A}$ is a family of charts on ξ satisfying properties (i), (ii) above, then $\mathscr{A}' = \mathscr{A}$.

Such a bundle $\xi = (E, \pi, M, \mathscr{A})$, or simply $\xi = (E, \pi, M)$, is called a (real) *vector bundle* of rank n. The C^∞ maps $g_{\alpha\beta} : M \to \mathrm{GL}(V)$ are called the *changes of charts* of the atlas $\mathscr{A}$.

Proposition 2.2 *The changes of charts of a vector bundle have the property* (*called the* cocycle condition)

$$g_{\alpha\gamma}(p) g_{\gamma\beta}(p) = g_{\alpha\beta}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 2.3 Two vector bundles of rank n are said to be *equivalent* if they are isomorphic and have the same base space B.

One has the following converse to Proposition 2.2:

Theorem 2.4 *Let* $\mathscr{U} = \{U_\alpha\}$ *be an open covering of a differentiable manifold* M, *and let* V *be a finite-dimensional real vector space. Let* $g_{\alpha\beta} : M \to \mathrm{GL}(V)$, $U_\alpha \cap U_\beta \neq \emptyset$, *be a family of* C^∞ *maps satisfying the cocycle condition in* Proposition 2.2. *Then there exists a real vector bundle* $\xi = (E, \pi, M, \mathscr{A})$, *unique up to equivalence, such that the maps* $g_{\alpha\beta}$ *are the changes of charts of the atlas* $\mathscr{A}$.

Definition 2.5 The family $(U_\alpha, g_{\alpha\beta})$ is said to be a $\mathrm{GL}(V)$-valued *cocycle on* M *subordinated to the open covering* $\mathscr{U}$.

Definitions 2.6 Let $\mathscr{T}_s^r(M)$ be the set of tensor fields of type (r, s) on a differentiable manifold M and write $\mathscr{T}(M) = \bigoplus_{r,s=0}^{\infty} \mathscr{T}_s^r(M)$. A *derivation* D of $\mathscr{T}(M)$ is a map of $\mathscr{T}(M)$ into itself satisfying:

(i) D is linear and satisfies

$$D_X(T_1 \otimes T_2) = D_X T_1 \otimes T_2 + T_1 \otimes D_X T_2, \quad X \in \mathfrak{X}(M),\ T_1, T_2 \in \mathscr{T}(M).$$

(ii) D_X is type-preserving: $D_X(\mathscr{T}_s^r(M)) \subset \mathscr{T}_s^r(M)$.
(iii) D_X commutes with every contraction of a tensor field.

Let $\Lambda^r M$ be the space of differential forms of degree r on the n-manifold M, that is, skew-symmetric covariant tensor fields of degree r. With respect to the exterior product, $\Lambda^* M = \bigoplus_{r=0}^n \Lambda^r M$ is an algebra over $\mathbb{R}$. A *derivation* (resp. *anti-derivation*) of $\Lambda^* M$ is a linear map of $\Lambda^* M$ into itself satisfying

$$D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + \omega_1 \wedge D\omega_2, \quad \omega_1, \omega_2 \in \Lambda^* M$$

$$\big(\text{resp. } D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge D\omega_2, \quad \omega_1 \in \Lambda^r M,\ \omega_2 \in \Lambda^* M.\big)$$

A derivation or anti-derivation D of $\Lambda^* M$ is said to be *of degree* k if it maps $\Lambda^r M$ into $\Lambda^{r+k} M$ for every r.

Theorem 2.7 (Exterior Differentiation) *There exists a unique anti-derivation*

$$\mathrm{d}\colon \Lambda^* M \to \Lambda^* M$$

of degree $+1$ *such that*:

(i) $\mathrm{d}^2 = 0$.
(ii) *Whenever* $f \in C^\infty M = \Lambda^0 M$, $\mathrm{d}f$ *is the differential of* f.

Definitions 2.8 Fix a vector field X on M and let φ_t be the local one-parameter group of transformations associated with X. Let Y be another vector field on M. The *Lie derivative of* Y *with respect to* X *at* $p \in M$ is the vector $(L_X Y)_p$ defined by

$$(L_X Y)_p = \lim_{t \to 0} \frac{Y_p - \varphi_{t*} Y_{\varphi_t^{-1}(p)}}{t} = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_{t*} Y_{\varphi_t^{-1}(p)}).$$

The *Lie derivative of a differential form* ω *with respect to* X *at* p is defined by

$$(L_X \omega)_p = \lim_{t \to 0} \frac{\omega_p - \varphi_{-t}^*(\omega_{\varphi_t(p)})}{t}. \tag{2.1}$$

The *Lie derivative of a tensor field* T *of type* (r, s) *with respect to* X *at* p is defined by

$$(L_X T)_p = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t \cdot T)_p,$$

where the dot denotes, for an arbitrary diffeomorphism Φ of M,

$$\begin{aligned}&\Phi \cdot (X_1 \otimes \cdots \otimes X_r \otimes \theta_1 \otimes \cdots \otimes \theta_s)\\&\quad = \Phi \cdot X_1 \otimes \cdots \otimes \Phi \cdot X_r \otimes (\Phi^{-1})^* \theta_1 \otimes \cdots \otimes (\Phi^{-1})^* \theta_s,\end{aligned}$$

$X_i \in \mathfrak{X}(M)$, $\theta_j \in \Lambda^1 M$.

In particular, the action of Φ on a differential form $\theta \in \Lambda^1 M$ is given by

$$(\Phi \cdot \theta)_p = \theta_{\Phi^{-1}(p)} \circ (\Phi^{-1})_* = ((\Phi^{-1})^* \theta)_p, \quad p \in M.$$

For each $X \in \mathfrak{X}(M)$, the *interior product with respect to* X is the unique anti-derivation i of degree -1 defined by $i_X f = 0$, $f \in C^\infty M$, and $i_X \theta = \theta(X)$, $\theta \in \Lambda^1 M$. We shall use sometimes, to avoid confusion, ι instead of i to denote the interior product.

Theorem 2.9 *Let $X \in \mathfrak{X}(M)$. Then*:

(i) $L_X f = Xf$, $f \in C^\infty M$.
(ii) $L_X Y = [X, Y]$, $Y \in \mathfrak{X}(M)$.
(iii) *L_X maps $\Lambda^* M$ to $\Lambda^* M$, and it is a derivation which commutes with the exterior differentiation* d.
(iv) *On $\Lambda^* M$, we have*

$$L_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X,$$

where i_X denotes the interior product with respect to X.

Proposition 2.10 *Let φ_t be a local one-parameter group of local transformations generated by a vector field X on M. For any tensor field T on M, we have*

$$\varphi_s \cdot (L_X T) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t \cdot T)\right)_{t=s}.$$

In particular, $L_X T = 0$ if and only if $\varphi_t \cdot T = T$ for all t.

Definitions 2.11 Let m, n be integers, $1 \leqslant m \leqslant n$. An *$m$-dimensional distribution* $\mathscr{D}$ on an n-dimensional manifold M is a choice of an m-dimensional subspace $\mathscr{D}_p$ of $T_p M$ for each $p \in M$. $\mathscr{D}$ is C^∞ if for each $p \in M$, there are a neighbourhood U of p and m vector fields $X_1, \dots, X_m$ on U which span $\mathscr{D}$ at each point in U. A vector field is said to *belong to* (or *lie in*) the distribution $\mathscr{D}$ if $X_p \in \mathscr{D}_p$ for each $p \in M$. Then one writes $X \in \mathscr{D}$. A C^∞ distribution is called *involutive* (or *completely integrable*) if $[X, Y] \in \mathscr{D}$ whenever X and Y are vector fields lying in $\mathscr{D}$.

A submanifold (N, ψ) of M is an *integral manifold* of a distribution $\mathscr{D}$ on M if

$$\psi_*(T_q N) = \mathscr{D}_{\psi(q)}, \quad q \in N.$$

Definitions 2.12 Let $\mathscr{D}$ be an r-dimensional C^∞ distribution on M. A differential s-form ω is said to *annihilate* $\mathscr{D}$ if, for each $p \in M$,

$$\omega_p(v_1, \dots, v_s) = 0, \quad v_1, \dots, v_s \in \mathscr{D}_p.$$

A differential form $\omega \in \Lambda^* M$ is said to annihilate $\mathscr{D}$ if each of the homogeneous parts of ω annihilates $\mathscr{D}$. Let

$$\mathscr{I}(\mathscr{D}) = \{\omega \in \Lambda^* M : \omega \text{ annihilates } \mathscr{D}\}.$$

A function $f \in C^\infty M$ is said to be a *first integral of* $\mathscr{D}$ if $\mathrm{d}f$ annihilates $\mathscr{D}$. An ideal $\mathscr{I} \subset \Lambda^* M$ is called a *differential ideal* if it is closed under exterior differentiation d, that is, $\mathrm{d}\mathscr{I} \subset \mathscr{I}$.

Proposition 2.13 *A C^∞ distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.*

Theorem 2.14 (Frobenius' Theorem) *Let $\mathscr{D}$ be an $(n-q)$-dimensional, involutive, C^∞ distribution on the n-dimensional manifold M. Let $p \in M$. Then through p there passes a unique maximal connected integral manifold of $\mathscr{D}$, and every connected integral manifold of $\mathscr{D}$ through p is contained in the maximal one.*

Definitions 2.15 In the conditions of Theorem 2.14 it is said that the involutive distribution $\mathscr{D}$ defines a *foliation*, M is said to be *a foliated manifold*, the unique maximal connected integral manifold of $\mathscr{D}$ through each point is called a *leaf* of the foliation, and the foliation is said to be of *codimension* q.

Definition 2.16 A *codimension q foliation* $\mathcal{F}$ on a differentiable manifold M of dimension n is a collection of disjoint, connected, $(n-q)$-dimensional submanifolds of M (the leaves of the foliation), whose union is M, and such that for each point $p \in M$, there is a chart (U, φ) containing p such that each leaf of the foliation intersects U in either the empty set or a countable union of $(n-q)$-dimensional slices of the form $x^{n-q+1} = c^{n-q+1}, \dots, x^n = c^n$. More formally, a foliation $\mathcal{F}$ consists of a covering $\mathcal{U}$ of M by charts (U_i, φ_i) such that on each intersection $U_i \cap U_j$, the changes of charts $\Phi_{ij} = \varphi_j \circ \varphi_i^{-1}$ are of the form

$$\Phi_{ij}(x^1, \dots, x^q, x^{q+1}, \dots, x^n) = (\varphi_{ij}(x^1, \dots, x^q), \psi_{ij}(x^1, \dots, x^q, x^{q+1}, \dots, x^n))$$

with

$$\varphi_{ij} : \mathbb{R}^q \to \mathbb{R}^q, \qquad \psi_{ij} : \mathbb{R}^n \to \mathbb{R}^{n-q}.$$

2.2 Vector Bundles

Problem 2.17 Let (E, π, M) be a C^∞ vector bundle with fibre $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Prove that the homotheties

$$h : \mathbb{F} \times E \to E, \qquad (\lambda, y) \mapsto h(\lambda, y) = \lambda y,$$

are C^∞.

Solution Let U be an open subset of M. Let $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{F}$ be a trivialisation of (E, π, M), that is, a fibre-preserving diffeomorphism linear on the fibres, and φ a chart, that is, a diffeomorphism of the open subset $E_U = \pi^{-1}(U)$ of E onto $U \times \mathbb{F}^n$, linear on the fibres.

Then, $h|_U$ is the composition map

$$\begin{array}{ccccccc} \mathbb{F} \times E_U & \xrightarrow{\mathrm{id}_{\mathbb{F}} \times \varphi} & \mathbb{F} \times U \times \mathbb{F}^n & \xrightarrow{h'} & U \times \mathbb{F}^n & \xrightarrow{\varphi^{-1}} & E_U \\ (\lambda, y) & \longmapsto & (\lambda, p, x) & \longmapsto & (p, \lambda x) & \longmapsto & \lambda y. \end{array}$$

Since φ is a diffeomorphism and h' is C^∞, the map $h|_U$ is C^∞.

Problem 2.18 Show that for a C^∞ vector bundle $\xi = (E, \pi, M)$ with fibre $\mathbb{R}^n$, triviality is equivalent to the existence of n C^∞ global sections, linearly independent at each point.

Solution Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^n$. If we have a global trivialisation

$$\begin{array}{ccc} E & \xrightarrow{u} & M \times \mathbb{R}^n \\ \pi \big\downarrow & & \big\downarrow \mathrm{pr}_1 \\ M & \xrightarrow{\mathrm{id}} & M \end{array}$$

then we have sections $\tilde{e}_i$ of $M \times \mathbb{R}^n$ given by $\tilde{e}_i = (\mathrm{id}, e_i)$. Thus, we have sections ξ_i of E defined by $\xi_i = u^{-1} \circ \tilde{e}_i$, which are linearly independent because u^{-1} is an isomorphism on each fibre.

Conversely, if ξ_i are such linearly independent sections of E, we define the trivialisation u by $u(\alpha) = (\pi(\alpha), \alpha^1, \dots, \alpha^n)$ with $\alpha = \sum_i \alpha^i \xi_i(\pi(\alpha)) \in E$. Its inverse map is given by $u^{-1}(p, \alpha^1, \dots, \alpha^n) = \sum_i \alpha^i \xi_i|_p$, $p \in M$.

Problem 2.19 Prove that the infinite Möbius strip M (see Problem 1.31) can be considered as the total space of a vector bundle over S^1. Specifically:

(i) Determine the base space, the fibre and the projection map π.
(ii) Prove that the vector bundle (M, π, S^1) is locally trivial but not trivial.

Solution

(i) With the notations of Problem 1.31, we have that the base space is $S^1 \equiv ([0, 1] \times \{0\})/\!\sim\, \subset M$, the fibre is $\mathbb{R}$ (see Fig. 2.1), and the projection map is defined by

$$\pi\big([(x, y)]\big) = \begin{cases} [(x, 0)] & \text{if } 0 < x < 1, \\ [(0, 0)] = [(1, 0)] & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(ii) The charts in Problem 1.31 are in fact trivialisations that cover S^1 entirely. Now suppose that there exists a non-vanishing global section $\sigma : S^1 \to M$, i.e. a continuous map such that $\pi \circ \sigma = \mathrm{id}_{S^1}$. This is equivalent to a continuous function

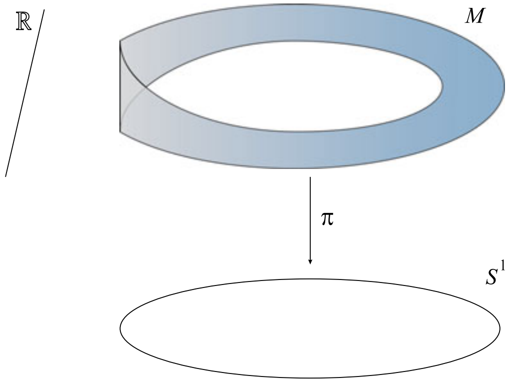

Fig. 2.1 The Möbius strip as the total space of a vector bundle

$s\colon [0,1] \to \mathbb{R}$ such that $s(0) = -s(1)$. Since s must vanish somewhere, σ must also vanish somewhere, so getting a contradiction.

Problem 2.20

(i) Consider

$$E = \left\{(u, v) = (x, y, z, a, b, c) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1, \langle u, v\rangle = 0\right\}$$

and the projection map on the unit sphere S^2 given by $\pi\colon E \to S^2$, $\pi(u, v) = u$. Prove that $\xi = (E, \pi, S^2)$ is a locally trivial bundle over S^2 with fibre $\mathbb{R}^2$.

(ii) Let $\mathscr{A} = \{(U_i, \Phi_i)\}$, $i = 1, 2, 3$, be as in the solution of (i) below. Prove that $TS^2 = (E, \pi, S^2, \mathscr{A})$ is a vector bundle (see Definitions 2.1) with fibre $\mathbb{R}^2$.

Solution

(i) The open subsets U_1, U_2, U_3 of S^2 given by $|x| < 1$, $|y| < 1$, $|z| < 1$, respectively, are an open covering of S^2. Define local trivialisations by

$$\Phi_1\colon \pi^{-1}(U_1) \to U_1 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, bz - cy, a),$$

$$\Phi_2\colon \pi^{-1}(U_2) \to U_2 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, cx - az, b),$$

$$\Phi_3\colon \pi^{-1}(U_3) \to U_3 \times \mathbb{R}^2, \qquad (x, y, z, a, b, c) \mapsto (x, y, z, ay - bx, c).$$

It is immediate that they are diffeomorphisms.

(ii) As a computation shows, the changes of charts are given, for each $u = (x, y, z) \in S^2$, by

$$g_{21}(u) = \frac{-1}{y^2 + z^2}\begin{pmatrix} xy & z \\ -z & xy \end{pmatrix}, \qquad g_{32}(u) = \frac{-1}{z^2 + x^2}\begin{pmatrix} yz & x \\ -x & yz \end{pmatrix},$$

$$g_{13}(u) = \frac{-1}{x^2 + y^2}\begin{pmatrix} zx & y \\ -y & zx \end{pmatrix}.$$

The cocycle condition is thus satisfied. Indeed, one has

$$g_{21}(u)g_{13}(u) = \frac{1}{x^2+y^2}\begin{pmatrix} -yz & x \\ -x & -yz \end{pmatrix} = \big(g_{32}(u)\big)^{-1} = g_{23}(u)$$

and the similar identities for $g_{12}(u)g_{23}(u)$ and $g_{13}(u)g_{32}(u)$.

Moreover, for

$$\widehat{E} = \big\{\big((u,v),\big(u',v'\big)\big) \in E \times E : u = u',\, \langle u,v\rangle = \big\langle u,v'\big\rangle = 0\big\},$$

the maps

$$\begin{aligned} s\colon \widehat{E} \to E, &\qquad \big((u,v),\big(u',v'\big)\big) \mapsto \big(u, v+v'\big), \\ h\colon \mathbb{R}\times E \to E, &\qquad \big(\lambda,(u,v)\big) \mapsto (u,\lambda v), \end{aligned}$$

are C^∞ (as for h, see Problem 2.17), and they induce a structure of two-dimensional vector space on each fibre of TS^2.

Problem 2.21

(i) Let $\{(U_\alpha,\varphi_\alpha)\}$ be an atlas on a manifold M, where $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$, $\varphi_\alpha = (x_\alpha^1,\dots,x_\alpha^n)$, $n = \dim M$. Let $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ be the map

$$\big(g_{\alpha\beta}(p)\big)_i^h = \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p), \qquad p \in U_\alpha \cap U_\beta.$$

Prove that $\{g_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the tangent bundle TM.

(ii) Similarly, if the map $g^*_{\alpha\beta}\colon U_\alpha\cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ is given by

$$\big(g^*_{\alpha\beta}(p)\big)_i^h = \frac{\partial x_\beta^i}{\partial x_\alpha^h}(p), \qquad p \in U_\alpha \cap U_\beta,$$

prove that $\{g^*_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the cotangent bundle T^*M.

Solution

(i) Let us define two linear frames at p:

$$u_\alpha = \left(\frac{\partial}{\partial x_\alpha^1}\bigg|_p,\dots,\frac{\partial}{\partial x_\alpha^n}\bigg|_p\right), \qquad u_\beta = \left(\frac{\partial}{\partial x_\beta^1}\bigg|_p,\dots,\frac{\partial}{\partial x_\beta^n}\bigg|_p\right).$$

According to the definition of $g_{\alpha\beta}(p)$, we have

$$\frac{\partial}{\partial x_\beta^i}\bigg|_p = \sum_{h=1}^n \big(g_{\alpha\beta}(p)\big)_i^h \frac{\partial}{\partial x_\alpha^h}\bigg|_p.$$

Hence $u_\beta = u_\alpha \cdot g_{\alpha\beta}(p)$, where the dot on the right-hand side stands for the right action of $\mathrm{GL}(n,\mathbb{R})$ on the bundle of linear frames FM (see Definitions 5.3). Accordingly,

$$u_\beta = u_\gamma \cdot g_{\gamma\beta}(p) = \big(u_\alpha \cdot g_{\alpha\gamma}(p)\big) \cdot g_{\gamma\beta}(p) = u_\alpha \cdot \big(g_{\alpha\gamma}(p) g_{\gamma\beta}(p)\big).$$

As $\mathrm{GL}(n,\mathbb{R})$ acts freely on FM, we conclude that

$$g_{\alpha\beta}(p) = g_{\alpha\gamma}(p) g_{\gamma\beta}(p),$$

thus proving that $\{g_{\alpha\beta}\}$ is a cocycle.

Moreover, if $\pi : TM \to M$ is the tangent bundle, for every index α, we have a trivialisation

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \qquad \Phi_\alpha(X) = \big(p, \lambda^1, \dots, \lambda^n\big),$$

$X = \sum_{i=1}^n \lambda^i (\partial/\partial x_\alpha^i)_p \in T_p U_\alpha$, or in other words,

$$\Phi_\alpha(X) = \big(p, u_\alpha^{-1}(X)\big),$$

where u_α is understood as a linear isomorphism $u_\alpha : \mathbb{R}^n \to T_p M$.

In order to prove that the cocycle $\{g_{\alpha\beta}\}$ defines TM, it suffices to see that the cocycle associated to these trivialisations is $\{g_{\alpha\beta}\}$. In fact, if $\{e_i\}$ is the standard basis of $\mathbb{R}^n$, for $v = \sum_i \lambda^i e_i$, $p \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
\big(\Phi_\alpha \circ \Phi_\beta^{-1}\big)(p, v) &= \Phi_\alpha\big(u_\beta(v)\big) = \big(p, u_\alpha^{-1}\big(u_\beta(v)\big)\big) \\
&= \left(p, u_\alpha^{-1}\left(\sum_{i=1}^n \lambda^i \left.\frac{\partial}{\partial x_\beta^i}\right|_p\right)\right) \\
&= \left(p, u_\alpha^{-1}\left(\sum_{i,h=1}^n \lambda^i \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p) \left.\frac{\partial}{\partial x_\alpha^h}\right|_p\right)\right) \\
&= \left(p, \sum_{i,h=1}^n \lambda^i \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p) u_\alpha^{-1}\left(\left.\frac{\partial}{\partial x_\alpha^h}\right|_p\right)\right) \\
&= \left(p, \sum_{i,h} \lambda^i \big(g_{\alpha\beta}(p)\big)_i^h e_h\right) = \big(p, g_{\alpha\beta}(p) \cdot v\big).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\sum_{j=1}^n \big(g_{\alpha\beta}^*(p)\big)_j^h \big({}^t g_{\alpha\beta}(p)\big)_i^j &= \sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^h}(p) \frac{\partial x_\alpha^i}{\partial x_\beta^j}(p) = \sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^h}(p) \left.\frac{\partial}{\partial x_\beta^j}\right|_p \big(x_\alpha^i\big) \\
&= \left.\frac{\partial}{\partial x_\alpha^h}\right|_p \big(x_\alpha^i\big) = \delta_h^i.
\end{aligned}$$

Hence $g^*_{\alpha\beta}(p) = ({}^t g_{\alpha\beta})^{-1}(p)$, and then

$$g^*_{\alpha\gamma}(p) g^*_{\gamma\beta}(p) = \left({}^t g_{\alpha\gamma}\right)^{-1}(p)\left({}^t g_{\gamma\beta}\right)^{-1}(p) = \left({}^t g_{\alpha\beta}\right)^{-1}(p) = g^*_{\alpha\beta}(p),$$

thus proving that $\{g^*_{\alpha\beta}\}$ is a cocycle.

Finally, by proceeding as in (i) above, it is easily checked that $\{g^*_{\alpha\beta}\}$ is the cocycle attached to the trivialisations of the cotangent bundle $\pi\colon T^*M \to M$ defined as follows:

$$\begin{aligned}
&\Psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n,\\
&\Psi_\alpha(\omega) = \left(p, u^*_\alpha(\omega)\right) = \left(p, \omega\left(\left.\frac{\partial}{\partial x^1_\alpha}\right|_p\right), \dots, \omega\left(\left.\frac{\partial}{\partial x^n_\alpha}\right|_p\right)\right),\\
&\quad \omega \in T^*_p M,\ p \in U_\alpha \cap U_\beta,
\end{aligned}$$

where $u^*_\alpha\colon T^*_p M \to (\mathbb{R}^n)^*$ is the dual map to $u_\alpha\colon \mathbb{R}^n \to T_p M$.

Problem 2.22 (The Tautological Bundle Over the Real Grassmannian) Denote by $\gamma^k(\mathbb{R}^n)$ the subset of pairs $(V, v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n$ such that $v \in V$ and let $\pi\colon \gamma^k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ be the projection $\pi(V, v) = V$. Prove that $\gamma^k(\mathbb{R}^n)$ is a C^∞ vector bundle of rank k.

Solution The fibres of π are endowed with a natural structure of vector space as $\pi^{-1}(V) = V$. Hence $\operatorname{rank} \pi^{-1}(V) = k$ for all $V \in G_k(\mathbb{R}^n)$. The maps

$$\begin{aligned}
\gamma^k\left(\mathbb{R}^n\right) \times_{G_k(\mathbb{R}^n)} \gamma^k\left(\mathbb{R}^n\right) \to \gamma^k\left(\mathbb{R}^n\right), &\qquad \left((V, v), (V, w)\right) \mapsto (V, v + w),\\
\mathbb{R} \times \gamma^k\left(\mathbb{R}^n\right) \to \gamma^k\left(\mathbb{R}^n\right), &\qquad \left(\lambda, (V, v)\right) \mapsto (V, \lambda v),
\end{aligned}$$

are differentiable as they are induced by the corresponding operations in $\mathbb{R}^n$. It remains to prove that $\gamma^k(\mathbb{R}^n)$ is locally trivial. Let us fix a point $V_0 \in G_k(\mathbb{R}^n)$, and let $\mathscr{U}$ be the set of k-planes V such that $\ker p|_V = 0$, where p is the orthogonal projection onto V_0 relative to the decomposition $\mathbb{R}^n = V_0 \oplus V_0^\perp$. Certainly, $V_0 \in \mathscr{U}$ as $p|_{V_0} = \mathrm{id}$.

If $\{v^0_1, \dots, v^0_k\}$ is an orthonormal basis of V_0 and $\{v_1, \dots, v_k\}$ is a basis of V, then $V \in \mathscr{U}$ if and only if

$$\det\left(\langle v^0_i, v_j\rangle\right)_{i,j=1,\dots,k} \neq 0,$$

thus proving that $\mathscr{U}$ is an open neighbourhood of V_0. For every $V \in \mathscr{U}$, the restriction $p|_V\colon V \to V_0$ is an isomorphism as $\ker p|_V = 0$ and $\dim V = \dim V_0$. Hence we can define a C^∞ trivialisation

$$\begin{aligned}
\mathscr{U} \times V_0 &\xrightarrow{\tau} \pi^{-1}(\mathscr{U}) \subset \gamma^k\left(\mathbb{R}^n\right)\\
(V, v_0) &\mapsto \left(V, (p|_V)^{-1}(v_0)\right).
\end{aligned}$$

Problem 2.23 Let $\Phi\colon E \to E'$ be a homomorphism of vector bundles over M with constant rank. Prove that $\ker\Phi$ and $\operatorname{im}\Phi$ are vector sub-bundles of E and E', respectively.

Solution As the problem is local, we can assume that E, E' are trivial: $E = M \times \mathbb{R}^n$, $E' = M \times \mathbb{R}^m$. Then Φ is given by

$$\Phi(p, v) = \bigl(p, A(p)v\bigr),$$

where $A = (a^i_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $a^i_j \in C^\infty M$, is a C^∞ $m \times n$ matrix. Set $r = \operatorname{rank}_p \Phi$ for all $p \in M$. Given $p_0 \in M$, by permuting rows and columns in A, we can suppose that

$$\det\begin{pmatrix} a^1_1(p_0) & \dots & a^1_r(p_0) \\ \vdots & & \vdots \\ a^r_1(p_0) & \dots & a^r_r(p_0) \end{pmatrix} \neq 0.$$

Hence there exists an open neighbourhood U of p_0 such that

$$\det\begin{pmatrix} a^1_1(p) & \dots & a^1_r(p) \\ \vdots & & \vdots \\ a^r_1(p) & \dots & a^r_r(p) \end{pmatrix} \neq 0, \quad p \in U.$$

As $\operatorname{rank} A(p) = r$ for all $p \in U$, it is clear that $\ker(\Phi|_U)$ is defined by the equations

$$\sum_{j=1}^n a^i_j(p)v^j = 0, \quad 1 \leqslant i \leqslant r,$$

where $v = \sum_j v^j e_j$, $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$. By using Cramer's formulas we conclude that the previous system is equivalent to

$$v^h = \sum_{k=r+1}^n b^h_k(p)v^k, \quad 1 \leqslant h \leqslant r.$$

Hence $(p, v) \in \ker\Phi$ if and only if

$$v = \sum_{k=r+1}^n v^k \left(e_k + \sum_{h=1}^r b^h_k(p)e_h\right).$$

Define sections of E over U by

$$\sigma_k(p) = \begin{cases} e_k, & 1 \leqslant k \leqslant r, \\ e_k + \sum_{h=1}^r b^h_k(p)e_h, & r+1 \leqslant k \leqslant n. \end{cases}$$

Then, $\{\sigma_1(p), \dots, \sigma_n(p)\}$ is a basis of E_p, and $\{\sigma_{r+1}(p), \dots, \sigma_n(p)\}$ is a basis of $(\ker\Phi)_p$ for all $p \in U$, thus proving that $\ker\Phi$ is a sub-bundle of E.

Moreover, if $F \subset E$ is a sub-bundle, then $F^0 = \{w \in E^* : w|_F = 0\}$ is a sub-bundle of E^*, as if $\{\sigma_1, \dots, \sigma_n\}$ is a basis of sections of E over U and $\{\sigma_{r+1}, \dots, \sigma_n\}$ is a basis of sections of F, then the dual basis $\{\sigma_1^*, \dots, \sigma_n^*\}$ is a basis of sections of $E^*|_U$, and $\{\sigma_1^*, \dots, \sigma_r^*\}$ is a basis of sections of F^0. Furthermore, as Φ has constant rank, then the same holds for $\Phi^* \colon E'^* \to E^*$, as a matrix and its transpose have the same rank. We can conclude by remarking that $\operatorname{im}\Phi = (\ker \Phi^*)^0$.

Finally, we give the following counterexample. Let $E = E' = \mathbb{R} \times \mathbb{R}$ be the trivial bundle over $\mathbb{R}$ with fibre $\mathbb{R}$, and let $\Phi \colon E \to E'$ be defined by $\Phi(p, \lambda) = (p, \lambda p)$. Then

$$(\ker \Phi)_p = \begin{cases} 0 & \text{if } p \neq 0, \\ \mathbb{R} & \text{if } p = 0. \end{cases}$$

2.3 Tensor and Exterior Algebras. Tensor Fields

Problem 2.24 Let V be a finite-dimensional vector space. An element $\theta \in \Lambda^* V^*$ is said to be *homogeneous* of degree k if $\theta \in \Lambda^k V^*$, and a homogeneous element of degree $k \geqslant 1$ is said to be *decomposable* if there exist $\theta^1, \dots, \theta^k \in \Lambda^1 V^*$ such that $\theta = \theta^1 \wedge \dots \wedge \theta^k$.

(i) Assume that $\theta \in \Lambda^k V^*$ is decomposable. Calculate $\theta \wedge \theta$.
(ii) If $\dim V > 3$ and $\theta^1, \theta^2, \theta^3, \theta^4$ are linearly independent, is $\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$ decomposable?
(iii) Prove that if $\dim V = n \leqslant 3$, then every homogeneous element of degree $k \geqslant 1$ is decomposable.
(iv) If $\dim V = 4$, give an example of a non-decomposable homogeneous element of $\Lambda^* V^*$.

Solution

(i) It is immediate that $\theta \wedge \theta = 0$.
(ii) No, since

$$\left(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4\right) \wedge \left(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4\right) = 2\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \neq 0,$$

so by virtue of (i) it is not decomposable.
(iii) If $\dim V = 1$ or 2, the result is trivial. Suppose then that $\dim V = 3$, and let $\{\alpha^1, \alpha^2, \alpha^3\}$ be a basis of V^*. If $\theta \in \Lambda^1 V^*$, the result follows trivially. If $\theta \in \Lambda^3 V^*$, then $\theta = a\alpha^1 \wedge \alpha^2 \wedge \alpha^3$, and hence it is decomposable. Then suppose that $\theta \in \Lambda^2 V^*$, so that $\theta = a\alpha^1 \wedge \alpha^2 + b\alpha^1 \wedge \alpha^3 + c\alpha^2 \wedge \alpha^3$. Assume that $a \neq 0$. Then

$$\theta = a\alpha^1 \wedge \left(\alpha^2 + \frac{b}{a}\alpha^3\right) + c\alpha^2 \wedge \alpha^3 = \left(a\alpha^1 - c\alpha^3\right) \wedge \left(\alpha^2 + \frac{b}{a}\alpha^3\right).$$

If $a = 0$, then $\theta = (b\alpha^1 + c\alpha^2) \wedge \alpha^3$.
(iv) The one given in (ii) in the statement is such an example.

Problem 2.25

1. Let A, B be two $(1,1)$ tensor fields on a C^∞ manifold M. Define S by
$$S(X,Y) = [AX,BY] + [BX,AY] + AB[X,Y] + BA[X,Y] - A[X,BY] \\ - A[BX,Y] - B[X,AY] - B[AX,Y], \quad X,Y \in \mathfrak{X}(M).$$
Prove that S is a $(1,2)$ skew-symmetric tensor field on M, called the *Nijenhuis torsion* of A and B.
2. Let J be a tensor field of type $(1,1)$ on the C^∞ manifold M. The *Nijenhuis tensor* of J is defined by
$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y], \quad X,Y \in \mathfrak{X}(M).$$
   (a) Prove that N_J is a tensor field of type $(1,2)$ on M.
   (b) Find its local expression in terms of that of J.

The relevant theory is developed, for instance, in Kobayashi and Nomizu [2, vol. 2, Chap. IX]. However, for the sake of simplicity we omit the factor 2 of the Nijenhuis tensor in that reference.

Solution

1. From the formula
$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$
it follows that $S(fX,gY) = fgS(X,Y)$, $f,g \in C^\infty M$. Since the Lie bracket is skew-symmetric, so is S.
2. (a) The proof is similar to the one in the case 1.
   (b) Let $x^1,\dots,x^n$ be local coordinates in which $J = \sum_{i,j} J^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ and $N_J = \sum_{i,j,k} N^i_{jk} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$, so
$$J\frac{\partial}{\partial x^k} = \sum_{i=1}^n J^i_k \frac{\partial}{\partial x^i}, \qquad N_J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{k=1}^n N^k_{ij} \frac{\partial}{\partial x^k}.$$
From the definition of the Nijenhuis tensor we obtain
$$N^i_{jk} = \sum_{l=1}^n \left(J^l_j \frac{\partial J^i_k}{\partial x^l} - J^l_k \frac{\partial J^i_j}{\partial x^l} + J^i_l \frac{\partial J^l_j}{\partial x^k} - J^i_l \frac{\partial J^l_k}{\partial x^j} \right).$$

Problem 2.26 Write the tensor field $J \in \mathscr{T}^1_1\mathbb{R}^3$ given by
$$J = \mathrm{d}x \otimes \frac{\partial}{\partial x} + \mathrm{d}y \otimes \frac{\partial}{\partial y} + \mathrm{d}z \otimes \frac{\partial}{\partial z}$$
in the system of spherical coordinates given by
$$x = r\cos\varphi\cos\theta, \qquad y = r\cos\varphi\sin\theta, \qquad z = \sin\varphi,$$
$$r > 0,\ \varphi \in (-\pi/2, \pi/2),\ \theta \in (0, 2\pi).$$

Solution We have

$$J = \mathrm{d}r \otimes \frac{\partial}{\partial r} + \mathrm{d}\varphi \otimes \frac{\partial}{\partial \varphi} + \mathrm{d}\theta \otimes \frac{\partial}{\partial \theta},$$

as J represents the identity map in the natural isomorphism $T^*\mathbb{R}^3 \otimes T\mathbb{R}^3 \cong \operatorname{End} T\mathbb{R}^3$, and hence it has the same expression in any coordinate system.

2.4 Differential Forms. Exterior Product

Problem 2.27 Consider on $\mathbb{R}^2$:

$$X = \left(x^2 + y\right)\frac{\partial}{\partial x} + \left(y^2 + 1\right)\frac{\partial}{\partial y}, \qquad Y = (y-1)\frac{\partial}{\partial x},$$
$$\theta = \left(2xy + x^2 + 1\right)\mathrm{d}x + \left(x^2 - y\right)\mathrm{d}y,$$

and let f be the map

$$f \colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad (u, v, w) \mapsto (x, y) = \left(u - v, v^2 + w\right).$$

Compute:

(i) $[X, Y]_{(0,0)}$.
(ii) $\theta(X)(0,0)$.
(iii) $f^*\theta$.

Solution

(i)

$$[X, Y] = \left(y^2 - 2xy + 2x + 1\right)\frac{\partial}{\partial x}, \quad \text{so} \quad [X, Y]_{(0,0)} = \left.\frac{\partial}{\partial x}\right|_{(0,0)}.$$

(ii)

$$\theta(X)(0,0) = \left(\left(2xy + x^2 + 1\right)\left(x^2 + y\right) + \left(x^2 - y\right)\left(y^2 + 1\right)\right)(0,0) = 0.$$

(iii)

$$\begin{aligned} f^*\theta = {} & \left\{2(u-v)\left(v^2 + w\right) + (u-v)^2 + 1\right\}\mathrm{d}u \\ & + \left\{2v\left((u-v)^2 - v^2 - w\right) - 2(u-v)\left(v^2 + w\right) - (u-v)^2 - 1\right\}\mathrm{d}v \\ & + \left\{(u-v)^2 - v^2 - w\right\}\mathrm{d}w. \end{aligned}$$

Problem 2.28 Consider the vector fields on $\mathbb{R}^2$:

$$X = x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial y},$$

and let ω be the differential form on $\mathbb{R}^2$ given by

$$\omega = \bigl(x^2 + 2y\bigr)\,\mathrm{d}x + \bigl(x + y^2\bigr)\,\mathrm{d}y.$$

Show that ω satisfies the relation

$$\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega\bigl([X, Y]\bigr),$$

between the bracket product and the exterior differential.

Solution We have $[X, Y] = 0$ and

$$\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial(x^2+2y)}{\partial x}\,\mathrm{d}x + \frac{\partial(x^2+2y)}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial(x+y^2)}{\partial x}\,\mathrm{d}x + \frac{\partial(x+y^2)}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\
&= -\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

From

$$\begin{aligned}
&X\omega(Y) = xy + 2x^2y + 6xy^3, \qquad Y\omega(X) = 2xy + 2x^2y + 6xy^3, \\
&\mathrm{d}\omega(X, Y) = -(\mathrm{d}x \wedge \mathrm{d}y)\left(x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right) = -xy
\end{aligned}$$

one easily concludes.

Problem 2.29 Find the subset of $\mathbb{R}^2$ where the differential forms

$$\alpha = x\,\mathrm{d}x + y\,\mathrm{d}y, \qquad \beta = y\,\mathrm{d}x + x\,\mathrm{d}y$$

are linearly independent and determine the field of dual frames $\{X, Y\}$ on this set.

Solution We have $\det\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right) = x^2 - y^2 \neq 0$ on $\mathbb{R}^2 \setminus \{(x, y) : x = \pm y\}$. Thus α and β are linearly independent on the subset of $\mathbb{R}^2$ complementary to the diagonals $x + y = 0$ and $x - y = 0$.

The dual field of frames

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, \qquad Y = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}, \quad a, b, c, d \in C^\infty\mathbb{R}^2$$

must satisfy $X(\alpha) = Y(\beta) = 1$, $X(\beta) = Y(\alpha) = 0$. Hence,

$$\begin{cases} ax + by = 1 \\ ay + bx = 0 \end{cases} \quad \text{and} \quad \begin{cases} cx + dy = 0 \\ cy + dx = 1. \end{cases}$$

Solving these systems, we obtain

$$X = \frac{x}{x^2 - y^2}\frac{\partial}{\partial x} - \frac{y}{x^2 - y^2}\frac{\partial}{\partial y}, \qquad Y = -\frac{y}{x^2 - y^2}\frac{\partial}{\partial x} + \frac{x}{x^2 - y^2}\frac{\partial}{\partial y}.$$

Remark The result also follows (here and in other problems below) from the general fact that, $\{e_i = \sum_k \lambda_i^k \frac{\partial}{\partial x^k}\}$ being a basis of vector fields on a manifold and $\{\theta^j = \sum_l \mu_l^j \mathrm{d}x^l\}$ its dual basis, from $(\sum_l \mu_l^j \mathrm{d}x^l)(\sum_k \lambda_i^k \frac{\partial}{\partial x^k}) = \delta_{ij}$ one has $(\mu_j^i) = {}^t(\lambda_j^i)^{-1}$.

Problem 2.30 Consider the three vector fields on $\mathbb{R}^3$:

$$e_1 = (2 + y^2)\mathrm{e}^z \frac{\partial}{\partial x}, \qquad e_2 = 2xy\frac{\partial}{\partial x} + (2 + y^2)\frac{\partial}{\partial y},$$
$$e_3 = -2xy^2\frac{\partial}{\partial x} - y(2 + y^2)\frac{\partial}{\partial y} + (2 + y^2)\frac{\partial}{\partial z}.$$

(i) Show that these vector fields are a basis of the module of C^∞ vector fields on $\mathbb{R}^3$.
(ii) Write the elements θ^i of its dual basis in terms of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$.
(iii) Compute the Lie brackets $[e_i, e_j]$ and express them in the basis $\{e_i\}$.

Solution

(i) The determinant of the matrix of coefficients is $(2 + y^2)^3\mathrm{e}^z$, which is never null; hence the three fields are indeed a basis of $\mathfrak{X}(\mathbb{R}^3)$.
(ii) We proceed by direct computation. One has $\theta^i(e_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. Hence, if

$$\theta^1 = A(x, y, z)\,\mathrm{d}x + B(x, y, z)\,\mathrm{d}y + C(x, y, z)\,\mathrm{d}z,$$

we have

$$1 = \theta^1(e_1) = A(2 + y^2)\mathrm{e}^z, \qquad 0 = \theta^1(e_2) = A2xy + B(2 + y^2),$$
$$0 = \theta^1(e_3) = A(-2xy^2) + B(-y(2 + y^2)) + C(2 + y^2).$$

Solving the system, we have

$$A = \frac{1}{(2 + y^2)\mathrm{e}^z}, \qquad B = -\frac{2xy}{(2 + y^2)\mathrm{e}^z}, \qquad C = 0.$$

Similarly, if $\theta^2 = D(x, y, z)\,\mathrm{d}x + E(x, y, z)\,\mathrm{d}y + F(x, y, z)\,\mathrm{d}z$, we deduce

$$D = 0, \qquad E = \frac{1}{2 + y^2}, \qquad F = \frac{y}{2 + y^2}.$$

Finally, if $\theta^3 = G(x,y,z)\,\mathrm{d}x + H(x,y,z)\,\mathrm{d}y + I(x,y,z)\,\mathrm{d}z$, we similarly obtain

$$G = 0, \qquad H = 0, \qquad I = \frac{1}{2+y^2}.$$

Hence,

$$\theta^1 = \frac{1}{(2+y^2)\mathrm{e}^z}\,\mathrm{d}x - \frac{2xy}{(2+y^2)^2\mathrm{e}^z}\,\mathrm{d}y,$$

$$\theta^2 = \frac{1}{2+y^2}\,\mathrm{d}y + \frac{y}{2+y^2}\,\mathrm{d}z, \qquad \theta^3 = \frac{1}{2+y^2}\,\mathrm{d}z.$$

(iii) Applying the formula

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y],$$

we deduce $[e_1, e_2] = 0$. Similarly, one gets

$$[e_1, e_3] = -(2+y^2)e_1, \qquad [e_2, e_3] = (y^2-2)e_2 + 2ye_3.$$

Problem 2.31 Consider the three vector fields on $\mathbb{R}^3$:

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (1+x^2)\frac{\partial}{\partial z}.$$

(i) Show that these vector fields are a basis of the module of C^∞ vector fields on $\mathbb{R}^3$.
(ii) Write the elements of the dual basis $\{\theta^i\}$ of $\{e_i\}$ in terms of $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$.

Solution

(i)

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1+x^2 \end{pmatrix} = 1 + x^2 \neq 0.$$

(ii)

$$1 = \theta^1(e_1) = (A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z)(e_1) = A, \qquad 0 = \theta^1(e_2) = A + B,$$

$$0 = \theta^1(e_3) = A + B + (1+x^2)C.$$

Solving the system, we have $A = 1$, $B = -1$, $C = 0$. Hence $\theta^1 = \mathrm{d}x - \mathrm{d}y$. Similarly, we obtain $\theta^2 = \mathrm{d}y - \mathrm{d}z/(1+x^2)$ and $\theta^3 = \mathrm{d}z/(1+x^2)$.

Problem 2.32 Consider the vector fields

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$, and let $\psi : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by

$$u = x^2 - y^2, \qquad v = x^2 + y^2, \qquad w = x + y, \qquad t = x - y.$$

(i) Compute $[X, Y]$.
(ii) Show that X, Y are linearly independent on the open subset $\mathbb{R}^2 \setminus \{(0,0)\}$ of $\mathbb{R}^2$ and write the basis $\{\alpha, \beta\}$ dual to $\{X, Y\}$ in terms of the standard basis $\{\mathrm{d}x, \mathrm{d}y\}$.
(iii) Find vector fields on $\mathbb{R}^4$, ψ-related to X and Y, respectively.

Solution

(i) $[X, Y] = 0$.
(ii)

$$\det\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0, \quad (x, y) \in \mathbb{R}^2 \setminus \bigl\{(0,0)\bigr\}.$$

Let

$$\alpha = a(x, y)\,\mathrm{d}x + b(x, y)\,\mathrm{d}y, \qquad \beta = c(x, y)\,\mathrm{d}x + d(x, y)\,\mathrm{d}y.$$

We thus have

$$1 = \alpha(X) = a(x, y)\,\mathrm{d}x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + b(x, y)\,\mathrm{d}y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

$$0 = \alpha(Y) = a(x, y)\,\mathrm{d}x\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + b(x, y)\,\mathrm{d}y\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right).$$

That is, $1 = a(x, y)x + b(x, y)y$ and $0 = a(x, y)(-y) + b(x, y)x$, and one has $a(x, y) = x/(x^2 + y^2)$, $b(x, y) = y/(x^2 + y^2)$. Hence,

$$\alpha = \frac{x}{x^2 + y^2}\,\mathrm{d}x + \frac{y}{x^2 + y^2}\,\mathrm{d}y.$$

Similarly, we obtain $\beta = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$.

(iii)

$$\begin{aligned}
\psi_* X &\equiv \begin{pmatrix} 2x & -2y \\ 2x & 2y \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\equiv \bigl(2x^2 - 2y^2\bigr)\left(\frac{\partial}{\partial u} \circ \psi\right) + \bigl(2x^2 + 2y^2\bigr)\left(\frac{\partial}{\partial v} \circ \psi\right) \\
&\quad + (x + y)\left(\frac{\partial}{\partial w} \circ \psi\right) + (x - y)\left(\frac{\partial}{\partial t} \circ \psi\right),
\end{aligned}$$

$$\psi_* Y = -4xy\left(\frac{\partial}{\partial u} \circ \psi\right) + (x - y)\left(\frac{\partial}{\partial w} \circ \psi\right) + (-y - x)\left(\frac{\partial}{\partial t} \circ \psi\right).$$

Taking

$$\widetilde{X} = 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} + t\frac{\partial}{\partial t}, \qquad \widetilde{Y} = \left(t^2 - w^2\right)\frac{\partial}{\partial u} + t\frac{\partial}{\partial w} - w\frac{\partial}{\partial t},$$

we have

$$\psi_* X = \widetilde{X} \circ \psi, \qquad \psi_* Y = \widetilde{Y} \circ \psi.$$

Problem 2.33 Prove that the differential 1-forms $\omega^1, \dots, \omega^k$ on an n-manifold M are linearly independent if and only if $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$.

Solution If $\omega^1, \dots, \omega^k$ are linearly independent, then each $T_p M$, $p \in M$, has a basis $\{v_1, \dots, v_k, \dots, v_n\}$ such that its dual basis $\{\varphi^1, \dots, \varphi^k, \dots, \varphi^n\}$ satisfies $\varphi^i = \omega^i|_p$, $1 \leqslant i \leqslant k$; hence $\omega^1 \wedge \cdots \wedge \omega^k$ is an element of a basis of $\Lambda^k M$, and so it does not vanish.

Conversely, suppose that such differential forms are linearly dependent. Then there exist a point $p \in M$ and $i \in \{1, \dots, n\}$ such that $\omega^i|_p = \sum_{j \neq i} a_j \omega^j|_p$, and thus, at the point p,

$$\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^i \wedge \cdots \wedge \omega^k = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \sum_{j \neq i} a_j \omega^j \wedge \cdots \wedge \omega^k = 0.$$

Problem 2.34 Prove that the restriction to the sphere S^3 of the differential form

$$\alpha = x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z$$

on $\mathbb{R}^4$, does not vanish.

Solution Given $p \in S^3$, $(\alpha|_{S^3})_p = 0$ if and only if $\alpha_p(X) = 0$ for all

$$X \in T_p S^3 = \left\{X \in T_p\mathbb{R}^4 : \langle X, N\rangle = 0\right\},$$

where $\langle\,,\rangle$ stands for the Euclidean metric of $\mathbb{R}^4$, and

$$N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}$$

is the outward-pointing unit normal vector field to S^3. Define the differential form β by $\beta(X) = \langle X, N\rangle$. Thus $\beta = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z + t\,\mathrm{d}t$.

If $(\alpha|_{S^3})_p = 0$, then α_p and β_p vanish on $T_p S^3$. But two linear forms vanishing on the same hyperplane are proportional, and thus $\alpha_p = \lambda\beta_p$, $\lambda \in \mathbb{R}$, or equivalently,

$$\frac{-y}{x} = \frac{x}{y} = \frac{-t}{z} = \frac{z}{t} = \lambda.$$

We find $x^2 + y^2 = 0$, $z^2 + t^2 = 0$, and hence $x = y = z = t = 0$, which is not possible because $p \in S^3$.

Problem 2.35 Let $\omega^1, \dots, \omega^r$ be differential 1-forms on a C^∞ n-manifold M that are independent at each point. Prove that a differential form θ belongs to the ideal $\mathscr{I}$ generated by $\omega^1, \dots, \omega^r$ if and only if

$$\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

Solution If $\theta \in \mathscr{I}$, then θ is a linear combination of exterior products where those forms appear as factors, and hence $\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$.

Conversely, given a fixed point, complete $\omega^1, \dots, \omega^r$ to a basis

$$\omega^1, \dots, \omega^r, \omega^{r+1}, \dots, \omega^n,$$

so

$$\theta = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

If $\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$, then for each $\{i_1, \dots, i_k\}$, we have

$$f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

Then

$$\begin{aligned} \{1, \dots, r\} \cap \{i_1, \dots, i_k\} \neq \emptyset \quad &\Longrightarrow \quad \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0, \\ \{1, \dots, r\} \cap \{i_1, \dots, i_k\} = \emptyset \quad &\Longrightarrow \quad f_{i_1 \dots i_k} = 0. \end{aligned}$$

Hence,

$$\theta = \sum_{\{1, \dots, r\} \cap \{i_1, \dots, i_k\} \neq \emptyset} f_{i_1 \dots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

Problem 2.36 Let M be a C^∞ manifold. If $\{\omega^1, \dots, \omega^n\}$ is a basis of $T_p^* M$, $p \in M$, prove that there are coordinate functions $x^1, \dots, x^n$ around p such that $(\mathrm{d}x^i)_p = \omega^i$ for all i.

Solution Let $(U, y^1, \dots, y^n)$ be a coordinate system around p. Since the differentials $\{(\mathrm{d}y^1)_q, \dots, (\mathrm{d}y^n)_q\}$ are a basis of $T_q^* M$ for each $q \in U$, we can write $\omega^i = \sum_j f_j^i (\mathrm{d}y^j)_p$. Since $\{\omega^1, \dots, \omega^n\}$ is a basis of $T_p^* M$, we have $\det(f_j^i) \neq 0$. Thus the system $(U, x^1, \dots, x^n)$ defined by $x^i(q) = \sum_j f_j^i y^j(q)$ is a coordinate system, and one has $(\mathrm{d}x^i)_p = \sum_j f_j^i (\mathrm{d}y^j)_p = \omega^i$.

Problem 2.37 Determine which of the following differential forms on $\mathbb{R}^3$ are closed and which are exact:

(i) $\alpha = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z$. (ii) $\beta = x\,\mathrm{d}x + x^2 y^2\,\mathrm{d}y + yz\,\mathrm{d}z$.

(iii) $\gamma = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$.

Solution

(i) $\alpha = \mathrm{d}(xyz)$; thus α is exact and hence closed.
(ii) $\mathrm{d}\beta = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$; thus β is not closed, hence it is not exact.
(iii) $\gamma = \mathrm{d}\omega$, where $\omega = (x^2y^2 - \frac{1}{2}z^2)\,\mathrm{d}y$; thus γ is exact, hence closed.

Recall that, by the Poincaré lemma, every closed differential form on $\mathbb{R}^n$ is exact. Thus, another way to prove (i) and (iii) is:

(i) $\mathrm{d}\alpha = 0$, and thus α is closed and hence exact.
(iii) $\mathrm{d}\gamma = 0$, and thus γ is closed and hence exact.

Problem 2.38 Let $\pi : M \to M'$ be a surjective submersion of manifolds M and M'. Suppose that the set $\pi^{-1}(p')$ is connected for all $p' \in M'$. Let $\omega \in \Lambda^*(M)$.

Prove that there exists a unique differential form $\omega' \in \Lambda^*(M')$ such that $\omega = \pi^*(\omega')$ if and only if $i_Y\omega = 0$ and $L_Y\omega = 0$ for all vector fields Y belonging to the smooth distribution $\ker \pi_* \subset TM$ of vectors annihilated by π_*.

Solution The distribution $\ker \pi_*$ is an involutive smooth distribution (see Problem 2.57). Since the map $\pi : M \to M'$ is a submersion, by the Theorem of the Rank 1.11, for any point $p \in M$, there exist a connected neighbourhood U of p, coordinates $x^1, \dots, x^n$ on U and coordinates $x^1, \dots, x^{n'}$ ($n \geqslant n'$) on the open set $U' = \pi(U) \subset M'$ such that the restriction $\pi|_U$ in these coordinates has the form

$$\pi : \left(x^1, x^2, \dots, x^n\right) \to \left(x^1, x^2, \dots, x^{n'}\right),$$

i.e. in the neighbourhood U the restriction $\ker \pi_*|_U$ is spanned by the vector fields $\partial/\partial x^{n'+1}, \dots, \partial/\partial x^n$. Now let $\omega \in \Lambda^q M$. Let $i_Y\omega = 0$ and $L_Y\omega = 0$ for all vector fields $Y \in \ker \pi_*$. Since

$$L_Y = i_Y \circ \mathrm{d} + \mathrm{d} \circ i_Y$$

(see formula (7.3)), we obtain that $i_Y\,\mathrm{d}\omega = 0$. Then in the local coordinates $(x^1, \dots, x^n)$ on U we have

$$\mathrm{d}\omega|_U = \sum_{1 \leqslant j_1 < \cdots < j_{q+1} \leqslant n'} b_{j_1 \dots j_{q+1}}\left(x^1, \dots, x^{n'}, \dots, x^n\right) \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{q+1}}$$

and, consequently,

$$\omega|_U = \sum_{1 \leq i_1 < \cdots < i_q \leq n'} a_{i_1 \dots i_q}\left(x^1, \dots, x^{n'}\right) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q},$$

where $a_{i_1 \dots i_q}$ are functions only of the variables $x^1, \dots, x^{n'}$. Hence, there is a unique local differential q-form $\omega' \in \Lambda^q U'$ such that $\omega|_U = p^*\omega'$.

Let $p_1 \in U$ and $p_2 \in \pi^{-1}(\pi(p_1))$, i.e. $\pi(p_1) = \pi(p_2)$. Since the set $\pi^{-1}(\pi(p_1))$ is a connected closed submanifold of M (by the Implicit map Theorem for Submersions), the points p_1, p_2 belong to the same leaf of the distribution $\ker \pi_*$. Then there exists a smooth vector field $Z \in \ker \pi_*$ such that for the corresponding (local) one-parameter group φ_t we have $\varphi_T(p_1) = p_2$, $T \in \mathbb{R}$. But $L_Z\omega = 0$, and therefore $\varphi_t^*\omega = \omega$ for all t (see Proposition 2.10). Thus, for $\varphi_{-T} = \varphi_T^{-1}$,

$$\omega_{p_2} = \varphi_{-T}^*\omega_{p_1} = \varphi_{-T}^*\big(\pi^*\omega'_{\pi(p_1)}\big) = (\pi \circ \varphi_{-T})^*\omega'_{\pi(p_2)} = \pi^*\omega'_{\pi(p_2)},$$

i.e. $\omega|_{\pi^{-1}(U')} = \pi^*(\omega')$. From the uniqueness of the local form $\omega' \in \Lambda^q U'$ it follows that there is a smooth global differential q-form $\omega' \in \Lambda^q M'$ such that $\omega = \pi^*\omega'$. Since the map π is a surjective submersion, such a form ω' is unique.

Problem 2.39 Let α be a closed differential 2-form of constant rank 2ℓ on a manifold M. Denote by $\ker\alpha$ the kernel of α, i.e. the distribution on M which is formed by the set of all vector fields $X \in \mathfrak{X}(M)$ satisfying $i_X\alpha = 0$.

Prove that the distribution $\ker\alpha$ is a smooth involutive distribution.

Solution Let $p_0 \in M$ be an arbitrary point. Locally, in a coordinate system $(U, x^1, \dots, x^n)$, where $U \subset M$ is an open subset containing p_0, the form α is determined by the expression

$$\sum_{1 \leqslant i < j \leqslant n} a_{ij}\big(x^1, \dots, x^n\big)\, \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Since the two-form α is smooth, the map $p \mapsto A(p) = (a_{ij}(x^1(p), \dots, x^n(p)))$ $(a_{ij} = -a_{ji})$ determines a smooth matrix function on the set U. Moreover, there exists some $2\ell \times 2\ell$ minor of the matrix $A(p)$ nowhere vanishing on some open subset $O \subset U$ containing p_0. Therefore in O the kernel of the form α_p, which coincides with the kernel of $A(p)$, is generated by $n - 2\ell$ smooth vector fields. Thus $\ker\alpha$ is a smooth distribution of dimension $n - 2\ell$.

By the definition of $\mathrm{d}\alpha$ (see formula (7.2)), for arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned}\mathrm{d}\alpha(X, Y, Z) = {} & X\big(\alpha(Y, Z)\big) - Y\big(\alpha(X, Z)\big) + Z\big(\alpha(X, Y)\big) \\ & - \alpha\big([X, Y], Z\big) + \alpha\big([X, Z], Y\big) - \alpha\big([Y, Z], X\big).\end{aligned}$$

Suppose now, in addition, that $X, Y \in \ker\alpha$. Then in the right-hand side of the expression above all terms vanish with the exception of the fourth term. Since $\mathrm{d}\alpha = 0$, we obtain that $\alpha([X, Y], Z) = 0$. Thus

$$X, Y \in \ker\alpha \quad \Rightarrow \quad [X, Y] \in \ker\alpha,$$

i.e. $\ker\alpha$ is involutive.

Problem 2.40 Let $\widetilde{M}$ be a submanifold of a manifold M. Suppose that X, Y are smooth vector fields on M which are tangent to $\widetilde{M}$ at each point belonging to $\widetilde{M}$, i.e. $X_p, Y_p \in T_p\widetilde{M} \subset T_pM$ if $p \in \widetilde{M}$.

Prove:

(i) The map $\widetilde{M} \to T\widetilde{M}$, $p \mapsto X_p$ (resp. $p \mapsto Y_p$), defines a smooth vector field $\widetilde{X}$ (resp. $\widetilde{Y}$) on $\widetilde{M}$.
(ii) The bracket $[X, Y]$ of the vector fields X, Y has the same property as X and Y: $[X, Y]_p \in T_p\widetilde{M} \subset T_pM$ if $p \in \widetilde{M}$.
(iii) We have $[\widetilde{X}, \widetilde{Y}]_p = [X, Y]_p$ for each $p \in \widetilde{M} \subset M$.

Solution It is clear that it is only necessary to prove all three assertions (i), (ii), (iii) locally.

(i) Fix some point $p_0 \in \widetilde{M} \subset M$. Since $\widetilde{M}$ is a submanifold of M, by the Theorem of the Rank 1.11 there exist neighbourhoods $O \subset M$ and $\widetilde{O} \subset \widetilde{M} \cap O$ of the point p_0, and coordinates $(x^1, \dots, x^n)$ in O such that a point $p \in O$ is an element of the subset $\widetilde{O}$ if and only if $x^i(p) = 0$ for all $i > \dim \widetilde{M}$, $i \leqslant n$. In particular, $x^1(p), \dots, x^l(p)$, where $l = \dim \widetilde{M}$, are coordinate functions in the open subset $\widetilde{O}$. We have

$$(X|_O)_p = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}.$$

But for each $\widetilde{p} \in \widetilde{O} \subset O$, the vector $X_{\widetilde{p}}$ is an element of $T_{\widetilde{p}}\widetilde{M}$, i.e.

$$a_i(\widetilde{p}) = a_i\bigl(x^1(\widetilde{p}), \dots, x^l(\widetilde{p}), 0, \dots, 0\bigr) = 0, \quad i > l, \tag{$\star$}$$

and, consequently,

$$\frac{\partial a_i}{\partial x^j}(\widetilde{p}) = \frac{\partial a_i}{\partial x^j}\bigl(x^1(\widetilde{p}), \dots, x^l(\widetilde{p}), 0, \dots, 0\bigr) = 0, \quad i > l, \ j \leqslant l. \tag{$\star\star$}$$

Thus,

$$(\widetilde{X}|_{\widetilde{O}})_{\widetilde{p}} = \sum_{i=1}^{l} \widetilde{a}_i(\widetilde{p}) \frac{\partial}{\partial x^i}$$

for all $\widetilde{p} \in \widetilde{O} \subset O$, where $\widetilde{a}_i = a_i|_{\widetilde{O}}$. Since $\widetilde{a}_i(x^1, \dots, x^l) = a_i(x^1, \dots, x^l, 0, \dots, 0)$, the vector field $\widetilde{X}|_{\widetilde{O}}$ is smooth.

Similarly,

$$(Y|_O)_p = \sum_{i=1}^{n} b_i(p) \frac{\partial}{\partial x^i}$$

for all $p \in O \subset M$, and the vector field $\widetilde{Y}|_{\widetilde{O}}$,

$$(\widetilde{Y}|_{\widetilde{O}})_{\widetilde{p}} = \sum_{i=1}^{l} \widetilde{b}_i(\widetilde{p}) \frac{\partial}{\partial x^i},$$

is smooth.

(ii) and (iii) We shall see (ii) and (iii) giving two proofs. The first proof using the local representations of the vector fields:

$$[X,Y]_{\widetilde{p}} = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by definition)}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{l}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by }(\star))$$

$$= \sum_{j=1}^{l}\sum_{i=1}^{l}\left(a_i(\widetilde{p})\frac{\partial b_j}{\partial x^i}(\widetilde{p}) - b_i(\widetilde{p})\frac{\partial a_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j} \quad \text{(by }(\star\star))$$

$$= \sum_{j=1}^{l}\sum_{i=1}^{l}\left(\widetilde{a}_i(\widetilde{p})\frac{\partial \widetilde{b}_j}{\partial x^i}(\widetilde{p}) - \widetilde{b}_i(\widetilde{p})\frac{\partial \widetilde{a}_j}{\partial x^i}(\widetilde{p})\right)\frac{\partial}{\partial x^j}.$$

Hence assertions (ii) and (iii) hold.

As to the second proof, consider the one-to-one immersion $\pi\colon \widetilde{M} \to M$, $p \mapsto p$, defining the submanifold $\widetilde{M} \subset M$. Then the vector fields $\widetilde{X}$ and X, $\widetilde{Y}$ and Y are π-related, i.e.

$$\pi_* \circ \widetilde{X} = X \circ \pi, \qquad \pi_* \circ \widetilde{Y} = Y \circ \pi.$$

By Theorem 1.21 the vector fields (brackets) $[\widetilde{X},\widetilde{Y}]$ and $[X,Y]$ are also π-related. Thus,

$$[X,Y]_p = \pi_*\big([\widetilde{X},\widetilde{Y}]_p\big) = [\widetilde{X},\widetilde{Y}]_p, \quad p \in \widetilde{M},$$

and, in particular, $[X,Y]_p \in T_p\widetilde{M}$.

Problem 2.41 Let ω be a differential 1-form on a manifold M and consider a nowhere-vanishing function $f\colon M \to \mathbb{R}$ such that $\mathrm{d}(f\omega) = 0$. Prove that $\omega \wedge \mathrm{d}\omega = 0$.

Solution We have $\mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}\omega$, and since $f(x) \neq 0$ for all $x \in M$, one has $\mathrm{d}\omega = -(1/f)\,\mathrm{d}f \wedge \omega$. As ω is a differential 1-form, we have $\omega \wedge \mathrm{d}\omega = -(1/f)\omega \wedge \mathrm{d}f \wedge \omega = 0$.

2.5 Lie Derivative. Interior Product

Problem 2.42 Let X and Y be vector fields on a C^∞ manifold M. Prove that if φ_t is the local 1-parameter group generated by X, we have for all $p \in M$:

$$\varphi_{s*}\big((L_X Y)_{\varphi_s^{-1}(p)}\big) = \lim_{t\to 0}\frac{1}{t}\big(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*}Y_{\varphi_{s+t}^{-1}(p)}\big). \qquad (\star)$$

Solution Since φ_t is the local one-parameter group of X, one has $\varphi_s \cdot X = X$, where by definition $(\varphi_s \cdot X)_p = \varphi_{s*}(X_{\varphi_s^{-1}(p)})$. Then, applying Problem 1.107, we have

$$\varphi_s \cdot L_X Y = \varphi_s \cdot [X, Y] = [\varphi_s \cdot X, \varphi_s \cdot Y] = [X, \varphi_s \cdot Y] = L_X(\varphi_s \cdot Y).$$

Thus,

$$\begin{aligned}\varphi_{s*}\big((L_X Y)_{\varphi_s^{-1}(p)}\big) &= L_X(\varphi_{s*} Y_{\varphi_s^{-1}(p)}) \\ &= \lim_{t\to 0} \frac{1}{t}\big(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\big((\varphi_{s*} Y_{\varphi_s^{-1}(p)})_{\varphi_t^{-1}(p)}\big)\big) \\ &= \lim_{t\to 0} \frac{1}{t}(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\varphi_{s*} Y_{\varphi_s^{-1}(\varphi_t^{-1}(p))}) \\ &= \lim_{t\to 0} \frac{1}{t}(\varphi_{s*} Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*} Y_{\varphi_{s+t}^{-1}(p)}).\end{aligned}$$

Problem 2.43 Let f denote a diffeomorphism of the C^∞ manifold M. Prove that

$$i_X\big(f^*\alpha\big) = f^*(i_{f\cdot X}\alpha), \quad X \in \mathfrak{X}(M),\ \alpha \in \Lambda^* M.$$

Solution If $\alpha \in \Lambda^r M$, then for $X_1, \dots, X_{r-1} \in \mathfrak{X}(M)$, one has

$$\begin{aligned}&\big(i_X\big(f^*\alpha\big)\big)_p(X_1|_p, \dots, X_{r-1}|_p) \\ &= \big(f^*\alpha\big)_p(X_p, X_1|_p, \dots, X_{r-1}|_p) \\ &= \alpha_{f(p)}\big(f_* X_p, f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)\big) \\ &= \alpha_{f(p)}\big((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}\big)\end{aligned}$$

and

$$\begin{aligned}&\big(f^*(i_{f\cdot X}\alpha)\big)_p(X_1|_p, \dots, X_{r-1}|_p) \\ &= (i_{f\cdot X}\alpha)_{f(p)}\big(f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)\big) \\ &= \alpha_{f(p)}\big((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}\big).\end{aligned}$$

Problem 2.44 Consider on an open subset of $\mathbb{R}^3$ the differential 1-form

$$\alpha = P_1(x)\,\mathrm{d}x^1 + P_2(x)\,\mathrm{d}x^2 + P_3(x)\,\mathrm{d}x^3,$$

where $x = (x^1, x^2, x^3)$.

(i) Find the conditions under which $i_X\,\mathrm{d}\alpha = 0$ for

$$X = X_1\partial/\partial x + X_2\partial/\partial y + X_3\partial/\partial z.$$

(ii) When do we have $i_X\alpha = 0$ and $i_X\,\mathrm{d}\alpha = 0$?

Solution

(i) Let us compute $\mathrm{d}\alpha$. If we write $P_{ij} = \partial P_i/\partial x^j$ and $Q_{ji} = P_{ji} - P_{ij}$, then

$$\begin{aligned}\mathrm{d}\alpha &= (P_{21} - P_{12})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + (P_{31} - P_{13})\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &\quad + (P_{32} - P_{23})\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j.\end{aligned}$$

Hence,

$$\begin{aligned} i_X\,\mathrm{d}\alpha = 0 \quad &\Leftrightarrow \quad i_X\,\mathrm{d}\alpha(Y) = 0, \quad Y \in \mathfrak{X}\big(\mathbb{R}^3\big) \\ &\Leftrightarrow \quad \mathrm{d}\alpha\left(X, \frac{\partial}{\partial x^k}\right) = 0, \quad k = 1, 2, 3 \\ &\Leftrightarrow \quad \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \left(\sum_l X_l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) \\ &\qquad = \sum_l \sum_{i<j} Q_{ji}\big(X_l \delta^i_l \delta^j_k - X_l \delta^i_k \delta^j_l\big) \\ &\qquad = \sum_l \left(\sum_{l<k} Q_{kl} X_l - \sum_{k<l} Q_{lk} X_l\right) \\ &\qquad = \sum_l Q_{kl} X_l = 0, \quad k = 1, 2, 3.\end{aligned}$$

(ii) By (i),

$$i_X\,\mathrm{d}\alpha = 0 \quad \Leftrightarrow \quad \sum_{l=1}^{3} Q_{kl} X_l = 0, \quad k = 1, 2, 3,$$

and

$$i_X \alpha = \alpha(X) = 0 \quad \Leftrightarrow \quad \left(\sum_i P_i\,\mathrm{d}x^i\right)\left(\sum_j X^j \frac{\partial}{\partial x^j}\right) = 0 \quad \Leftrightarrow \quad \sum_i P_i X^i = 0.$$

2.6 Distributions and Integral Manifolds. Frobenius Theorem. Differential Ideals

Problem 2.45 Consider on the octant of $\mathbb{R}^3$ of positive coordinates the vector fields

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \qquad Y = xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}.$$

(i) Prove that they span an involutive distribution on this octant of $\mathbb{R}^3$.
(ii) Find the integral surfaces.

Hint (to (ii)) Substitute Y by $x^{-1}Y$.

Solution

(i) $[X, Y] = Y$.
(ii) Since in the given domain x does not vanish, we can substitute $x^{-1}Y$ for Y, which, jointly with X, determines the same distribution. The integral curves of X are $(x_0 e^t, y_0 e^{-2t}, z_0)$, and those of $x^{-1}Y$ are $(x_0, y_0 e^s, z_0 e^{-s})$, so that the respective local flows are

$$\varphi_t(x, y, z) = \left(x e^t, y e^{-2t}, z\right), \qquad \psi_s(x, y, z) = \left(x, y e^s, z e^{-s}\right).$$

The map

$$\begin{aligned}(t, s) \in \mathbb{R}^2 \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0) &= \psi_s\left(x_0 e^t, y_0 e^{-2t}, z_0\right)\\ &= \left(x_0 e^t, y_0 e^{-2t+s}, z_0 e^{-s}\right)\end{aligned}$$

is the integral surface through (x_0, y_0, z_0). In fact, the point $(\psi_s \circ \varphi_t)(x_0, y_0, z_0)$ is obtained from (x_0, y_0, z_0) as follows: We first run an interval "t" from $p = (x_0, y_0, z_0)$ along the integral curve of X through p for $t = 0$ and then an interval "s" from $\varphi_t(p)$ along the integral curve of $x^{-1}Y$ through $\varphi_t(p)$ for $s = 0$. If we put

$$x(t, s) = x_0 e^t, \qquad y(t, s) = y_0 e^{-2t+s}, \qquad z(t, s) = z_0 e^{-s},$$

then we see that x^2yz is constant. Hence the integral surfaces are defined by $x^2yz = \text{const}$. As a verification, observe that $X(x^2yz) = Y(x^2yz) = 0$.

Problem 2.46 Consider on $\mathbb{R}^3$ the distribution $\mathscr{D}$ determined by

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1 + x^2 + y^2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{2yz}{1 + x^2 + y^2}\frac{\partial}{\partial z}.$$

(i) Calculate $[X, Y]$ and find whether $\mathscr{D}$ is involutive or not.
(ii) Calculate the local flows of X and Y.
(iii) If $\mathscr{D}$ is involutive, find its integral surfaces.

Solution

(i) $[X, Y] = 0$, and thus $\mathscr{D}$ is involutive.
(ii) We have

$$\begin{cases} x' = 1 \\ y' = 0 \end{cases} \Leftrightarrow \quad \begin{cases} x = x_0 + t \\ y = y_0 \end{cases}$$

and

$$\frac{z'}{z} = \frac{2(x_0 + t)}{1 + (x_0 + t)^2 + y_0^2}$$

if and only if $\log z = \log A(1 + (x_0 + t)^2 + y_0^2)$ if and only if $z = A(1 + (x_0 + t)^2 + y_0^2)$. For $t = 0$, $z_0 = A(1 + x_0^2 + y_0^2)$, so

$$z = z_0 \frac{1 + (x_0 + t)^2 + y_0^2}{1 + x_0^2 + y_0^2}.$$

Hence the local flow of X is

$$\varphi_t(x, y, z) = \left(x + t, y, z \frac{1 + (x + t)^2 + y^2}{1 + x^2 + y^2} \right).$$

Similarly, the local flow of Y is

$$\psi_s(x, y, z) = \left(x, y + s, z \frac{1 + x^2 + (y + s)^2}{1 + x^2 + y^2} \right).$$

(iii) The integral manifolds can be written as $\psi(t, s) \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0)$. But let us see a better solution. We are looking for a differential 1-form annihilating X and Y. For example, we have as a solution:

$$\begin{aligned}\alpha &= 2xz\,\mathrm{d}x + 2yz\,\mathrm{d}y - \left(1 + x^2 + y^2\right)\mathrm{d}z \\ &= z\,\mathrm{d}\left(1 + x^2 + y^2\right) - \left(1 + x^2 + y^2\right)\mathrm{d}z \\ &= -\left(1 + x^2 + y^2\right)^2 \mathrm{d}\left(\frac{z}{1 + x^2 + y^2} \right).\end{aligned}$$

Hence, the integral manifolds are $\frac{z}{1+x^2+y^2} = \text{const.}$

Problem 2.47 The vector field $X = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, defined on $x > 0$, $y > 0$, $z > 0$ in $\mathbb{R}^3$, determines a two-dimensional distribution given by the vector fields orthogonal to X. Is this distribution involutive?

Solution The vector fields $U = -y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $V = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ are orthogonal to X and linearly independent at each point. They span that distribution, but $[U, V] = -y\frac{\partial}{\partial z}$. Since

$$\begin{vmatrix} -y & 1 & 0 \\ -z & 0 & x \\ 0 & 0 & -y \end{vmatrix} = -yz$$

is not identically zero, we have $[U, V]_p \notin \langle U_p, V_p \rangle$. Hence the distribution is not involutive.

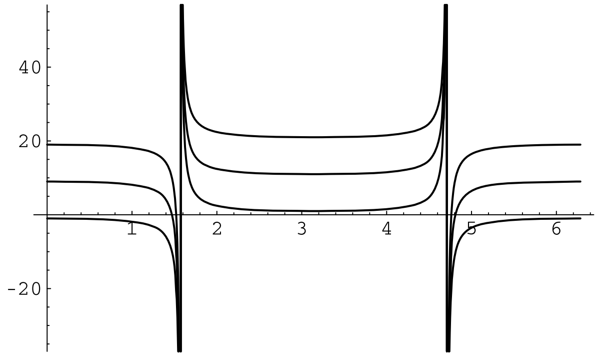

Fig. 2.2 An example of foliation with non-Hausdorff quotient manifold

Problem 2.48 Prove that

$$X = -\cos^2 x \frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y}$$

determines a foliation with non-Hausdorff quotient.

Solution This vector field determines an integrable distribution of codimension 1 of $\mathbb{R}^2$. We have two kind of solutions:

Integrating the equation that X determines, i.e.

$$\frac{\mathrm{d}x}{\cos^2 x} = -\frac{\mathrm{d}y}{\sin x},$$

we obtain the curves

$$y = -\sec x + A$$

(see Fig. 2.2) for $x \neq (2k+1)\pi/2$, $k \in \mathbb{Z}$.

Moreover, we have the solutions with initial conditions of the type $((2k+1)\pi/2, y_0)$, that is, the straight lines $t \mapsto ((2k+1)\pi/2, (-1)^k t)$. Actually, if p and q are two non-separable points of the quotient, then each of them corresponds to a solution of this kind.

Take, for instance, the integral curve $x = -\pi/2$; a point on it, say $(-\pi/2, y_0)$; and an open disk around this point. This open disk intersects all the integral curves intersecting the y-axis at the points with ordinate greater than or equal to $A_0 > 0$. This is also true for open disks around the point $(\pi/2, y_1)$. Such an open disk intersects all the integral curves that intersect the y-axis at points with ordinate greater than or equal to $A_1 > 0$. Now, the integral curves intersecting the y-axis at points with ordinate greater than $\max(A_0, A_1)$ intersect both open disks. Hence the projections of the two open disks on the quotient intersect, so that the projections of $x = -\pi/2$ and of $x = \pi/2$ cannot be separated. Consequently, the quotient manifold is not Hausdorff.

Problem 2.49 Consider on $\mathbb{R}^3$ the vector fields

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad Z = z\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

(i) Prove that X, Y, Z define a C^∞ distribution $\mathscr{D}$ on $\mathbb{R}^3$. Which dimension is it? Is it involutive?

(ii) Compute the set $\mathscr{I}(\mathscr{D})$ of forms which annihilate $\mathscr{D}$. Is it a differential ideal? Is the ideal $\mathscr{I}$ generated by $\mathrm{e}^x\,\mathrm{d}y$ a differential ideal?

Solution

(i) X, Y, Z are not linearly independent because $Z = X - Y$. Hence $\mathscr{D}$ is a two-dimensional C^∞ distribution spanned, for instance, by X and Y, which are linearly independent. $\mathscr{D}$ is not involutive, as $[X, Y] = -\frac{\partial}{\partial x}$ and $-\frac{\partial}{\partial x} \notin \mathscr{D}$, since if it were

$$-\frac{\partial}{\partial x} = az\frac{\partial}{\partial x} + a\frac{\partial}{\partial z} + b\frac{\partial}{\partial y} + b\frac{\partial}{\partial z},$$

we would have $az = -1$, $b = 0$, $b + a = 0$, which would lead us to a contradiction.

(ii) $\{X, Y, \partial/\partial x\}$ is a basis of $\mathfrak{X}(\mathbb{R}^3)$. Therefore, if $\{\alpha, \beta, \omega\}$ is its dual basis of 1-forms, then $\mathscr{I}(\mathscr{D}) = \langle\omega\rangle$, where $\langle\omega\rangle$ stands for the ideal generated by ω.

Let us determine $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$, $f, g, h \in C^\infty\mathbb{R}^3$. From

$$0 = \omega(X) = fz + h, \qquad 0 = \omega(Y) = g + h, \qquad 1 = \omega\left(\frac{\partial}{\partial x}\right) = f$$

it follows that $f = 1$. Thus $h = -z$, and hence $g = z$; that is, $\omega = \mathrm{d}x + z\,\mathrm{d}y - z\,\mathrm{d}z$. Since $\mathscr{D}$ is not involutive, $\mathscr{I}(\mathscr{D})$ cannot be a differential ideal.

We can also prove this directly. One has $\mathrm{d}\omega = \mathrm{d}z \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}z$. If it were, for $a, b, c \in C^\infty\mathbb{R}^3$,

$$\begin{aligned}\mathrm{d}\omega &= \omega \wedge (a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z)\\ &= (b - az)\,\mathrm{d}x \wedge \mathrm{d}y + (c + az)\,\mathrm{d}x \wedge \mathrm{d}z + (zc + zb)\,\mathrm{d}y \wedge \mathrm{d}z,\end{aligned}$$

we would have $b - az = 0$, $c + az = 0$, $zc + zb = -1$. From the first and second equations one has $b + c = 0$, in contradiction with the third equation. One can also conclude by applying Problem 2.35, as $\omega \wedge \mathrm{d}\omega = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$. Finally, $\mathscr{I}$ is a differential ideal since

$$\mathrm{d}\big(\mathrm{e}^x\,\mathrm{d}y\big) = \mathrm{e}^x\,\mathrm{d}x \wedge \mathrm{d}y = \mathrm{e}^x\,\mathrm{d}y \wedge (-\mathrm{d}x).$$

Problem 2.50 Given on $\mathbb{R}^4 = \{(x, y, z, t)\}$ the 1-forms $\alpha = \mathrm{d}x + z\,\mathrm{d}t$ and $\beta = \mathrm{d}z + \mathrm{d}t$, let $\mathscr{I}$ be the ideal generated by α and β, and let $\mathscr{D}$ be the distribution associated to $\mathscr{I}$.

(i) Compute a basis for $\mathscr{D}$.
(ii) Is $\mathscr{D}$ involutive?
(iii) If $p = (1, 0, 1, 0) \in \mathbb{R}^4$, do we have

$$v_p = -3\frac{\partial}{\partial y}\bigg|_p + z\frac{\partial}{\partial x}\bigg|_p \in \mathscr{D}_p?$$

(iv) If $\omega = \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}z \wedge \mathrm{d}t$, is $\omega \in \mathscr{I}$?
(v) Is $y = \text{const}$, $z = \text{const}$ an integral manifold of $\mathscr{D}$?

Solution

(i) For $X, Y \in \mathscr{D}$ given by

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d\frac{\partial}{\partial t}, \qquad Y = e\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\frac{\partial}{\partial z} + h\frac{\partial}{\partial t},$$

for $a, b, c, d, e, f, g, h \in C^\infty\mathbb{R}^4$, it must be

$$\alpha(X) = a + zd = 0, \qquad \alpha(Y) = e + zh = 0,$$
$$\beta(X) = c + d = 0, \qquad \beta(Y) = g + h = 0.$$

Thus, for instance, we can consider

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y}.$$

(ii) $[X, Y] = 0$, and hence $\mathscr{D}$ is involutive.
(iii) No, as

$$\alpha_p(v_p) = (\mathrm{d}x + z\,\mathrm{d}t)_p\left(-3\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}\right)_p = 1 \neq 0.$$

(iv) $\omega = \mathrm{d}x \wedge \beta + \mathrm{d}z \wedge \beta$, and hence $\omega \in \mathscr{I}$.
(v) The tangent space is $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \rangle$, but $\alpha(\frac{\partial}{\partial x}) = 1$, so $y = \text{const}$, $z = \text{const}$ is not an integral manifold of $\mathscr{D}$.

Problem 2.51 Prove that the 1-form $\alpha = (1 + y^2)(x\,\mathrm{d}y + y\,\mathrm{d}x)$, defined on $\mathbb{R}^2 \setminus \{0\}$, generates a rank-1 differential ideal and find the integral manifolds.

Solution Since $1 + y^2$ does not vanish, α generates the same annihilator ideal as

$$\frac{\alpha}{1 + y^2} = x\,\mathrm{d}y + y\,\mathrm{d}x = \mathrm{d}(xy).$$

As $\mathrm{d}(x\,\mathrm{d}y + y\,\mathrm{d}x) = 0$, the ideal is differential.

The integral manifolds are $xy = \text{const}$ (see Fig. 2.3).

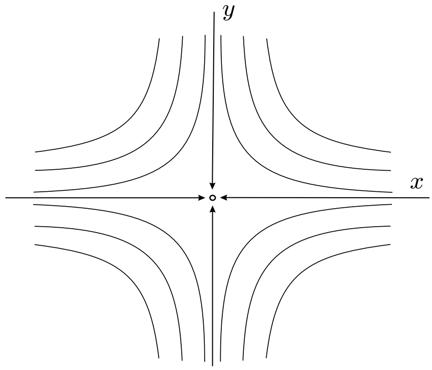

Fig. 2.3 Integral manifolds of $\alpha = (1 + y^2)(x\,\mathrm{d}y + y\,\mathrm{d}x)$

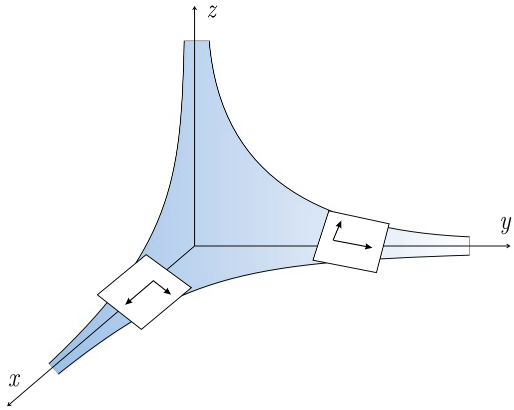

Fig. 2.4 The component in the first octant of an integral surface of the distribution $\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z$

Problem 2.52 Let $U = \mathbb{R}^3 \setminus \{\text{axes}\}$. Compute the integral surfaces of the distribution determined by the ideal of $\Lambda^* U$ generated by

$$\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z.$$

Solution We have $\alpha = \mathrm{d}(xyz)$. If X is annihilated by α, then we have $\alpha(X) = X(xyz) = 0$. Thus the integral surfaces are the surfaces $xyz = \text{const}$ (see Fig. 2.4).

Problem 2.53 Consider the $(1, 1)$ tensor field

$$J = \frac{1}{\cosh x}\frac{\partial}{\partial y} \otimes \mathrm{d}x + \cosh x \frac{\partial}{\partial x} \otimes \mathrm{d}y$$

on $\mathbb{R}^2$ and the distribution $\mathscr{D}$ defined by the condition: $X \in \mathscr{D}$ if and only if $JX = X$.

(i) Compute the integral curves of $\mathscr{D}$.
(ii) Compute the fields $X \in \mathscr{D}$ for which $L_X J = 0$.

Solution

(i) If $X = f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y} \in \mathscr{D}$, $f, h \in C^\infty\mathbb{R}^2$, then

$$\begin{aligned}&\left(\frac{1}{\cosh x}\frac{\partial}{\partial y}\otimes \mathrm{d}x + \cosh x\frac{\partial}{\partial x}\otimes \mathrm{d}y\right)\left(f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}\right)\\ &= \frac{f}{\cosh x}\frac{\partial}{\partial y} + h\cosh x\frac{\partial}{\partial x} = f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}.\end{aligned}$$

Thus $f = h\cosh x$. Denoting by (x, y) the integral curves of $\mathscr{D}$, we have $\mathrm{d}x/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}t)\cosh x$. Hence $\mathrm{d}y = \mathrm{d}x/\cosh x$, and thus

$$y = \arctan\sinh x + A. \tag{$\star$}$$

That is, the integral curves of $\mathscr{D}$ are given by ($\star$).

(ii)

$$\begin{aligned}L_X J = &\left(h_x\cosh x - \frac{f_y}{\cosh x}\right)\left(\frac{\partial}{\partial x}\otimes \mathrm{d}x - \frac{\partial}{\partial y}\otimes \mathrm{d}y\right)\\ &+ (h_y\cosh x + f\sinh x - f_x\cosh x)\left(\frac{\partial}{\partial x}\otimes \mathrm{d}y - \frac{1}{\cosh^2 x}\frac{\partial}{\partial y}\otimes \mathrm{d}x\right)\\ = &\,0.\end{aligned} \tag{$\star\star$}$$

Moreover, if $X \in \mathscr{D}$, then we have $f = g\cosh x$, and from this equation and from ($\star\star$) we conclude that we have to solve only the following equation:

$$\frac{\partial h}{\partial x}\cosh x = \frac{\partial h}{\partial y}.$$

Let $u = 2\arctan \mathrm{e}^x$. Then we have

$$\frac{\partial h}{\partial x} = \frac{1}{\cosh x}\frac{\partial h}{\partial u},$$

and hence $\frac{\partial h}{\partial u} = \frac{\partial h}{\partial y}$. Taking $t = u + y$, $w = u - y$, we obtain

$$0 = \frac{\partial h}{\partial u} - \frac{\partial h}{\partial y} = 2\frac{\partial h}{\partial w}.$$

Thus $h = h(u + y) = h(2\arctan \mathrm{e}^x + y)$, and we finally have

$$f = h\big(2\arctan \mathrm{e}^x + y\big)\cosh x,$$

where $h(2\arctan \mathrm{e}^x + y)$ is an arbitrary differentiable function in that argument.

Problem 2.54 (A Reeb Foliation of S^3) The three-sphere S^3 can be decomposed as two solid 2-tori joint along their common 2-torus boundary. In fact, if one removes

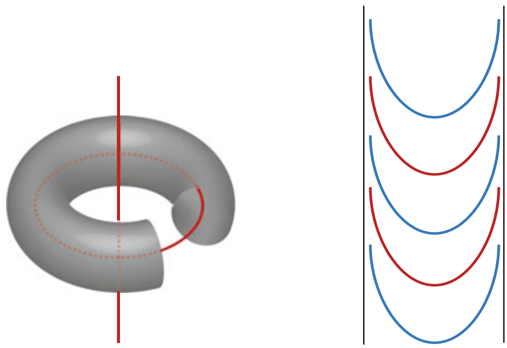

Fig. 2.5 *Left:* The two core circles of S^3 (here actually the part in $S^3 \setminus \{\infty\}$), this viewed as the union of two solid 2-tori. *Right:* Some curves $y = f(x) + c'$

the solid torus of rotation from $\mathbb{R}^3 = S^3 \setminus \{\infty\}$, what remains is homeomorphic to a solid torus minus an interior point. Consider the vertical coordinate axis as the core circle (see Fig. 2.5, left).

Find a foliation of the strip

$$\left\{(x, y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1\right\}$$

originating a foliation in (each) solid torus and so a codimension 1 foliation of the three-sphere S^3.

For the development of the relevant theory, see Reeb [4] and Lawson [3].

Solution Consider the C^∞-foliation of the (x, y)-plane given by the lines $x = c$ for $|c| \geqslant 1$ together with the graphs of the functions

$$y = f(x) + c', \quad -1 < x < 1,\ c' \in \mathbb{R},$$

where f has the property that its derivatives $f^{(r)}$ satisfy $\lim_{|x| \to 1} f^{(r)} = \infty$ for all r (see Fig. 2.5, right).

Consider now the foliation of the solid cylinder obtained by rotating the strip given in the statement about the y-axis in $\mathbb{R}^3$. This foliation is invariant by vertical translations, and so we can obtain a foliation of the solid torus where each noncompact leaf has the form that one can see in Fig. 2.6, left. Gluing together two copies of the foliated solid torus gives a Reeb foliation of S^3 (see Fig. 2.6, right, showing part of a transversal cutting of two leaves). Note that both the "interior" and the "exterior" leaves approach their common 2-torus boundary after turning around it.

Problem 2.55 Let M be a C^∞ n-manifold, and let $\mathscr{D} \subset TM$ be an integrable distribution of rank p. By Frobenius' theorem, $\mathscr{D}$ is spanned by $\partial/\partial x^1, \dots, \partial/\partial x^p$ on an open subset U of M, for a certain coordinate system (U, x^i). We can consider local frames of M of the type

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}, X_1, \dots, X_q\right), \quad p + q = n = \dim M,$$

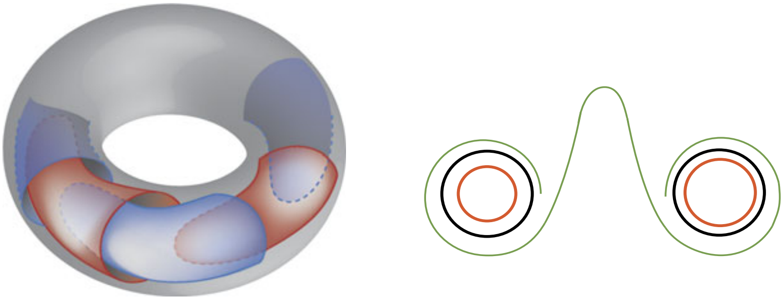

Fig. 2.6 *Left:* The foliation (generated by curves as the previous ones) of the "interior" solid torus in S^3. *Right:* Transversal cut of a Reeb foliation of S^3 showing two sections of an "interior" leaf and part of an "exterior" leaf

where

$$X_u = \frac{\partial}{\partial x^{p+u}} - \sum_a f_u^a \frac{\partial}{\partial x^a}, \quad 1 \leqslant a \leqslant p,\ 1 \leqslant u \leqslant q,\ f_u^a \in C^\infty M.$$

Write the integrability condition of the complementary distribution $\mathscr{H}$ generated by $X_1, \dots, X_q$ on the open subset where these vector fields are defined.

Solution In order for $\mathscr{H}$ to be integrable, it must be $[X_u, X_v] \in \mathscr{H}$ for any $X_u, X_v \in \mathscr{H}$, $u, v = 1, \dots, q$. Then

$$\begin{aligned}
[X_u, X_v] &= \left[\frac{\partial}{\partial x^{p+u}} - \sum_a f_u^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{p+v}} - \sum_b f_v^b \frac{\partial}{\partial x^b}\right] \\
&= \sum_a \left(\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + \sum_b \left(f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b} \right) \right) \frac{\partial}{\partial x^a} \in \mathscr{D}.
\end{aligned}$$

As $[X_u, X_v] \in \mathscr{H}$, the last expression in parentheses must be zero, that is, the condition is

$$\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + \sum_b \left(f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b} \right) = 0.$$

Problem 2.56 Let X be a vector field on a smooth manifold M, and let φ_t be its local one-parameter group (local flow) on M. Let $\mathscr{D} \subset TM$ be a smooth distribution.

Prove that the following conditions are equivalent:

(i) For any vector field Y lying in $\mathscr{D}$, the bracket $[X, Y]$ belongs to $\mathscr{D}$ (the distribution $\mathscr{D}$ is preserved by the vector field X).
(ii) For any vector field Y lying in $\mathscr{D}$, the local vector field $\varphi_t \cdot Y$ belongs to $\mathscr{D}$ (the distribution $\mathscr{D}$ is preserved by the local flow φ_t of X).

For a development of the relevant theory, see, for instance, Gawedzki [1].

Solution Let $p \in M$ and suppose that $\mathscr{D}$ is preserved by the vector field X. Let us choose a scalar product in T_pM. Let P_t be the orthogonal projection onto $\varphi_{t*}(\mathscr{D}_{\varphi_{-t}(p)}) \subset T_pM$. The operator function $P_t\colon T_pM \to T_pM$ smoothly depends on the parameter t. Let $Y \in \mathscr{D}$, and let

$$Y_t = (\varphi_t \cdot Y)_p, \quad \text{i.e. by definition} \quad Y_t = \varphi_{t*}(Y_{\phi_{-t}(p)}).$$

We have by Proposition 2.10

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -\big(\varphi_t \cdot [X,Y]\big)_p \in P_t(T_pM),$$

because $[X,Y] \in \mathscr{D}$. Now $Y_t = P_t(Y_t)$ by the definition of P_t, and consequently,

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = \frac{\mathrm{d}(P_tY_t)}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t + P_t\frac{\mathrm{d}Y_t}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t + \frac{\mathrm{d}Y_t}{\mathrm{d}t}.$$

Thus,

$$\frac{\mathrm{d}P_t}{\mathrm{d}t}Y_t = 0.$$

Since varying Y, Y_t span the range of P_t, we get

$$\frac{\mathrm{d}P_t}{\mathrm{d}t}P_t = 0.$$

Let P_t^* denote the transpose operator of P_t (with respect to the scalar product in T_pM). From $P_t = P_t^*$ and $P_t^2 = P_t$ one obtains

$$\left(\frac{\mathrm{d}P_t}{\mathrm{d}t}\right)^* = \frac{\mathrm{d}P_t}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}P_t}{\mathrm{d}t}P_t + P_t\frac{\mathrm{d}P_t}{\mathrm{d}t} = \frac{\mathrm{d}P_t}{\mathrm{d}t}.$$

Hence,

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = P_t\frac{\mathrm{d}P_t}{\mathrm{d}t} = \left(\frac{\mathrm{d}P_t}{\mathrm{d}t}P_t\right)^* = 0.$$

Consequently, $P_t = P_0$, and φ_t preserves $\mathscr{D}$.

Clearly, from the definition of the Lie bracket (see also Proposition 2.10) we have that if $\varphi_t \cdot Y \in \mathscr{D}$, then $[X,Y] \in \mathscr{D}$.

Problem 2.57 Let $\pi\colon M \to M'$ be a surjective submersion of manifolds M and M'.

(i) Prove that $\ker \pi_* \subset TM$ (the set of vectors annihilated by π_*) is an involutive smooth distribution on M.

Let the set $\pi^{-1}(p')$ be connected for all $p' \in M'$, and let $\mathscr{D} \subset TM$ be a smooth distribution on M containing the distribution $\ker \pi_*$. Suppose that $\mathscr{D}$ is preserved by $\ker \pi_*$, i.e. $[Z,Y] \in \mathscr{D}$ for all vector fields $Z \in \ker \pi_*$ and $Y \in \mathscr{D}$.

Prove:

(ii) There exists a unique smooth distribution $\mathscr{D}'$ on M' such that $\mathscr{D}'_{\pi(p)} = \pi_* \mathscr{D}_p$ for all $p \in M$. Moreover, for any point $p \in M$, there exist a neighbourhood $U \subset M$ and vector fields $\{Y_l\}$ lying in $\mathscr{D}|_U$ such that the restriction $\mathscr{D}'|_{U'}$, where $U' = \pi(U)$, is spanned by vector fields $\{Y'_l\}$, and the vector fields Y_l, Y'_l are π-related for each l.

(iii) If the distribution $\mathscr{D}$ is involutive, then so is $\mathscr{D}'$.

Solution

(i) Since the map $\pi : M \to M'$ is a submersion, by the Theorem of the Rank 1.11, for any point $p \in M$, there exist a neighbourhood U of p, coordinates $x^1, \dots, x^n$ on U, $-1 < x^j < 1$, $j = 1, \dots, n$, and coordinates $x^1, \dots, x^{n'}$ ($n' \leqslant n$) on the open subset $U' = \pi(U) \subset M'$ such that the point p has coordinates $(0, \dots, 0)$ and the restriction $\pi|_U$ in these coordinates has the form

$$\pi : (x^1, x^2, \dots, x^n) \to (x^1, x^2, \dots, x^{n'}), \tag{$\star$}$$

i.e. in the neighbourhood U the restriction $\ker \pi_*|_U$ is spanned by the commuting vector fields $\partial/\partial x^{n'+1}, \dots, \partial/\partial x^n$. Therefore $\mathscr{D}$ is an involutive smooth distribution on M.

(ii) Let $p_1, p_2 \in \pi^{-1}(p') \subset M$ for some point $p' \in M'$. Since the set $\pi^{-1}(p')$ is connected, the points p_1, p_2 belong to the same leaf of the distribution $\ker \pi_*$. Then there exists a smooth vector field $Z \in \ker \pi_*$ such that for a corresponding (local) one-parameter group φ_t, we have $\varphi_{t_0}(p_1) = p_2, t_0 \in \mathbb{R}$ (we can use a partition of unity to construct such a field). But $\pi \circ \varphi_t = \pi$ for all t, and therefore it follows (see Problem 2.56) that

$$\pi_*(\mathscr{D}_{p_1}) = (\pi_* \circ \varphi_{t_0 *})(\mathscr{D}_{p_1}) = \pi_*(\mathscr{D}_{p_2}).$$

Hence the distribution $\mathscr{D}'$ is well defined. To prove the smoothness of $\mathscr{D}'$, choose a point $p \in M$ and neighbourhoods $U \subset M$, $U' \subset M'$, with the coordinates as above. Let Y be any local vector field belonging to $\mathscr{D}|_U$:

$$Y(x^1, \dots, x^n) = \sum_{j=1}^{n} a_j(x^1, \dots, x^n) \frac{\partial}{\partial x^j}.$$

The sub-bundle $\ker \pi_*$ is spanned on U by $\partial/\partial x^k, k = n' + 1, \dots, n$, and the distribution $\mathscr{D}|_U$ is preserved by these vector fields $\partial/\partial x^k$ and, consequently (see Problem 2.56), by the corresponding local flows

$$\varphi^k_t : (x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots, x^n) \mapsto (x^1, \dots, x^{k-1}, x^k + t, x^{k+1}, \dots, x^n).$$

Therefore the vector field

$$Y''(x^1, \dots, x^n) = \sum_{j=1}^{n} a_j(x^1, \dots, x^{n'}, 0, \dots, 0) \frac{\partial}{\partial x^j} = \sum_{j=1}^{n} b_j(x^1, \dots, x^{n'}) \frac{\partial}{\partial x^j}$$

(recall that $x^j(p)=0$, $j=1,\dots,n$) is a smooth vector field belonging to $\mathscr{D}|_U$. Thus,

$$Y'=\pi_* Y''\big(x^1,\dots,x^{n'}\big)=\sum_{j=1}^{n'} b_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}$$

is a smooth vector field belonging to $\mathscr{D}'|_{U'}$, and the vector fields Y'', Y' are π-related. Thus there are vector fields $\{Y_l''\}$ and $\{Y_l'\}$ belonging to the restrictions $\mathscr{D}|_U$ and $\mathscr{D}'|_{U'}$, respectively, such that $\mathscr{D}'|_{U'}$ is spanned by the vector fields $\{Y_l'\}$ and the vector fields Y_l'', Y_l' are π-related for each l.

(iii) By Proposition 1.21, if the distribution $\mathscr{D}$ is involutive, then so is $\mathscr{D}'$.

Problem 2.58 Let $\pi: M\to M'$ be a surjective submersion of manifolds M and M'. Let the set $\pi^{-1}(p')$ be connected for all $p'\in M'$, and let $X\in\mathfrak{X}(M)$ be a smooth vector field which preserves the distribution $\ker\pi_*$.

Prove that there exists a unique smooth vector field X' on M' such that the vector fields X, X' are π-related.

Solution We will use the notation of the solution of the previous Problem 2.57. As above, consider the vector field Z (belonging to the distribution $\ker\pi_*$) with its local one-parametric group φ_t connecting points p_1, p_2 for which $\pi(p_1)=\pi(p_2)$. For the vector field X, we have (see Proposition 2.10)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t\cdot X)=-\varphi_t\cdot[Z,X].$$

Since the bracket $[Z,X]$ belongs to the distribution $\ker\pi_*$ and the local flow φ_t of $Z\in\ker\pi_*$ preserves the (involutive) distribution $\ker\pi_*$, the difference $\varphi_t\cdot X-X$ is a vector field belonging to $\ker\pi_*$ for all t. Thus $\pi_*X_{p_1}=\pi_*X_{p_2}$ and X', $X'_{\pi(p)}=\pi_*X_p$ is a well-defined vector field on the manifold M'. Therefore in the coordinate system $(U,x^1,\dots,x^n)$ around $p\in M$, the smooth vector field $X|_U$ has the following form (see the local expression ($\star$) for π in the solution of Problem 2.57):

$$\begin{aligned}(X|_U)\big(x^1,\dots,x^n\big)=&\sum_{j=1}^{n'} a_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}\\ &+\sum_{j=n'+1}^{n} a_j\big(x^1,\dots,x^{n'},\dots,x^n\big)\frac{\partial}{\partial x^j}.\end{aligned}$$

Now it is clear that the vector field

$$X'\big|_{U'}=\sum_{j=1}^{n'} a_j\big(x^1,\dots,x^{n'}\big)\frac{\partial}{\partial x^j}$$

is also smooth. The vector fields X, X' are π-related.

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Chapter 3
Integration on Manifolds

Abstract After giving some definitions and results on orientability of smooth manifolds, the problems treated in the present chapter are concerned with orientation of smooth manifolds; especially the orientation of several manifolds introduced in the previous chapter, such as the cylindrical surface, the Möbius strip, and the real projective space $\mathbb{R}\mathrm{P}^2$. Some attention is paid to integration on chains and integration on oriented manifolds, by applying Stokes' and Green's Theorems. Some calculations of de Rham cohomology are proposed, such as the cohomology groups of the circle and of an annular region in the plane. This cohomology is also used to prove that the torus T^2 and the sphere S^2 are not homeomorphic. The chapter ends with an application of Stokes' Theorem to a certain structure on the complex projective space $\mathbb{C}\mathrm{P}^n$.

> Dans le domain des paramètres $a_1, \dots, a_r$ d'un groupe continu quelconque d'ordre r, il existe en effet un élément de volume qui se conserve par une transformation quelconque du groupe des paramètres (...) Le premier groupe des paramètres, par example, est formé de l'ensemble des transformations que laissent invariantes r expressions de Pfaff $\omega_1, \dots, \omega_r$; l'élément de volume $d\tau$ est $\omega_1\omega_2\cdots\omega_r$. Si l'on désigne par S_0 une transformation fixe du groupe et si l'on pose $S_0S_a = S_b$, à un domaine (a) correspond un domain (b) de même volume. Or il existe des groupes continus dont le domain est fermé et de volume total fini; si alors on parte d'un function quelconque des variables et qu'on fasse l'integral des functiones transformées par les differents transformations du groupe, on obtient une function invariante par le groupe.[1]

[1] "In the domain of the parameters $a_1, \dots, a_r$ of any continuous group of order r, there exists indeed a volume element which is preserved under any transformation of the group of parameters (...) The first group of parameters consists, for instance, of the set of transformations preserving r Pfaff expressions $\omega_1, \dots, \omega_r$; the volume element $d\tau$ is $\omega_1\omega_2\cdots\omega_r$. Denoting by S_0 a fixed transformation of the group and putting $S_0S_a = S_b$, to a domain (a) corresponds a domain (b) with the same volume. Now, there are continuous groups with closed domain and finite total volume; hence starting with any function of the variables and integrating the functions obtained by the different transformations of the group, one obtains a function invariant under the group."

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_3,

ÉLIE CARTAN, "*Les tenseurs irreductibles et les groupes linéaires simples et semi-simples*," Boll. Sc. Math. *49* (1925), p. 131. Oeuvres Complètes, vol. I, part I, Gauthier-Villars, avec le concours du C.N.R.S, Paris, 1952, p. 532. (Reproduced with kind permission from Dunod Éditeur, Paris. Not for re-use elsewhere.)

The extension to manifolds involves two steps: first, we define integrals over the entire manifold M of suitable exterior n-forms and second, for those M which have a predetermined volume element (e.g. Riemannian manifolds), integrals of functions over domains are defined. All the standard properties of integrals follow readily from the corresponding facts in the Euclidean space. As an illustration of the use of integration on manifolds an application is made to compact Lie groups. It is shown that by averaging a left-invariant metric on a compact group one may obtain a bi-invariant Riemannian metric. With the same techniques—due to Weyl—it is shown that any representation of a compact group as a matrix group acting on a vector space leaves invariant some inner product on that vector space, from which it follows that any invariant subspace has a complementary invariant subspace.

WILLIAM M. BOOTHBY, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd Revised Ed., Academic Press, 2003, p. 222. (With kind permission from Elsevier.)

3.1 Some Definitions and Theorems on Integration on Manifolds

Definitions 3.1 Let V be a real vector space of dimension n. An *orientation* of V is a choice of component of $\Lambda^n V \setminus \{0\}$.

A connected differentiable manifold M of dimension n is said to be *orientable* if it is possible to choose in a consistent way an orientation on $T_p^* M$ for each $p \in M$. More precisely, let O be the "0-section" of the exterior n-bundle $\Lambda^n M^*$, that is,

$$O = \bigcup_{p \in M} \{0 \in \Lambda^n T_p^* M\}.$$

Then since $\Lambda^n T_p^* M \setminus \{0\}$ has exactly two components, it follows that $\Lambda^n T^* M \setminus \{O\}$ has at most two components. It is said that M is *orientable* if $\Lambda^n T^* M \setminus \{O\}$ has two components; and if M is orientable, an *orientation* is a choice of one of the two components of $\Lambda^n T^* M \setminus \{O\}$. It is said that M is *non-orientable* if $\Lambda^n T^* M \setminus \{O\}$ is connected.

Let M and N be two orientable differentiable n-manifolds, and let $\Phi\colon M \to N$ be a differentiable map. It is said that Φ *preserves orientations* or that it is *orientation-preserving* if $\Phi_*\colon T_p M \to T_{\Phi(p)} N$ is an isomorphism for every $p \in M$, and the induced map $\Phi^*\colon \Lambda^n T^* N \to \Lambda^n T^* M$ maps the component $\Lambda^n T^* M \setminus \{O\}$ determining the orientation of N into the component $\Lambda^n T^* M \setminus \{O\}$ determining the orientation of M. Equivalently, Φ is orientation-preserving if Φ_* sends oriented bases of the tangent spaces to M to oriented bases of the tangent spaces to N.

Proposition 3.2 *Let M be a connected differentiable manifold of dimension n. Then the following are equivalent*:

(i) *M is orientable*;
(ii) *There is a collection $\mathscr{C} = \{(U, \varphi)\}$ of coordinate systems on M such that*

$$M = \bigcup_{(U,\varphi)\in\mathscr{C}} U \quad \textit{and} \quad \det\left(\frac{\partial x^i}{\partial y^j}\right) > 0 \quad \textit{on } U \cap V$$

whenever $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ belong to $\mathscr{C}$;
(iii) *There is a nowhere-vanishing differential n-form on M.*

Theorem 3.3 (Stokes' Theorem I) *Let c be an r-chain in M, and let ω be a C^∞ $(r-1)$-form defined on a neighbourhood of the image of c. Then*

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega.$$

Theorem 3.4 (Green's Theorem) *Let $\sigma(t) = (x(t)), y(t))$, $t \in [a, b]$, be a simple, closed plane curve. Suppose that σ is positively oriented (that is, $\sigma|_{(a,b)}$ is orientation-preserving) and let D denote the bounded, closed, connected domain whose boundary is σ. Let $f = f(x, y)$ and $g = g(x, y)$ be real functions with continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial g/\partial x$, $\partial g/\partial y$ on D. Then*

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \int_\sigma \left(f\frac{\mathrm{d}x}{\mathrm{d}t} + g\frac{\mathrm{d}y}{\mathrm{d}t}\right) \mathrm{d}t.$$

Definition 3.5 Let M be a differentiable manifold. A subset $D \subseteq M$ is said to be a *regular domain* if for every $p \in \partial D$ there exists a chart $(U, \varphi) = (U, x^1, \dots, x^n)$ centred at p such that

$$\varphi(U \cap D) = \left\{x \in \varphi(U) : x^n \geqslant 0\right\}.$$

Theorem 3.6 (Stokes' Theorem II) *Let D be a regular domain in an oriented n-dimensional manifold M, and let ω be a differential $(n-1)$-form on M such that* $\operatorname{supp}(\omega) \cap \bar{D}$ *is compact. Then*

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

Definitions 3.7 A differential r-form α on M is said to be *closed* if $\mathrm{d}\alpha = 0$. It is called *exact* if there is an $(r-1)$-form β such that $\alpha = \mathrm{d}\beta$. Since $\mathrm{d}^2 = 0$, every exact form is closed. The quotient space of closed r-forms modulo the space of exact r-forms is called the rth *de Rham cohomology group of M*:

$$H^r_{dR}(M, \mathbb{R}) = \{\text{closed } r\text{-forms}\}/\{\text{exact } r\text{-forms}\}.$$

If $\Phi \colon M \to N$ is differentiable, then $\Phi^* \colon \Lambda^* N \to \Lambda^* M$ transforms closed (resp., exact) forms into closed (resp., exact) forms. Hence Φ induces a linear map

$$\Phi^* \colon H^r_{dR}(N, \mathbb{R}) \to H^r_{dR}(M, \mathbb{R}).$$

3.2 Orientable Manifolds. Orientation-Preserving Maps

Problem 3.8 Prove:

(i) The product of two orientable manifolds is orientable.
(ii) The total space of the tangent bundle over any manifold is an orientable manifold.

Solution

(i) A C^∞ manifold M is orientable if and only if (see Proposition 3.2(ii)) there is a collection Φ of coordinate systems on M such that

$$M = \bigcup_{(U,\varphi)\in\Phi} U \quad \text{and} \quad \det\left(\frac{\partial x^i}{\partial y^j}\right) > 0 \quad \text{on } U \cap V$$

whenever $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ belong to Φ.

Suppose M_1 and M_2 are orientable. Denote by (U_1, x_1^i) and (U_2, x_2^j) two such coordinate systems on M_1 and M_2, respectively. With a little abuse of notation (that is, dropping the projection maps pr_1 and pr_2 from $M_1 \times M_2$ onto the factors M_1 and M_2), we can write the corresponding coordinate systems on $M_1 \times M_2$ as $(U_1 \times U_2, x_1^i, x_2^j)$. As the local coordinates on each factor manifold do not depend on the local coordinates on the other one, the Jacobian matrix of the corresponding change of charts of the product manifold $M_1 \times M_2$ can be expressed in block form as

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1^i}{\partial y_1^k} & 0 \\ 0 & \frac{\partial x_2^j}{\partial y_2^l} \end{pmatrix}.$$

Since $\det J_1$ and $\det J_2$ are positive, we have $\det J > 0$.

Alternatively, the question can be solved more intrinsically as follows: Given the non-vanishing differential forms of maximum degree ω_1 and ω_2 determining the respective orientations on M_1 and M_2, it suffices to consider the form $\omega = \mathrm{pr}_1^* \omega_1 \wedge \mathrm{pr}_2^* \omega_2$ on $M_1 \times M_2$.

(ii) Let M be a differentiable n-manifold and let π be the projection map of the tangent bundle TM. For any coordinates $\{x^i\}$ on an open subset $U \subset M$, denote by $\{x^i, y^i\} = \{x^i \circ \pi, \mathrm{d}x^i\}$ the usual coordinates on $\pi^{-1}(U)$. Let $\{x'^i\}$ be another set of coordinates defined on a open subset $U' \subset M$ such that $U \cap U' \neq \emptyset$. The change of coordinates $x'^i = x'^i(x^j)$ on $U \cap U'$ induces the change of coordinates on $\pi^{-1}(U \cap V)$ given by

$$x'^i = x'^i(x^1, \dots, x^n), \qquad y'^i = \sum_{j=1}^n \frac{\partial x'^i}{\partial x^j} y^j, \quad i = 1, \dots, n.$$

The Jacobian matrix of this change of coordinates is

$$J = \begin{pmatrix} \frac{\partial x'^i}{\partial x^j} & 0 \\ \frac{\partial^2 x'^i}{\partial x^k \partial x^j} y^k & \frac{\partial x'^i}{\partial x^j} \end{pmatrix}.$$

Since $\det J = \det(\frac{\partial x'^i}{\partial x^j})^2 > 0$, it follows that TM is orientable.

Problem 3.9 Prove that if a C^∞ manifold M admits an atlas formed by two charts (U, φ), (V, ψ), and $U \cap V$ is connected, then M is orientable. Apply this result to the sphere S^n, $n > 1$, with the atlas formed by the stereographic projections from the poles (see Problem 1.28).

Solution Let $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$ be the coordinate maps. If $\det(\partial x^i / \partial y^j) \neq 0$ on $U \cap V$ and $U \cap V$ is connected, we have either (a) $\det(\partial x^i / \partial y^j) > 0$ for all $U \cap V$; or (b) $\det(\partial x^i / \partial y^j) < 0$ for all $U \cap V$. In the case (a), it follows that M is orientable with the given atlas. In the case (b), we should only have to consider as coordinate maps $\varphi = (x^1, \dots, x^n)$ and $\psi = (-y^1, y^2, \dots, y^n)$.

For S^n, $n > 1$, considering the stereographic projections, we have the coordinate domains

$$U_N = \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq 1\},$$
$$U_S = \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq -1\}.$$

As

$$U_N \cap U_S = \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq \pm 1\} = \varphi_N^{-1}(\mathbb{R}^n \setminus \{0\})$$

is connected, we conclude that S^n is orientable.

Problem 3.10 Study the orientability of the following C^∞ manifolds:

(i) A cylindrical surface of $\mathbb{R}^3$, with the atlas given in Problem 1.30.
(ii) The Möbius strip, with the atlas given in Problem 1.31.
(iii) The real projective space $\mathbb{RP}^2$, with the atlas given in Problem 1.81.

Solution

(i) The Jacobian matrix J of the change of the charts given in Problem 1.30 always has positive determinant; in fact, equal to 1. Thus the manifold is orientable.

(ii) For the given atlas, the open subset $U \cap V$ decomposes into two connected open subsets W_1 and W_2, such that on W_1 (resp., W_2) the Jacobian of the change of coordinates has positive (resp., negative) determinant. Hence M is not orientable.

(iii) With the notations in (ii) in Problem 1.81, we have in the case of $\mathbb{R}\mathrm{P}^2$ three charts (U_1, φ_1), (U_2, φ_2) and (U_3, φ_3), such that, for instance,

$$\begin{aligned}\varphi_1(U_1 \cap U_2) &= \varphi_1\left(\left\{\left[x^1, x^2, x^3\right] : x^1 \neq 0, x^2 \neq 0\right\}\right) = \left\{\left(t^1, t^2\right) \in \mathbb{R}^2 : t^1 \neq 0\right\} \\ &= V_1 \cup V_2,\end{aligned}$$

where $V_1 = \{(t^1, t^2) \in \mathbb{R}^2 : t^1 > 0\}$ and $V_2 = \{(t^1, t^2) \in \mathbb{R}^2 : t^1 < 0\}$ are connected. The change of coordinates on $\varphi_1(U_1 \cap U_2)$ is given by

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)\left(t^1, t^2\right) = \varphi_2\left(\left[1, t^1, t^2\right]\right) = \left(\frac{1}{t^1}, \frac{t^2}{t^1}\right),$$

and the determinant of its Jacobian matrix is easily seen to be equal to $-1/(t^1)^3$, which is negative on V_1 and positive on V_2. Hence, $\mathbb{R}\mathrm{P}^2$ is not orientable.

Problem 3.11 Consider the map

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (u, v) = \left(x\,\mathrm{e}^y + y, x\,\mathrm{e}^y + \lambda y\right), \quad \lambda \in \mathbb{R}.$$

(i) Find the values of λ for which φ is a diffeomorphism.
(ii) Find the values of λ for which the diffeomorphism φ is orientation-preserving.

Solution

(i) Suppose that

$$x\,\mathrm{e}^y + y = x'\mathrm{e}^{y'} + y', \qquad x\,\mathrm{e}^y + \lambda y = x'\mathrm{e}^{y'} + \lambda y'. \tag{$\star$}$$

Subtracting, we have $(1-\lambda)y = (1-\lambda)y'$. Hence, for $\lambda \neq 1$, we have $y = y'$. And from any of the two equations ($\star$), we deduce that $x = x'$.

The map φ is clearly C^∞ and its inverse map, given by

$$y = \frac{u-v}{1-\lambda}, \qquad x = \frac{\lambda u - v}{\lambda - 1}\mathrm{e}^{\frac{u-v}{\lambda-1}},$$

is a C^∞ map if and only if $\lambda \neq 1$. Thus φ is a diffeomorphism if and only if $\lambda \neq 1$.

(ii) Consider the canonical orientation of $\mathbb{R}^2$ given by $\mathrm{d}x \wedge \mathrm{d}y$, or by $\mathrm{d}u \wedge \mathrm{d}v$. We have

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} \mathrm{e}^y & x\,\mathrm{e}^y + 1 \\ \mathrm{e}^y & x\,\mathrm{e}^y + \lambda \end{pmatrix}.$$

Therefore,

$$\mathrm{d}u \wedge \mathrm{d}v = \frac{\partial(u, v)}{\partial(x, y)} \mathrm{d}x \wedge \mathrm{d}y = \mathrm{e}^y(\lambda - 1)\, \mathrm{d}x \wedge \mathrm{d}y.$$

That is, φ is orientation-preserving if $\lambda > 1$.

3.3 Integration on Chains. Stokes' Theorem I

For the theory relevant to the next problem and others in this chapter, see, among others, Spivak [1] and Warner [2].

Problem 3.12 Compute the integral of the differential 1-form

$$\alpha = \left(x^2 + 7y\right) \mathrm{d}x + \left(-x + y \sin y^2\right) \mathrm{d}y \in \Lambda^1\mathbb{R}^2$$

over the 1-cycle given by the oriented segments going from $(0, 0)$ to $(1, 0)$, then from $(1, 0)$ to $(0, 2)$, and then from $(0, 2)$ to $(0, 0)$.

Solution Denoting by c the 2-chain (with the usual counterclockwise orientation) whose boundary is the triangle above, by Stokes' Theorem I (Theorem 3.3), we have

$$\int_{\partial c} \alpha = \int_c \mathrm{d}\alpha = -8 \int_c \mathrm{d}x \wedge \mathrm{d}y = -8 \int_0^1 \left(\int_0^{2(1-x)} \mathrm{d}y\right) \mathrm{d}x = -8.$$

Problem 3.13 Deduce from Green's Theorem 3.4:

(i) The formula for the area of the interior D of a simple, closed positively oriented plane curve $[a, b] \to (x(t), y(t)) \in \mathbb{R}^2$:

$$A(D) = \int_D \mathrm{d}x\, \mathrm{d}y = \frac{1}{2} \int_a^b \left(x(t)\frac{\mathrm{d}y}{\mathrm{d}t} - y(t)\frac{\mathrm{d}x}{\mathrm{d}t}\right) \mathrm{d}t.$$

(ii) The formula of change of variables for double integrals:

$$\iint_D F(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_{\varphi^{-1}D} F\left(x(u, v), y(u, v)\right) \frac{\partial(x, y)}{\partial(u, v)}\, \mathrm{d}u\, \mathrm{d}v,$$

corresponding to the coordinate transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $x \circ \varphi = x(u, v)$, $y \circ \varphi = y(u, v)$.

Solution

(i) It follows directly from Green's Theorem 3.4 by letting $g = x$, $f = -y$ in the formula mentioned there.

(ii) First, we let $f=0$, $\partial g/\partial x = F$ in Green's formula. Then, from the formula for change of variables and again from Green's Theorem 3.4, we obtain

$$\begin{aligned}
\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\partial D} g\,\mathrm{d}y = \pm\int_{\varphi^{-1}(\partial D)} \varphi^*(g\,\mathrm{d}y)\\
&= \pm\int_{\varphi^{-1}(\partial D)} (g\circ\varphi)\left(\frac{\partial y}{\partial u}u'(t) + \frac{\partial y}{\partial v}v'(t)\right)\mathrm{d}t\\
&= \pm\int_{\varphi^{-1}(\partial D)} \left\{\left((g\circ\varphi)\frac{\partial y}{\partial u}\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \left((g\circ\varphi)\frac{\partial y}{\partial v}\right)\frac{\mathrm{d}v}{\mathrm{d}t}\right\}\mathrm{d}t\\
&= \pm\iint_{\varphi^{-1}D}\left\{\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u}\right)\right\}\mathrm{d}u\,\mathrm{d}v,
\end{aligned} \tag{$\star$}$$

where one takes $+$ if φ preserves orientation and $-$ if not. Moreover

$$\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right) = \left(\left(\frac{\partial g}{\partial x}\circ\varphi\right)\frac{\partial x}{\partial u} + \left(\frac{\partial g}{\partial y}\circ\varphi\right)\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v} + (g\circ\varphi)\frac{\partial^2 y}{\partial u\partial v},$$

$$\frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u}\right) = \left(\left(\frac{\partial g}{\partial x}\circ\varphi\right)\frac{\partial x}{\partial v} + \left(\frac{\partial g}{\partial y}\circ\varphi\right)\frac{\partial y}{\partial v}\right)\frac{\partial y}{\partial u} + (g\circ\varphi)\frac{\partial^2 y}{\partial v\partial u}.$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u}\right) &= (F\circ\varphi)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\\
&= (F\circ\varphi)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|.
\end{aligned}$$

Substituting this equality in ($\star$), we have

$$\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y = \pm\iint_{\varphi^{-1}D}(F\circ\varphi)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v.$$

Problem 3.14 Let c_2 be a 2-chain in $\mathbb{R}^2$ and $f\in C^\infty\mathbb{R}^2$. Prove that

$$\int_{\partial c_2}\left(\frac{\partial f}{\partial y}\,\mathrm{d}x - \frac{\partial f}{\partial x}\,\mathrm{d}y\right) = 0$$

if f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Solution From Stokes' Theorem I, we have

$$\int_{\partial c_2}\left(\frac{\partial f}{\partial y}\,\mathrm{d}x-\frac{\partial f}{\partial x}\mathrm{d}y\right)=\int_{c_2}\mathrm{d}\left(\frac{\partial f}{\partial y}\,\mathrm{d}x-\frac{\partial f}{\partial x}\,\mathrm{d}y\right)=-\int_{c_2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\wedge\mathrm{d}y=0.$$

Problem 3.15 Consider the 1-chain

$$c_{r,n}\colon [0,1]\to\mathbb{R}^2\setminus\{0\},\qquad c_{r,n}(t)=\big(x(t)=r\cos 2\pi nt,\ y(t)=r\sin 2\pi nt\big),$$

for $r\in\mathbb{R}^+$, $n\in\mathbb{Z}^+$.

Prove that $c_{r,n}$ is not the boundary of any 2-chain in $\mathbb{R}^2\setminus\{0\}$.

Solution Let θ be the angle function on $C=c_{r,n}([0,1])$. Then, $\mathrm{d}\theta$ is a globally defined differential 1-form on C, and we have

$$\int_{c_{r,n}}\mathrm{d}\theta=\int_{c_{r,n}}\mathrm{d}\arctan\left(\frac{y}{x}\right)=2\pi n.$$

Suppose $c_{r,n}=\partial c_2$ for a 2-chain $c_2\in\mathbb{R}^2\setminus\{0\}$. Then, from Stokes' Theorem I, it follows that

$$\int_{c_{r,n}}\mathrm{d}\theta=\int_{c_2}\mathrm{d}\,(\mathrm{d}\theta)=0,$$

thus leading us to a contradiction.

3.4 Integration on Oriented Manifolds. Stokes' Theorem II

Problem 3.16 Given on $\mathbb{R}^3$ the differential form

$$\omega=\big(z-x^2-xy\big)\,\mathrm{d}x\wedge\,\mathrm{d}y-\mathrm{d}y\wedge\mathrm{d}z-\mathrm{d}z\wedge\mathrm{d}x,$$

compute $\int_D i^*\omega$, where i denotes the inclusion map of

$$D=\big\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leqslant 1,\ z=0\big\}$$

in $\mathbb{R}^3$.

Solution We have

$$\int_D i^*\omega=-\int_D\big(x^2+xy\big)\,\mathrm{d}x\wedge\,\mathrm{d}y.$$

Taking polar coordinates

$$x=\rho\cos\theta,\qquad y=\rho\sin\theta,\quad \rho\in(0,1),\ \theta\in(0,2\pi),$$

one has

$$\frac{\partial(x,y)}{\partial(\rho,\theta)} = \det\begin{pmatrix}\cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta\end{pmatrix} = \rho.$$

Therefore, for $D_0 = D \setminus \{[0,1) \times \{0\}\}$, one has

$$\begin{aligned}\int_D i^*\omega &= -\int_{D_0}\big(x^2+xy\big)\,\mathrm{d}x\wedge\mathrm{d}y = -\int_{D_0}\rho^2\big(\cos^2\theta+\sin\theta\cos\theta\big)\rho\,\mathrm{d}\rho\wedge\mathrm{d}\theta \\ &= -\int_0^{2\pi}\int_0^1 \rho^3\big(\cos^2\theta+\sin\theta\cos\theta\big)\,\mathrm{d}\rho\,\mathrm{d}\theta \\ &\quad -\frac{1}{4}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}+\frac{\sin 2\theta}{2}\right)\mathrm{d}\theta \\ &= -\frac{\pi}{4}.\end{aligned}$$

Problem 3.17 Let (u,v,w) denote the usual coordinates on $\mathbb{R}^3$. Consider the parametrisation (see Remark 1.4)

$$u = \frac{1}{2}\sin\theta\cos\varphi, \qquad v = \frac{1}{2}\sin\theta\sin\varphi, \qquad w = \frac{1}{2}\cos\theta + \frac{1}{2}, \tag{$\star$}$$

where $\theta \in (0,\pi)$, $\varphi \in (0,2\pi)$, of the sphere

$$S^2 = \left\{(u,v,w)\in\mathbb{R}^3 : u^2+v^2+\left(w-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}.$$

Let $N=(0,0,1)$ be its north pole and let $\pi\colon S^2\setminus\{N\}\to\mathbb{R}^2$ be the stereographic projection onto the plane $\mathbb{R}^2 \equiv w = 0$. Let $v_{\mathbb{R}^2} = \mathrm{d}x\wedge\mathrm{d}y$ be the canonical volume form on $\mathbb{R}^2$ and let $v_{S^2} = \frac{1}{4}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi$ be the volume form on S^2 above. Write $\pi^* v_{\mathbb{R}^2}$ in terms of v_{S^2}.

Remark The 2-form

$$v_{S^2} = \frac{1}{4}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi$$

is called the *canonical volume form* on S^2 because one has $\sigma(X,Y)=1$, $X,Y\in\mathfrak{X}(S^2)$, for $\{X,Y,\mathbf{n}\}$ an orthonormal basis of $\mathbb{R}^3$, where $\mathbf{n}$ denotes the exterior (i.e. pointing outwards) unit normal field on S^2.

Solution The given stereographic projection is the restriction to $S^2\setminus\{N\}$ of the map

$$\widetilde{\pi}\colon \mathbb{R}^3\setminus\{w=1\}\to\mathbb{R}^2, \qquad (u,v,w)\mapsto\left(\frac{u}{1-w},\frac{v}{1-w}\right),$$

whose Jacobian matrix is

$$\begin{pmatrix} \frac{1}{1-w} & 0 & \frac{u}{(1-w)^2} \\ 0 & \frac{1}{1-w} & \frac{v}{(1-w)^2} \end{pmatrix}.$$

Hence

$$\begin{aligned}
\widetilde{\pi}^* v_{\mathbb{R}^2} &= \widetilde{\pi}^*(\mathrm{d}x \wedge \mathrm{d}y) \\
&= \widetilde{\pi}^* \mathrm{d}x \wedge \widetilde{\pi}^* \mathrm{d}y \\
&= \left(\frac{1}{1-w}\,\mathrm{d}u + \frac{u}{(1-w)^2}\,\mathrm{d}w\right) \wedge \left(\frac{1}{1-w}\,\mathrm{d}v + \frac{v}{(1-w)^2}\,\mathrm{d}w\right) \\
&= \frac{1}{(1-w)^2}\,\mathrm{d}u \wedge \mathrm{d}v + \frac{v}{(1-w)^3}\,\mathrm{d}u \wedge \mathrm{d}w - \frac{u}{(1-w)^3}\,\mathrm{d}v \wedge \mathrm{d}w. \qquad (\star\star)
\end{aligned}$$

Thus, substituting ($\star$) into ($\star\star$), we obtain after an easy computation

$$\pi^* v_{\mathbb{R}^2} = \widetilde{\pi}^* v_{\mathbb{R}^2} = -\frac{\sin\theta}{(1-\cos\theta)^2}\,\mathrm{d}\theta \wedge \mathrm{d}\varphi = -\frac{4}{(1-\cos\theta)^2} v_{S^2}.$$

Problem 3.18 Compute the integral of $\omega = (x - y^3)\,\mathrm{d}x + x^3\,\mathrm{d}y$ along S^1 applying Stokes' Theorem II.

Solution Let D (resp., $\bar{D}$) be the open (resp., closed) unit disk of $\mathbb{R}^2$, and let $D_0 = D \setminus \{[0, 1) \times \{0\}\}$. Applying Stokes' Theorem II, we have

$$\int_{S^1} \omega = \int_{\partial \bar{D}} \omega = \int_{\bar{D}} \mathrm{d}\omega = \int_{D_0} \mathrm{d}\omega = \int_{D_0} 3\big(x^2 + y^2\big)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Taking polar coordinates, we have as in Problem 3.16 that

$$\int_{S^1} \omega = \int_{D_0} 3\rho^3\,\mathrm{d}\rho \wedge \mathrm{d}\theta = 3\int_0^{2\pi} \left(\int_0^1 \rho^3\,\mathrm{d}\rho\right) \mathrm{d}\theta = \frac{3\pi}{2}.$$

Problem 3.19 Let f be a C^∞ function on $\mathbb{R}^2$, and D a compact and connected subset of $\mathbb{R}^2$ with regular boundary ∂D such that $f|_{\partial D} = 0$.

(i) Prove the equality

$$\int_D f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \mathrm{d}x \wedge \mathrm{d}y = -\int_D \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\} \mathrm{d}x \wedge \mathrm{d}y.$$

(ii) Deduce from (i) that if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on D, then $f|_D = 0$.

Solution

(i) By Stokes' Theorem II, we have

$$\int_D \left\{ f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right\} \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial D} \psi,$$

where ψ is a differential 1-form so that $\mathrm{d}\psi$ is equal to the 2-form in the left-hand side. One solution is given by

$$\psi = -f\frac{\partial f}{\partial y}\,\mathrm{d}x + f\frac{\partial f}{\partial x}\,\mathrm{d}y.$$

Since $f|_{\partial D} = 0$, we have

$$\int_{\partial D} \psi = 0,$$

from which the wanted equality follows.

(ii) If $\Delta f = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$, by the equality we have just proved, one has

$$\int_D \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right\} \mathrm{d}x \wedge \mathrm{d}y = 0,$$

that is, $|\mathrm{d}f|$ being the modulus of $\mathrm{d}f$, we have $\int_D |\mathrm{d}f|^2\,\mathrm{d}x \wedge \mathrm{d}y = 0$; thus f is constant on D, but since $f|_{\partial D} = 0$ we have $f|_D = 0$.

Problem 3.20 Let $\alpha = \frac{1}{2\pi}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} \in \Lambda^1(\mathbb{R}^2 \setminus \{0\})$.

(i) Prove that α is closed.
(ii) Compute the integral of α on the unit circle S^1.
(iii) How does this result show that α is not exact?
(iv) Let $j\colon S^1 \hookrightarrow \mathbb{R}^2$ be the canonical embedding. How can we deduce from (iii) that $j^*\alpha$ is not exact?

Solution

(i) Immediate.
(ii) Parametrise S^1 (see Remark 1.4) as $x = \cos\theta$, $y = \sin\theta$, $\theta \in (0, 2\pi)$. Then

$$\int_{S^1} \alpha = \frac{1}{2\pi}\int_0^{2\pi} \left(\cos^2\theta + \sin^2\theta\right) \mathrm{d}\theta = 1.$$

(iii) If it were $\alpha = \mathrm{d}f$ for a given function f, applying Stokes' Theorem II, it would be

$$\int_{S^1} \alpha = \int_{S^1} \mathrm{d}f = \int_{\partial S^1} f = 0,$$

contradicting the result in (ii).

(iv) Let us suppose that $j^*\alpha$ is exact, i.e. $j^*\alpha = \mathrm{d}f$. Then we would have

$$\int_{S^1} j^*\alpha = \int_{j(S^1)} \alpha = \int_{S^1} \alpha = 1.$$

On the other hand, as $j^*\,\mathrm{d} = \mathrm{d}j^*$, and denoting by $\emptyset$ the empty set, we would have

$$\int_{S^1} j^*\alpha = \int_{S^1} j^*\,\mathrm{d}f = \int_{S^1} \mathrm{d}j^*f = \int_{\partial S^1} j^*f = \int_{\emptyset} j^*f = 0,$$

but this contradicts the previous calculation.

Problem 3.21 Consider

$$\alpha = \frac{x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2+y^2+z^2)^{3/2}} \in \Lambda^2\big(\mathbb{R}^3 \setminus \{0\}\big).$$

(i) Prove that α is closed.
(ii) Compute $\int_{S^2} \alpha$.
(iii) How does this prove that α is not exact?

Solution

(i) Immediate.
(ii) Consider the parametrisation of S^2 given (see Remark 1.4) by

$$x = \cos\theta\cos\varphi, \qquad y = \cos\theta\sin\varphi, \qquad z = \sin\theta,$$

$\theta \in (-\pi/2, \pi/2)$, $\varphi \in (0, 2\pi)$, which covers the surface up to a set of measure zero. We have $\alpha|_{S^2} = -\cos\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and

$$\int_{S^2} \alpha = \int_0^{2\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos\theta\,\mathrm{d}\theta \right) \mathrm{d}\varphi = -4\pi.$$

(iii) If $\alpha = \mathrm{d}\beta$, by Stokes' Theorem II, it would be

$$\int_{S^2} \alpha = \int_{S^2} \mathrm{d}\beta = \int_{\partial S^2} \beta = 0,$$

which contradicts the result in (ii).

3.5 De Rham Cohomology

Problem 3.22 Prove that the de Rham cohomology groups of the circle are

$$H^i_{dR}\big(S^1, \mathbb{R}\big) = \begin{cases} \mathbb{R}, & i = 0, 1, \\ 0, & i > 1. \end{cases}$$

The relevant theory is developed, for instance, in Warner [2].

Solution One has $H^0_{dR}(S^1,\mathbb{R})=\mathbb{R}$ because S^1 is connected. Since $\dim S^1=1$, one has $H^i_{dR}(S^1,\mathbb{R})=0$ if $i>1$.

As for $H^1_{dR}(S^1,\mathbb{R})=\mathbb{R}$, every 1-form on S^1 is closed. Now, let ω_0 be the restriction to S^1 of the differential form $(-y\,\mathrm{d}x+x\,\mathrm{d}y)/(x^2+y^2)$ on $\mathbb{R}^2\setminus\{(0,0)\}$. We locally have $\omega_0=\mathrm{d}\theta$, θ being the angle function. Hence $\mathrm{d}\theta$ is non-zero at every point of S^1. (In spite of the notation, $\mathrm{d}\theta$ is not exact, cf. Problem 3.20.) Hence, if ω is any 1-form on S^1, then we have $\omega=f(\theta)\,\mathrm{d}\theta$, where f is differentiable and periodic with period 2π. To prove this, we only have to see that there is a constant c and a differentiable and periodic function $g(\theta)$ such that

$$f(\theta)\,\mathrm{d}\theta=c\,\mathrm{d}\theta+\mathrm{d}g(\theta).$$

In fact, if this is so, integrating we have

$$c=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,\mathrm{d}\theta.$$

We then define

$$g(\theta)=\int_0^{\theta}\bigl(f(t)-c\bigr)\,\mathrm{d}t,$$

where c is the constant determined by the previous equality. One clearly has that g is differentiable. Finally, we must see that it is periodic. Indeed,

$$\begin{aligned}2g(\theta+2\pi)&=g(\theta)+\int_{\theta}^{\theta+2\pi}\bigl(f(t)-c\bigr)\,\mathrm{d}t\\&=g(\theta)+\int_{\theta}^{\theta+2\pi}f(t)\,\mathrm{d}t-\int_0^{2\pi}f(t)\,\mathrm{d}t\quad(f\text{ is periodic})\\&=g(\theta).\end{aligned}$$

Problem 3.23 Compute the de Rham cohomology groups of the annular region

$$M=\bigl\{(x,y)\in\mathbb{R}^2:1<\sqrt{x^2+y^2}<2\bigr\}.$$

Hint Apply the following general result: if two maps $f,g\colon M\to N$ between two C^∞ manifolds are C^∞ homotopic, that is, if there exists a C^∞ map $F\colon M\times[0,1]\to N$ such that $F(p,0)=f(p)$, $F(p,1)=g(p)$ for every $p\in M$, then the maps

$$f^*\colon H^k_{dR}(N,\mathbb{R})\to H^k_{dR}(M,\mathbb{R}),\qquad g^*\colon H^k_{dR}(N,\mathbb{R})\to H^k_{dR}(M,\mathbb{R}),$$

are equal for every $k=0,1,\dots$.

The relevant theory is developed, for instance, in Warner [2].

Solution Let $N = S^1(3/2)$ be the circle with centre at the origin and radius $3/2$ in $\mathbb{R}^2$. Let $j\colon N \to M$ be the inclusion map and let r be the retraction $r\colon M \to N$, $p \mapsto \frac{3}{2}(p/|p|)$. Then, $r \circ j\colon N \to N$ is the identity on $S^1(3/2)$. The map $j \circ r\colon M \to M$, $p \mapsto \frac{3}{2}(p/|p|)$, although not the identity of M, is homotopic to the identity. In fact, we can define the homotopy by

$$H\colon M \times [0,1] \to M, \qquad (p,t) \mapsto tp + (1-t)\frac{3}{2}\frac{p}{|p|}.$$

Thus, for $k = 0, 1, 2$, we have

$$j^*\colon H^k_{dR}(M,\mathbb{R}) \to H^k_{dR}\big(S^1(3/2),\mathbb{R}\big), \qquad r^*\colon H^k_{dR}\big(S^1(3/2),\mathbb{R}\big) \to H^k_{dR}(M,\mathbb{R}),$$

so, applying the general result quoted in the hint, we have

$$r^* \circ j^* = (j \circ r)^* = \text{identity on } H^k_{dR}(M,\mathbb{R}),$$
$$j^* \circ r^* = (r \circ j)^* = \text{identity on } H^k_{dR}\big(S^1(3/2),\mathbb{R}\big).$$

Hence, j^* and r^* are mutually inverse and it follows that

$$H^k_{dR}(M,\mathbb{R}) \cong H^k_{dR}\big(S^1(3/2),\mathbb{R}\big). \tag{$\star$}$$

Consequently, $H^0_{dR}(M,\mathbb{R}) = \mathbb{R}$ (as one can also deduce directly since M is connected). In fact, there are no exact 0-forms, and the closed 0-forms (that is, the differentiable functions f such that $\mathrm{d}f = 0$) are the constant functions, since M is connected.

As $\dim S^1(3/2) = 1$, from the isomorphism ($\star$) we obtain $H^k_{dR}(M,\mathbb{R}) = 0$, $k \geqslant 2$.

Finally, $H^1_{dR}(M,\mathbb{R}) \cong H^1_{dR}(S^1(3/2),\mathbb{R}) = \mathbb{R}$, hence

$$H^k_{dR}(M,\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$

Problem 3.24

(i) Prove that every closed differential 1-form on the sphere S^2 is exact.
(ii) Using de Rham cohomology, conclude that the torus T^2 and the sphere are not homeomorphic.

Hint Consider the parametrisation (see Remark 1.4)

$$x = (R + r\cos\theta)\cos\varphi, \qquad y = (R + r\cos\theta)\sin\varphi, \qquad z = r\sin\varphi,$$
$$R > r,\ \theta,\varphi \in (0, 2\pi),$$

of the torus T^2, and take the restriction to T^2 of the differential form $\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$ on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$.

The relevant theory is developed, for instance, in Warner [2].

Solution Let ω be a closed 1-form on the sphere. We shall prove that it is exact. Let U_1 and U_2 be the open subsets of S^2 obtained by removing two antipodal points, respectively. Then, writing $\omega_i = \omega|_{U_i}$, since U_i is homeomorphic to $\mathbb{R}^2$, there exist functions $f_i\colon U_i \to \mathbb{R}$, such that $\omega_i = \mathrm{d}f_i$. As $U_1 \cap U_2$ is connected, one has $f_1 = f_2 + \lambda$ on $U_1 \cap U_2$, for $\lambda \in \mathbb{R}$. The function $f\colon S^2 \to \mathbb{R}$ defined by $f|_{U_1} = f_1$, $f|_{U_2} = f_2 + \lambda$ is differentiable and $\mathrm{d}f = \omega$.

To prove that T^2 and S^2 are not homeomorphic, we only have to find a closed 1-form on the torus which is not exact. Let $j\colon T^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$ be the canonical injection map. The form

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$$

on $\mathbb{R}^3 \setminus \{0\}$ is closed. Since $\mathrm{d} \circ j^* = j^* \circ \mathrm{d}$, the form $j^*\omega$ on T^2 is also closed. To see that ω is not exact, by Stokes' Theorem, we only have to see that there exists a closed curve γ on the torus such that $\int_\gamma j^*\omega \neq 0$. In fact, let γ be the parallel obtained taking $\theta = 0$ in the parametric equations above of the torus. Hence,

$$\int_\gamma j^*\omega = \int_0^{2\pi} \mathrm{d}\varphi = 2\pi \neq 0.$$

Problem 3.25 Let $z^0, \dots, z^n$ be homogeneous coordinates on the complex projective space $\mathbb{C}\mathrm{P}^n$, and let U_α be the open subset defined by $z^\alpha \neq 0$, $\alpha = 0, \dots, n$. Let us fix two indices $0 \leqslant \alpha < \beta \leqslant n$. Set $u^j = z^j/z^\alpha$ on U_α, $v^j = z^j/z^\beta$ on U_β.

We define two differential 2-forms ω_α on U_α and ω_β on U_β, by setting

$$\omega_\alpha = \frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j}{\varphi} - \frac{\sum_{j,k} u^j \bar{u}^k\, \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j}{\varphi^2}\right),$$

$$\omega_\beta = \frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}v^j \wedge \mathrm{d}\bar{v}^j}{\psi} - \frac{\sum_{j,k} v^j \bar{v}^k\, \mathrm{d}v^k \wedge \mathrm{d}\bar{v}^j}{\psi^2}\right),$$

where $\varphi = \sum_{j=0}^n u^j \bar{u}^j$, $\psi = \sum_{j=0}^n v^j \bar{v}^j$. Prove:

(i) $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$.
(ii) There exists a unique differential 2-form ω on $\mathbb{C}\mathrm{P}^n$ such that $\omega|_{U_\alpha} = \omega_\alpha$, for all $\alpha = 0, \dots, n$.
(iii) $\mathrm{d}\omega = 0$.
(iv) $\omega \wedge \overset{(n)}{\cdots} \wedge \omega$ is a volume form.
(v) ω is not exact.

Remark Let $a = [\omega]$ be the (real) cohomology class of ω. It can be proved that a generates the real cohomology ring of $\mathbb{C}\mathrm{P}^n$; specifically, that $H^*_{dR}(\mathbb{C}\mathrm{P}^n, \mathbb{R}) \cong \mathbb{R}[a]/(a^{n+1})$.

Solution

(i) On $U_\alpha \cap U_\beta$ one has $v^j = u^j/u^\beta$, and hence $\varphi = \psi\, u^\beta \bar{u}^\beta$. We have

$$\begin{aligned}
\mathrm{d}v^k \wedge \mathrm{d}\bar{v}^j &= \frac{u^\beta\,\mathrm{d}u^k - u^k\,\mathrm{d}u^\beta}{(u^\beta)^2} \wedge \frac{\bar{u}^\beta\,\mathrm{d}\bar{u}^j - \bar{u}^j\,\mathrm{d}\bar{u}^\beta}{(\bar{u}^\beta)^2} \\
&= \frac{1}{(u^\beta)^2(\bar{u}^\beta)^2}\big(u^\beta\bar{u}^\beta\,\mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j - u^\beta\bar{u}^j\,\mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta \\
&\qquad - u^k\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^k\bar{u}^j\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\big),
\end{aligned}$$

and substituting into the expression of ω_β, on $U_\alpha \cap U_\beta$ we obtain

$$\begin{aligned}
\mathrm{i}\omega_\beta = {}& \frac{u^\beta\bar{u}^\beta}{\varphi}\sum_j \frac{1}{(u^\beta)^2(\bar{u}^\beta)^2}\big(u^\beta\bar{u}^\beta\,\mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j - u^\beta\bar{u}^j\,\mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta \\
& - u^j\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^j\bar{u}^j\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\big) \\
& - \frac{(u^\beta)^2(\bar{u}^\beta)^2}{\varphi^2}\sum_{j,k}\frac{u^j\bar{u}^k}{u^\beta\bar{u}^\beta}\frac{1}{(u^\beta)^2(\bar{u}^\beta)^2}\big(u^\beta\bar{u}^\beta\,\mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j \\
& - u^\beta\bar{u}^j\,\mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta - u^k\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^k\bar{u}^j\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\big).
\end{aligned}$$

Since the sum of the first and fifth summands above is $\mathrm{i}\omega_\alpha$, and moreover, the fourth and eighth summands are easily seen to cancel, we have

$$\begin{aligned}
\mathrm{i}\omega_\beta = {}& \mathrm{i}\omega_\alpha \\
& - \frac{1}{\varphi u^\beta\bar{u}^\beta}\sum_j\big(u^\beta\bar{u}^j\,\mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta + u^j\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j\big) \\
& + \frac{1}{\varphi^2 u^\beta\bar{u}^\beta}\sum_{j,k}u^j\bar{u}^k\big(u^\beta\bar{u}^j\,\mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta + u^k\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j\big). \qquad (\star)
\end{aligned}$$

Consider the last summand. Interchanging the indices j and k, since $\sum_j u^j\bar{u}^j = \sum_k u^k\bar{u}^k = \varphi$, this summand can be written as

$$\frac{1}{\varphi^2 u^\beta\bar{u}^\beta}\varphi\left(\sum_j \bar{u}^j u^\beta\,\mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta + \sum_k u^k\bar{u}^\beta\,\mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^k\right),$$

which is the opposite to the second summand in ($\star$). Hence $\omega_\beta = \omega_\alpha$ on $U_\alpha \cap U_\beta$.

(ii) Because of (i), we only need to prove that ω_α takes real values. In fact,

$$\bar{\omega}_\alpha = -\frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}\bar{u}^j \wedge \mathrm{d}u^j}{\varphi} - \frac{\sum_{j,k}\bar{u}^j u^k\,\mathrm{d}\bar{u}^k \wedge \mathrm{d}u^j}{\varphi^2}\right),$$

and permuting the indices j and k in the second summand, we obtain $\bar{\omega}_\alpha = \omega_\alpha$.

(iii) On U_α we easily get

$$\begin{aligned}
\mathrm{i}\,\mathrm{d}\omega &= \mathrm{i}\,\mathrm{d}\omega_\alpha \\
&= -\sum_{j,k}\frac{1}{\varphi^2}\big(u^k\,\mathrm{d}u^j\wedge \mathrm{d}\bar u^j\wedge \mathrm{d}\bar u^k + \bar u^k\,\mathrm{d}u^j\wedge \mathrm{d}\bar u^j\wedge \mathrm{d}u^k\big) \\
&\quad -\sum_{j,k}\frac{1}{\varphi^2}\big(\bar u^k\,\mathrm{d}u^j\wedge \mathrm{d}u^k\wedge \mathrm{d}\bar u^j + u^j\,\mathrm{d}\bar u^k\wedge \mathrm{d}u^k\wedge \mathrm{d}\bar u^j\big) \\
&\quad +\frac{2}{\varphi^3}\,\mathrm{d}\varphi\wedge\sum_{j,k}u^j\bar u^k\,\mathrm{d}u^k\wedge \mathrm{d}\bar u^j. \qquad (\star\star)
\end{aligned}$$

The first two summands at the right-hand side of ($\star\star$) cancel. The third summand vanishes, as

$$\mathrm{d}\varphi = \sum_h\big(u^h\,\mathrm{d}\bar u^h + \bar u^h\,\mathrm{d}u^h\big)$$

yields

$$\begin{aligned}
\mathrm{d}\varphi\wedge\sum_{j,k}u^j\bar u^k\,\mathrm{d}u^k\wedge \mathrm{d}\bar u^j &= -\mathrm{d}\varphi\wedge\sum_{j,k}\big(u^j\,\mathrm{d}\bar u^j\big)\wedge\big(\bar u^k\,\mathrm{d}u^k\big) \\
&= -\mathrm{d}\varphi\wedge\Big(\sum_j u^j\,\mathrm{d}\bar u^j\Big)\wedge\Big(\mathrm{d}\varphi-\sum_j u^j\,\mathrm{d}\bar u^j\Big).
\end{aligned}$$

(iv) As in (iii), we have

$$\sum_{j,k}u^j\bar u^k\,\mathrm{d}u^k\wedge \mathrm{d}\bar u^j = \sum_k \bar u^k\,\mathrm{d}u^k\wedge\sum_j u^j\,\mathrm{d}\bar u^j.$$

Set

$$\nu = \frac{1}{\varphi}\sum_j \bar u^j\,\mathrm{d}u^j, \qquad \mu = \frac{1}{\varphi}\sum_j \mathrm{d}u^j\wedge \mathrm{d}\bar u^j.$$

Then $\mathrm{i}\omega = \mu - \nu\wedge\bar\nu$. Thus

$$\mathrm{i}^n\omega^n = \mu^n - \binom{n}{1}\mu^{n-1}\wedge\nu\wedge\bar\nu.$$

Now,

$$\mu^n = \frac{n!}{\varphi^n}\,\mathrm{d}u^1\wedge \mathrm{d}\bar u^1\wedge\cdots\wedge \mathrm{d}u^n\wedge \mathrm{d}\bar u^n$$

(we suppose that $\alpha = 0$, so that only the coordinates $u^1, \bar{u}^1, \dots, u^n, \bar{u}^n$ are effective), and

$$\mu^{n-1} = \frac{(n-1)!}{\varphi^{n-1}} \sum_{k=1}^{n} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \widehat{\mathrm{d}u^k} \wedge \widehat{\mathrm{d}\bar{u}^k} \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n.$$

Hence

$$\begin{aligned} n\mu^{n-1} \wedge \nu \wedge \bar{\nu} &= \frac{n!}{\varphi^{n+1}} \sum_{k=1}^{n} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \\ &\qquad \wedge \widehat{\mathrm{d}u^k} \wedge \widehat{\mathrm{d}\bar{u}^k} \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n \wedge \bar{u}^k\,\mathrm{d}u^k \wedge u^k\,\mathrm{d}\bar{u}^k \\ &= \frac{n!}{\varphi^{n+1}} \sum_{k=1}^{n} u^k \bar{u}^k\,\mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n \\ &= \frac{n!(\varphi - 1)}{\varphi^{n+1}}\,\mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n, \end{aligned}$$

for

$$\sum_{k=0}^{n} u^k \bar{u}^k = u^0 \bar{u}^0 + \sum_{k=1}^{n} u^k \bar{u}^k = 1 + \sum_{k=1}^{n} u^k \bar{u}^k = \varphi.$$

Thus

$$\mathrm{i}^n \omega^n = \frac{n!}{\varphi^{n+1}}\,\mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n,$$

which does not vanish on U_0, hence on $\mathbb{C}\mathrm{P}^n$, as the same argument holds for any $\alpha = 0, \dots, n$.

(v) Immediate from (iv) and Stokes' Theorem.

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Chapter 4
Lie Groups

Abstract After giving some definitions and results on Lie groups, Lie algebras and homogeneous spaces, this chapter discusses problems on these topics. First, some specific examples of Lie groups and Lie algebras are introduced. Then, we consider homomorphisms, Lie subgroups and Lie subalgebras, integration on Lie groups, the exponential map exp and its differential map $\exp_*$, the adjoint representation Ad and its differential map ad, and Lie groups of transformations. We include several problems with the idea of introducing the reader to basic topics on Lie groups and Lie algebras, such as the determination of all the 2-dimensional Lie algebras, and all (up to isomorphism) 1-dimensional Lie groups. We also present some calculations that justify the name 'exponential,' the computation of the dimensions of some classical groups, and some properties of the Heisenberg group are studied. Some miscellaneous topics are considered as well. Special mention should also be paid of the fact that, in the present edition, two new problems have been added in the section concerning the exponential map, where the simply connected Lie group corresponding to a given Lie algebra is obtained, and that the section devoted to the adjoint representation, contains six new problems concerning somewhat more specialized topics such as Weyl group, Cartan matrix, Dynkin diagrams, etc. Similarly, the section devoted to Lie groups of transformations, has been increased in ten new application problems in Symplectic Geometry, Hamiltonian Mechanics, and other related topics. We then switch to homogeneous spaces, bringing in various classical examples, as the sphere, the real Stiefel manifold, and the real Grassmannian and other not so usual examples.

> (…) '**Satz 2**. *Sind $\delta x = X_1(x)\delta t, \dots, \delta x = X_r(x)\delta t$ r unabhängige infinitesimale Transformationen eine r-gliedrigen Gruppe, so befriedigen die X paarwise Relationen der Form*:
>
> $$X_i \frac{\mathrm{d}X_k}{\mathrm{d}x} - X_k \frac{\mathrm{d}X_i}{\mathrm{d}x} = c_{ik1}X_1 + \cdots + c_{ikr}X_r,$$
>
> *wo die c_{iks} Konstanten sind*.' Dieser Satz zusammen mit den Formeln (2) genügt zur Bestimmung aller Transformationgruppen einer einfach ausgedehnten Mannigfaltigkeit.

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_4,

(...) Meine Untersuchungen über Transformationgruppen beabsichtigen zunčhst die allegemeine des folgenden (...) **Problem**. *Man soll alle r-gliedrigen Transformationgruppen einer n-fach ausgedehnten Mannigfaltigkeit bestimmen.*

(...) Durch Verfolgung dieser Bemerkung kam ich zu dem überraschenden Resultate, daß alle Transformationgruppen einer einfach ausgedehnten Mannigfaltigkeit durch Einführung von zweckmäßingen Variablen auf die lineare Form reduzierte werden können, wie auch, daß die Bestimmung aller Gruppen einer n-fach ausgedehnten Mannigfaltigkeit durch die Integration von gewöhnlicehn Differentialgelichungen geleistet werder kann. Diese Entdeckund, deren esrte Spuren auf ABEL and HELMHOLTZ zurückfüheren sind, is der Ausgangspunkt meiner vieljähringen Untersuchungen über Transformationsgruppen gewesen.[1]

SOPHUS LIE, "*Theorie der Transformationgruppen*," Math. Ann. *16* (1880), 441–528. Gessammelte Abhandlungen, Sechster Band, ss. 1–94, B.G. Teubner, Leipzig und H. Aschehoug & Co., Oslo, 1927. Translated by Michael Ackerman, Comments by Robert Hermann in *Sophus Lie 1880 Transformation Group Paper*, Math Sci Press, Brookline, Massachusetts, 1975. (With kind permission from Springer.)

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformations groups of geometric objects. For example, the group of all affine transformations of a connected manifold with an affine connection and the group of all isometries of a pseudo-Riemannian manifold are known to be Lie groups in the compact open topology. However, the group of all diffeomorphisms of a manifold is too big to form a Lie group in any reasonable topology (...). In the early days of Lie group theory, the late nineteenth century, the notion of a Lie group had, in the hands of S. Lie, W. Killing, and É. Cartan, a primarily local character (...) Global Lie groups were not emphasized until during the 1920's through the work of H. Weyl, É. Cartan, and O. Schreier. These two viewpoints, the infinitesimal method and the integral method, were not completely coordinated until É. Cartan proved in 1930 that every Lie algebra over $\mathbb{R}$ is the Lie algebra of a Lie group.

[1] "(...) '**Theorem** 3.2.1. *If* $\delta x = X_1(x)\delta t, \dots, \delta x = X_r(x)\delta t$ *are* r *independent infinitesimal transformations of an* r*-term transformation group, then the* X*'s satisfy relations of the form* $X_i \mathrm{d}X_k/\mathrm{d}x - X_k \mathrm{d}X_i/\mathrm{d}x = c_{ik1}X_1 + \cdots + c_{ikr}X_r$, *where the* c_{iks} *are constants*.' This theorem, together with formulas 3.2.1, suffices for the determination of all transformation groups of a one-dimensional manifold. (...) My investigations of transformation groups are meant in the first place to settle the following (...) **Problem.** *To determine all* r*-term transformation groups of an* n*-dimensional manifold*. (...) I reached the astounding result that all transformation groups of a one-dimensional manifold can be reduced to linear form by introducing suitable variables and also that *all groups of an* n*-dimensional manifold can be determined by integrating ordinary differential equations*. This discovery, whose first traces back to ABEL and HELMHOLTZ, was the starting point of many years of my investigations on transformation groups."

SIGURDUR HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, 2001, pp. 87, 128. (With kind permission from the author.)

4.1 Some Definitions and Theorems on Lie Groups

In this section we denote by $\mathbb{F}$ the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Definitions 4.1 A *Lie group* G over $\mathbb{R}$ (resp. $\mathbb{C}$) is a differentiable (resp. complex) manifold endowed with a group structure such that the map $G \times G \to G$, $(s, t) \mapsto st^{-1}$, is C^∞ (resp. holomorphic).

Let G and H be Lie groups. A map $\Phi\colon G \to H$ is a *homomorphism of Lie groups* if it is a group homomorphism and a C^∞ (resp. holomorphic) map of differentiable (resp. complex) manifolds. Φ is said to be an *isomorphism* if it is moreover a diffeomorphism (resp. a biholomorphic map).

Let G and H be two Lie groups and consider a homomorphism of H into the abstract group of automorphisms of G, $\Psi\colon H \to \operatorname{Aut} G$. The *semi-direct product* $G \rtimes_\Psi H$ of G and H with respect to Ψ is the product manifold $G \times H$, endowed with the Lie group structure given (denoting $\Psi(h)$ by Ψ_h) by

$$(g, h)\bigl(g', h'\bigr) = \bigl(g\Psi_h\bigl(g'\bigr), hh'\bigr), \qquad (g, h)^{-1} = \bigl(\Psi_{h^{-1}}\bigl(g^{-1}\bigr), h^{-1}\bigr),$$

for $g, g' \in G$, $h, h' \in H$.

A *Lie algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ together with a bilinear operator $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the *bracket*) such that for all $X, Y, Z \in \mathfrak{g}$:

(i) $[X, Y] = -[Y, X]$ (anti-commutativity).
(ii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras* if it is linear and preserves brackets. φ is said to be an *isomorphism* if it is moreover one-to-one and surjective.

An endomorphism $D\colon \mathfrak{g} \to \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is called a *derivation* if

$$D\bigl([X, Y]\bigr) = [DX, Y] + [X, DY] \quad \text{for all } X, Y \in \mathfrak{g}.$$

The set of derivations of $\mathfrak{g}$ is denoted by $\operatorname{Der} \mathfrak{g}$.

Definition 4.2 Let $\operatorname{End}(n, \mathbb{F})$ be the vector space of all $n \times n$ matrices over $\mathbb{F}$. Let $\mathrm{GL}(n, \mathbb{F})$ be the set of all invertible elements of $\operatorname{End}(n, \mathbb{F})$. The subset $\mathrm{GL}(n, \mathbb{F})$ is open in $\operatorname{End}(n, \mathbb{F})$, and we regard it as an open submanifold of $\operatorname{End}(n, \mathbb{F})$. It is clear that under matrix multiplication $\mathrm{GL}(n, \mathbb{F})$ becomes a (real or complex) Lie group.

An endomorphism ϕ_V of a vector space V over $\mathbb{F}$ is called *semi-simple* if every invariant subspace of V admits a complementary invariant subspace of V or, equivalently, if V is a direct sum of ϕ_V-irreducible subspaces. If $\mathbb{F} = \mathbb{C}$, such irreducible subspaces are one-dimensional.

The *Lie algebra of the Lie group G* is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. There exists an isomorphism of vector spaces

$$\mathfrak{g} \to T_e G, \qquad X \mapsto X_e.$$

In other words, a left-invariant vector field is completely determined by its value at the identity. Using this isomorphism, we can identify the tangent space $T_e G$ with the Lie algebra $\mathfrak{g}$ of G.

For any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra of G, there exists a unique connected (not necessarily closed) subgroup H of G with Lie algebra $\mathfrak{h}$, i.e. H is a Lie group, and as a subset of G it is the image of some homomorphism (which is a natural immersion) $H \hookrightarrow G$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and let $\varphi\colon \mathfrak{h} \to \operatorname{End}\mathfrak{g}$ be a homomorphism such that every operator $\varphi(Y)$, $Y \in \mathfrak{h}$, is a derivation of $\mathfrak{g}$. The *semi-direct product* $\mathfrak{g} \rtimes_\varphi \mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$ with respect to φ is the direct sum vector space $\mathfrak{g} \oplus \mathfrak{h}$, endowed with the Lie algebra structure given by the bracket

$$\big[(X,Y),\big(X',Y'\big)\big] = \big(\big[X,X'\big] + \varphi(Y)X' - \varphi\big(Y'\big)X, \big[Y,Y'\big]\big) \tag{4.1}$$

for $X, X' \in \mathfrak{g}$, $Y, Y' \in \mathfrak{h}$.

Proposition 4.3 *Let G, H be simply connected Lie groups, and let $\mathfrak{g}, \mathfrak{h}$ be their respective Lie algebras. Then, given a homomorphism*

$$\psi : \mathfrak{h} \longrightarrow \operatorname{Der}\mathfrak{g},$$

there exists a unique map

$$\Psi : H \longrightarrow \operatorname{Aut} G$$

such that:

(i) $\Psi(h_1 h_2) = \Psi(h_1)\Psi(h_2)$.
(ii) *Denoting $\Psi(h)$ by Ψ_h, the map*

$$\begin{aligned} \Psi_h : G \times H &\longrightarrow G \\ (g,h) &\longmapsto \Psi_h(g) \end{aligned}$$

is C^∞, and, $g \rtimes_\psi \mathfrak{h}$ being the semi-direct product of $\mathfrak{g}$ and $\mathfrak{h}$ with respect to ψ, it is the Lie algebra of the Lie group $G \rtimes_\Psi H$.

Theorem 4.4 (Cartan's Criterion for Closed Subgroups) *Let G be a Lie group, and let H be a closed abstract subgroup of G. Then H has a unique manifold structure that makes H into a Lie subgroup of G.*

Definitions 4.5 A Lie group G acts on itself on the left by *inner automorphisms*, that is, automorphisms ι defined by

$$\iota\colon G \times G \to G, \qquad \iota(s,t) = sts^{-1}.$$

Letting $\iota_s(t) = \iota(s,t)$, the map $s \mapsto \iota_{*s}|_{T_eG}$ is, under the identification of the vector space T_eG with the Lie algebra $\mathfrak{g}$ of G, a homomorphism of G into the group of automorphisms $\operatorname{Aut}\mathfrak{g}$ of the vector space $\mathfrak{g}$, called the *adjoint representation of* G and denoted by

$$\mathrm{Ad}\colon G \to \operatorname{Aut}\mathfrak{g}.$$

The differential map of Ad, denoted by ad, is a homomorphism of $\mathfrak{g}$ into the Lie algebra $\operatorname{End}\mathfrak{g}$ of endomorphisms of the vector space $\mathfrak{g}$, called the *adjoint representation of the Lie algebra* $\mathfrak{g}$. One has

$$\mathrm{ad}_X Y = [X, Y], \quad X, Y \in \mathfrak{g}.$$

By the Jacobi identity each endomorphism ad_X, $X \in \mathfrak{g}$, is a derivation of $\mathfrak{g}$.

A bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ on a Lie algebra $\mathfrak{g}$ is called *invariant* if

$$B\big([X,Y], Z\big) = -B\big(Y, [X,Z]\big), \quad X, Y, Z \in \mathfrak{g}.$$

The connected subgroup $\operatorname{Int}\mathfrak{g}$ (not necessarily closed) of the Lie group $\operatorname{Aut}\mathfrak{g}$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}$ is called the *group of inner automorphisms* of $\mathfrak{g}$. Each invariant form B on $\mathfrak{g}$ is invariant with respect to the group $\operatorname{Int}\mathfrak{g}$.

The bilinear form

$$B_{\mathfrak{g}}(X,Y) = \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \quad X, Y \in \mathfrak{g},$$

is called the *Killing form* of the Lie algebra $\mathfrak{g}$. The Killing form is invariant with respect the group of automorphisms $\operatorname{Aut}\mathfrak{g}$ of $\mathfrak{g}$, i.e.

$$B_{\mathfrak{g}}\big(a(X), a(Y)\big) = B_{\mathfrak{g}}(X,Y), \quad X, Y \in \mathfrak{g},\ a \in \operatorname{Aut}\mathfrak{g}.$$

Each derivation of the Lie algebra $\mathfrak{g}$, in particular each operator ad_X, is skew-symmetric with respect to $B_{\mathfrak{g}}$: $B_{\mathfrak{g}}([X,Y],Z) = -B_{\mathfrak{g}}(Y,[X,Z])$, $X, Y, Z \in \mathfrak{g}$. In other words, the Killing form is an invariant bilinear form on $\mathfrak{g}$.

Since $\mathrm{GL}(n,\mathbb{F})$ is an open subset of $\mathrm{End}(n,\mathbb{F})$, the tangent space to the Lie group $\mathrm{GL}(n,\mathbb{F})$ at the identity element is identified naturally with the space $\mathrm{End}(n,\mathbb{F})$. It is clear that for any $g \in \mathrm{GL}(n,\mathbb{F})$, the linear map

$$\mathrm{End}(n,\mathbb{F}) \to \mathrm{End}(n,\mathbb{F}), \quad X \mapsto gXg^{-1},$$

is the adjoint operator Ad_g and determines the adjoint representation of $\mathrm{GL}(n,\mathbb{F})$. Therefore its differential map ad in this case has the following form:

$$\mathrm{ad}_X Y = XY - YX, \quad X, Y \in \mathrm{End}(n,\mathbb{F}).$$

In other words, the Lie algebra of $\mathrm{GL}(n,\mathbb{F})$ is the set of $n \times n$ matrices over $\mathbb{F}$ with bracket defined by

$$[X,Y] = XY - YX.$$

Moreover, the Lie algebra of an arbitrary Lie subgroup of $\mathrm{GL}(n,\mathbb{F})$ can be considered as a subalgebra of the Lie algebra $\mathrm{End}(n,\mathbb{F})$ with this bracket.

A real Lie algebra $\mathfrak{a}$ is called a *real form* of a complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g}$ is (isomorphic to) the complexification $\mathfrak{a}^{\mathbb{C}}$ of $\mathfrak{a}$. For instance, the Lie algebras $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{su}(n)$ are real forms of the complex Lie algebra $\mathfrak{sl}(n,\mathbb{C})$.

Definitions 4.6 Given a Lie algebra $\mathfrak{g}$, one defines its *derived algebra* by $\mathscr{D}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$, and one considers $\mathscr{D}^{k+1}(\mathfrak{g}) = \mathscr{D}(\mathscr{D}^k(\mathfrak{g}))$ for $k = 1,2,\dots$. Then $\mathfrak{g}$ is called *solvable* if there exists an integer $k \geqslant 1$ such that $\mathscr{D}^k(\mathfrak{g}) = 0$.

A Lie group over $\mathbb{F}$ is said to be *solvable* if its Lie algebra so is.

A solvable Lie algebra over $\mathbb{R}$ is said to be *completely solvable* (or *split solvable*) if all the eigenvalues of the adjoint representation belong to $\mathbb{R}$. A solvable Lie group over $\mathbb{R}$ is said completely solvable (or split solvable) if its Lie algebra so is. A completely solvable Lie algebra over $\mathbb{R}$ is also called *real solvable*.

A Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is said to be *nilpotent* if for each $X \in \mathfrak{g}$, ad_X is a nilpotent endomorphism of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a nilpotent subalgebra of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, and let $\lambda \in \mathfrak{h}^*$ be any linear function on $\mathfrak{h}$. The *generalised weight space* $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda(\mathfrak{h})$ of $\mathfrak{g}$ relative to $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ is defined as

$$\mathfrak{g}^\lambda = \big\{X \in \mathfrak{g} : \big(\mathrm{ad}_h - \lambda(h)I\big)^n(X) = 0\ \forall h \in \mathfrak{h} \text{ and some positive integer } n = n(h)\big\}.$$

It is evident that the *weight space* $\mathfrak{g}_\lambda \subset \mathfrak{g}$, where

$$\mathfrak{g}_\lambda = \big\{X \in \mathfrak{g} : [h, X] = \lambda(h)X,\ \ \forall h \in \mathfrak{h}\big\}$$

is a subspace of $\mathfrak{g}^\lambda$.

Proposition 4.7 *If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ a nilpotent Lie subalgebra, then the generalised weight spaces of $\mathfrak{g}$ relative to $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$ satisfy:*

(i) $\mathfrak{h} \subseteq \mathfrak{g}^0$.
(ii) $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subseteq \mathfrak{g}^{\lambda+\mu}$ *(with $\mathfrak{g}^{\lambda+\mu}$ understood to be zero if $\lambda+\mu$ is not a generalised weight) for any $\lambda, \mu \in \mathfrak{h}^*$.*
(iii) *If B is an invariant form on $\mathfrak{g}$, then the spaces $\mathfrak{g}^\lambda$ and $\mathfrak{g}^\mu$ such that $\lambda+\mu \neq 0$ are orthogonal with respect to B.*
(iv) *If an invariant form B on $\mathfrak{g}$ is non-degenerate, then its restriction to $\mathfrak{g}^\lambda \times \mathfrak{g}^{-\lambda}$, $\lambda \in \mathfrak{h}^*$, is non-degenerate, and, in particular, the restriction of B to $\mathfrak{g}^0 \times \mathfrak{g}^0$ is non-degenerate.*
(v) *If, in addition, $\mathbb{F} = \mathbb{C}$, then $\mathfrak{g} = \bigoplus_{\lambda\in\mathfrak{h}^*} \mathfrak{g}^\lambda$.*

In particular, the subspace $\mathfrak{g}^0(\mathfrak{h}) \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ over $\mathbb{F}$.

Definition 4.8 A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is called a *Cartan subalgebra* if $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$.

Proposition 4.9 *A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is a Cartan subalgebra if and only if $\mathfrak{h}$ coincides with its normaliser*:

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} : [X, \mathfrak{h}] \subseteq \mathfrak{h}\}.$$

Definitions 4.10 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$, and let ρ be a representation of $\mathfrak{g}$ on a vector space V. One can regard each $X \in \mathfrak{g}$ as generating a one-dimensional Abelian subalgebra, and we can form the generalised eigenspace

$$V^{0,X} = \{Y \in \mathfrak{g} : \rho(X)^n(Y) = 0 \text{ for some } n = n(\rho, X)\}$$

for the eigenvalue 0 under $\rho(X)$. Let

$$\ell_{\mathfrak{g}}(V) = \min_{X \in \mathfrak{g}} \dim V^{0,X}, \qquad R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} : \dim V^{0,X} = \ell_{\mathfrak{g}}(V)\}.$$

These are related to the characteristic polynomial

$$\det\bigl(tI - \rho(X)\bigr) = t^n - \sum_{i=0}^{n-1} d_i(X) t^i.$$

In any basis of $\mathfrak{g}$, the $d_i(X)$ are polynomial functions in $\mathfrak{g}$, as one can see by expanding $\det(tI - \sum_j x_j \rho(X_j))$. For a given X, if i is the smallest value for which $d_i(X) \neq 0$, then $i = \dim V^{0,X}$, since the degree of the last term of the characteristic polynomial is the multiplicity of 0 as a generalised eigenvalue of $\rho(X)$. Thus, $\ell_{\mathfrak{g}}(V)$ is the minimum value i such that $d_i \not\equiv 0$, and

$$R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} : d_{\ell_{\mathfrak{g}}(V)}(X) \neq 0\}.$$

Let ρ be the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$. The elements of $R_{\mathfrak{g}}(\mathfrak{g})$, relative to the adjoint representation, are called the *regular elements* of $\mathfrak{g}$. By definition, $X \in R_{\mathfrak{g}}(\mathfrak{g})$ if and only if $\dim \mathfrak{g}^{0,X} = \ell_{\mathfrak{g}}(\mathfrak{g})$.

We have the following:

Theorem 4.11 *If X is a regular element of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, then the Lie algebra $\mathfrak{g}^{0,X}$ (of dimension $\ell_{\mathfrak{g}}(\mathfrak{g})$) is a Cartan subalgebra of $\mathfrak{g}$ (this implies in particular that the set of Cartan subalgebras is not empty). Each Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of dimension $\ell_{\mathfrak{g}}(\mathfrak{g})$ has the form $\mathfrak{h} = \mathfrak{g}^{0,X}$ for some $X \in R_{\mathfrak{g}}(\mathfrak{g})$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then* $\dim \mathfrak{h} \geqslant \ell_{\mathfrak{g}}(\mathfrak{g})$.

Definition 4.12 A Lie algebra (over $\mathbb{F}$) is said to be *simple* if it is of dimension greater than one and has no proper ideals. A Lie algebra is *semi-simple* if it has no non-zero Abelian ideals. A Lie group G is semi-simple if its Lie algebra is semi-simple.

A Lie algebra is semi-simple if and only if it is the direct sum of simple Lie algebras.

Theorem 4.13 (Cartan's Criterion for Semisimplicity) *A Lie algebra is semisimple if and only its Killing form is non-degenerate.*

In the case of a complex semi-simple Lie algebra, we have the following:

Theorem 4.14 *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Then all Cartan subalgebras of $\mathfrak{g}$ are conjugate with respect to the group of inner automorphisms* $\mathrm{Int}\,\mathfrak{g}$ *of $\mathfrak{g}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is an Abelian Lie algebra, and for each $\lambda \in \mathfrak{h}^*$, we have $\mathfrak{g}^\lambda(\mathfrak{h}) = \mathfrak{g}_\lambda(\mathfrak{h})$, i.e. each endomorphism* ad_X, *$X \in \mathfrak{h}$, is semi-simple.*

In particular, each non-zero space $\mathfrak{g}^\lambda$ consists of eigenvectors of the operators ad_h, $h \in \mathfrak{h}$, with eigenvalues $\lambda(h)$, and in $\mathfrak{g}$ there exists a common eigenvector basis for all ad_h, $h \in \mathfrak{h}$.

Definitions 4.15 A *torus* is a complex Lie group T isomorphic to $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (n times). The integer n is called the *rank* of T. If G is a complex Lie group, then a torus $T \subset G$ is *maximal* if it is not contained in any larger torus of G. The *rank* of a complex Lie group G is defined as the rank of any maximal torus.

In a semi-simple complex connected Lie group G, all maximal toruses are conjugate with respect to the group of inner automorphisms $\mathrm{Ad}(G)$ of G, and a connected subgroup $H \subset G$ is a maximal torus if and only if its Lie algebra $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. In particular, each Cartan subalgebra of $\mathfrak{g}$ is the Lie algebra of some maximal torus.

Definitions 4.16 Let G be a complex semi-simple Lie group, let H be a maximal torus, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras ($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$). Let $\mathfrak{h}^*$ be the space dual to $\mathfrak{h}$. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is said to be a *root* and $\mathfrak{g}_\alpha$ is said to be a *root space* (of $\mathfrak{g}$ with respect to $\mathfrak{h}$). If α is a root, then a non-zero element of $\mathfrak{g}_\alpha$ is said to be a *root vector* for α. The set Δ of roots is said to be the *root system* of $\mathfrak{g}$. It depends on a choice of Cartan subalgebra (maximal torus), so one writes $\Delta(\mathfrak{g}, \mathfrak{h})$ to make the choice explicit.

Since by Proposition 4.7 the Killing form $B_{\mathfrak{g}}$ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$, it defines a bilinear form on $\mathfrak{h}^*$ that we denote by $\langle \alpha, \beta \rangle$. Let $\alpha \in \Delta$ and $\beta' \in (\Delta \cup \{0\})$. The *$\alpha$-series containing β'* is by definition the set of all elements of the set $\Delta \cup \{0\}$ of the form $\beta' + n\alpha$, where n is an integer.

Then we have the following:

Theorem 4.17 *The roots and root spaces of a complex semi-simple Lie algebra satisfy the following properties*:

(i) *Δ spans* $\mathfrak{h}^*$.

(ii) *If* $\alpha \in \Delta$, *then* $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$, *and there is a unique element* $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *such that* $\alpha(h) = B_\mathfrak{g}(h_\alpha, h)$ *for all* $h \in \mathfrak{h}$, *the real subspace* $\mathfrak{h}_\mathbb{R}$ *of* $\mathfrak{h}$ *generated by the vectors* h_α, $\alpha \in \Delta$, *has real dimension* $\dim_\mathbb{C} \mathfrak{h}$, *and it is a real form of* $\mathfrak{h}$. *Moreover, the Killing form* $B_\mathfrak{g}$ *is positive definite on* $\mathfrak{h}_\mathbb{R}$, *and*

$$[Y_\alpha, Y_{-\alpha}] = B_\mathfrak{g}(Y_\alpha, Y_{-\alpha})h_\alpha, \quad Y_\alpha \in \mathfrak{g}_\alpha,\ Y_{-\alpha} \in \mathfrak{g}_{-\alpha}.$$

(iii) *The quotient* $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ *is an integer for all* $\alpha, \beta \in \Delta$.

(iv) *Δ spans the real subspace* $\mathfrak{h}^*_\mathbb{R}$ *of* $\mathfrak{h}^*$, *in particular* $\alpha(h_\beta) \in \mathbb{R}$ *for all* $\alpha, \beta \in \Delta$.

(v) *The orthogonal transformations*

$$s_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}\alpha, \quad \beta \in \Delta,$$

carry Δ *to itself.*

(vi) *For* $\alpha \in \Delta$ *and* $\beta' \in (\Delta \cup \{0\})$, *the* α*-series containing* β' *has the form* $\beta' + n\alpha$ *with* $p \leqslant n \leqslant q$ *(it is an uninterrupted string* $\beta' + p_{\alpha,\beta}\alpha, \dots, \beta' + 0\alpha, \dots, \beta' + q_{\alpha,\beta}\alpha$*) such that* $p + q = -2\frac{\langle\beta',\alpha\rangle}{\langle\alpha,\alpha\rangle}$.

(vii) *If* $\alpha, \beta \in \Delta$ *and* $\alpha + \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(viii) *If* $Y_\alpha \in \mathfrak{g}_\alpha$, $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ *and* $Y_\beta \in \mathfrak{g}_\beta$ *with* $\alpha, \beta \in \Delta$, *then*

$$\big[Y_{-\alpha}, [Y_\alpha, Y_\beta]\big] = \frac{q_{\alpha,\beta}(1 - p_{\alpha,\beta})}{2}\alpha(h_\alpha)B_\mathfrak{g}(Y_{-\alpha}, Y_\alpha)Y_\beta,$$

where $\beta + n\alpha$, $p_{\alpha,\beta} \leqslant n \leqslant q_{\alpha,\beta}$, *is the* α*-series in* Δ *containing* β.

Note that by definition $\langle\alpha, \beta\rangle = B_\mathfrak{g}(h_\alpha, h_\beta)$ for $\alpha, \beta \in \Delta$.

Definition 4.18 An *abstract root system* in a finite-dimensional real vector space V with inner product $\langle\,,\rangle$ and squared norm $|\cdot|^2$ is a finite subset Δ of non-zero elements such that:

(i) Δ spans V.

(ii) The orthogonal transformations

$$s_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}\alpha, \quad \beta \in \Delta,$$

carry Δ to itself.

(iii) The quotient $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ is an integer $\forall\alpha, \beta \in \Delta$.

If $\alpha, \beta \in \Delta$ are proportional, $\beta = a\alpha$ ($a \in \mathbb{R}$), then $a = \pm\frac{1}{2}, \pm1, \pm2$. A root system Δ is said to be *reduced* if $\alpha, \beta \in \Delta$, $\beta = a\alpha$ implies $a = \pm1$.

It is evident that a root system $\Delta(\mathfrak{g}, \mathfrak{h})$ of a complex semi-simple Lie algebra $\mathfrak{g}$ is a reduced abstract root system.

Proposition 4.19 *Let Δ be a reduced root system in the inner product space V. Then*:

(i) *If $\alpha \in \Delta$, then $-\alpha \in \Delta$.*
(ii) *If $\alpha \in \Delta$, then the only members of $\Delta \cup \{0\}$ proportional to α are $0, \pm\alpha$.*
(iii) *If $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, then*

$$\frac{2\langle \beta, \alpha \rangle}{|\alpha|^2} = 0, \pm 1, \pm 2, \pm 3.$$

(iv) *If $\alpha \in \Delta$ and $\beta \in \Delta$ are such that $|\alpha| \leq |\beta|$, then*

$$\frac{2\langle \beta, \alpha \rangle}{|\beta|^2} = 0, \pm 1.$$

(v) *If $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta$ is a root or 0. If $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta \rangle < 0$, then $\alpha + \beta$ is a root or 0.*
(vi) *If $\alpha, \beta \in \Delta$ and neither $\alpha + \beta$ nor $\alpha - \beta$ belong to $\Delta \cup \{0\}$, then $\langle \alpha, \beta \rangle = 0$.*

Definitions 4.20 A *lexicographic ordering* on a vector space V, induced by the basis $\{v_1, \dots, v_n\}$ of V, is defined by putting $v > u$ if there is a (unique) integer k, with $1 \leqslant k \leqslant n$, such that $\langle v, v_k \rangle > \langle u, v_k \rangle$ and $\langle v, v_i \rangle = \langle u, v_i \rangle$ for all $i < k$. We say that an element v is positive if $v > 0$. The set of all *positive roots* of Δ (with respect to the above ordering) is denoted by Δ^+. It is clear that $\Delta = -\Delta$ since $s_\alpha(\alpha) = -\alpha$, and therefore $\Delta = \Delta^+ \cup -\Delta^+$ and $\Delta^+ \cap -\Delta^+ = \emptyset$.

A root α, i.e. a vector α in Δ, is said to be *simple* if it is positive, but not a sum of two positive roots. (Note that this definition and all the following developments depend on the chosen ordering.) Let $\Pi \subset \Delta^+$ be the set of all simple roots in Δ; this is called the *simple root system* of Δ. We have some elementary but basic properties of Π:

(i) $\Pi = \{\alpha_1, \dots, \alpha_n\}$ contains $n = \dim V$ linearly independent roots, i.e. Π is a basis of V.
(ii) If $\beta \in \Delta^+$, then $\beta = m_1\alpha_1 + \cdots + m_r\alpha_r$ where m_i are nonnegative integers.

It is evident that any subset $\Pi \subset \Delta$ satisfying conditions (i) and (ii) defines a lexicographic ordering on V. With respect to this ordering, Π is a simple root system.

Definitions 4.21 Let Δ be a reduced root system in a vector space V of dimension n. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots in Δ (determined by some lexicographic ordering). The *Cartan matrix* of Δ and Π is the $n \times n$-matrix C with entries

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}.$$

The *Dynkin diagram* of a set Π of simple roots is the graph obtained associating to each simple root α_i the vertex $\circ$ of a graph, attaching to that vertex a weight proportional to $|\alpha_i|^2$ and connecting each two such vertices corresponding to different simple roots α_i and α_j by $C_{ij}C_{ji}$ (no sum, and the possible values are $0, 1, 2, 3$) edges. When there is one edge, the roots have the same length, but if there are two or three edges, one often adds an arrow pointing from the longer to the shorter root.

Definition 4.22 Let G be a connected semi-simple complex Lie group, and let $H \subset G$ be a maximal torus. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the corresponding Lie algebras. The *Weyl group* W of G is the group of automorphisms of H (or, equivalently, of the Cartan subalgebra $\mathfrak{h}$) that are restrictions of inner automorphisms of G preserving H (of inner automorphisms of $\mathfrak{g}$ preserving $\mathfrak{h}$).

Since all maximal toruses of G are conjugate with respect to $\mathrm{Ad}(G)$ and all Cartan subalgebras of $\mathfrak{g}$ are conjugate with respect to $\mathrm{Int}\,\mathfrak{g}$, the Weyl group W does not depend on the choice of a maximal torus $H \subset G$ (or a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$). The real subspace $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ is invariant with respect of W, i.e. $s(h) \in \mathfrak{h}_{\mathbb{R}}$ for any $s \in W$ and $h \in \mathfrak{h}_{\mathbb{R}}$. The Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ as a group acting on $\mathfrak{h}_{\mathbb{R}}$ is generated by the reflections s_α defined by the roots $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.

A real Lie algebra $\mathfrak{a}$ is *compact* if there exists an invariant inner product on it, i.e. a symmetric bilinear positive (negative) definite form β on $\mathfrak{a}$ such that

$$\beta\big([X, Y], Z\big) = -\beta\big(Y, [X, Z]\big), \quad X, Y, Z \in \mathfrak{a}.$$

Theorem 4.23 *Let $\mathfrak{a}$ be a real Lie algebra. The following conditions are equivalent*:

(i) $\mathfrak{a}$ *is a compact Lie algebra.*
(ii) *The Lie group of inner automorphisms* $\mathrm{Int}\,\mathfrak{a}$ *of* $\mathfrak{a}$ *is compact.*
(iii) $\mathfrak{a}$ *is a direct sum of semi-simple and Abelian algebras*, *and for any* $X \in \mathfrak{a}$, *the endomorphism* ad_X *is semi-simple and has purely imaginary eigenvalues.*
(iv) *If* $\mathfrak{a}$ *is semi-simple*, *then the Killing form* $B_{\mathfrak{a}}$ *of* $\mathfrak{a}$ *is negative definite.*
(v) *If* $\mathfrak{a}$ *semi-simple*, *then* $\mathfrak{a}$ *is the Lie algebra of some compact Lie group.*

If a Lie algebra $\mathfrak{a}$ is compact and semi-simple, then any nilpotent subalgebra $\mathfrak{t}$ of $\mathfrak{a}$ is Abelian, and $\mathfrak{a}^0(\mathfrak{t}) = \mathfrak{a}_0(\mathfrak{t})$ because each endomorphism ad_X, $X \in \mathfrak{a}$, is semi-simple. Therefore each Cartan subalgebra of $\mathfrak{a}$ is the centraliser of some regular element of $\mathfrak{a}$, and it is a maximal Abelian subalgebra of $\mathfrak{a}$. All Cartan subalgebras of $\mathfrak{k}$ are conjugate with respect to the group of inner automorphisms $\mathrm{Int}\,\mathfrak{k}$.

Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$. Since a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ coincides with its normaliser in $\mathfrak{k}$, the connected Lie subgroup T of K with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$ is closed, and it is isomorphic to the real n-dimensional torus $S^1 \times \cdots \times S^1$, where $n = \dim \mathfrak{t}$. The group T is called a maximal torus of K.

Let G be a complex connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. There exists a compact real form $\mathfrak{k}$ of $\mathfrak{g}$. The corresponding (real) connected subgroup $K \subset G$ is closed in G, and, moreover, it is a maximal compact subgroup of G. All the compact real forms of $\mathfrak{g}$ are conjugate with respect to the group of inner automorphisms $\mathrm{Int}\,\mathfrak{g}$. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, then its complexification $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. Therefore the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$

of $\mathfrak{g}$ will also be called a *root system for the compact Lie algebra* $\mathfrak{k}$, and we will denote it by $\Delta = \Delta(\mathfrak{k}, \mathfrak{t})$.

Let T be a maximal torus of K with Lie algebra $\mathfrak{t}$. The *Weyl group* $W(\mathfrak{k}, \mathfrak{t})$ of K is the group of automorphisms of T (or, equivalently, of the Cartan subalgebra $\mathfrak{t}$) that are restrictions of inner automorphisms of K preserving T (of inner automorphisms of $\mathfrak{k}$ preserving $\mathfrak{t}$). If $\mathfrak{h}_{\mathbb{R}}$ is the space (real form of $\mathfrak{h}$) generated by the vectors $h_\alpha \in \mathfrak{h}$, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, then $\mathfrak{t} = \mathrm{i}\mathfrak{h}_{\mathbb{R}}$. Since the space $\mathfrak{h}_{\mathbb{R}}$ is invariant with respect to the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$, so is $\mathfrak{t} = \mathrm{i}\mathfrak{h}_{\mathbb{R}}$. Therefore the restriction map $s \mapsto s|_{\mathfrak{t}}$ of the group $W(\mathfrak{g}, \mathfrak{h})$ is well defined. This map induces an isomorphism of the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ onto the Weyl group $W(\mathfrak{k}, \mathfrak{t})$ of $\mathfrak{k}$. Using this map, we will identify the Weyl groups $W(\mathfrak{g}, \mathfrak{h})$ and $W(\mathfrak{k}, \mathfrak{t})$ and will denote them by the same symbol W.

We also have the following (cf. [2, Chap. V, §3, Lemma 2, (iii), Theorem 1, (iv); and Chap. VI, §1, Theorem 2, (vii)]):

Theorem 4.24 *Let K be a connected semi-simple compact Lie group, and let T be a maximal torus with Lie algebra $\mathfrak{t}$. The Weyl group W of K as a group of transformations of $\mathfrak{t}$ is generated by the transformations $s_{\alpha_i}|_{\mathfrak{t}}$, $i = 1, \dots, n$, where $\{\alpha_1, \dots, \alpha_n\}$ is the set Π of simple roots of the root system $\Delta = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \Delta(\mathfrak{k}, \mathfrak{t})$.*

If $s \in W$ and s is a reflection in $\mathfrak{t}$, then $s = s_\alpha|_{\mathfrak{t}}$ for some root $\alpha \in \Delta$.

We now give some more definitions.

Definitions 4.25 Suppose that $n = 2r$ is even. Let s_0 denote the $r \times r$ matrix

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

with 1 in the skew diagonal and 0 elsewhere. Set

$$J_+ = \begin{pmatrix} \mathbf{0} & s_0 \\ s_0 & \mathbf{0} \end{pmatrix}, \qquad J_- = \begin{pmatrix} \mathbf{0} & s_0 \\ -s_0 & \mathbf{0} \end{pmatrix},$$

and define the bilinear forms

$$B(z, w) = (z, J_+ w), \qquad \Omega(z, w) = (z, J_- w), \quad z, w \in \mathbb{C}^n.$$

The form B, with $B(z, w) = z^1 w^{2r} + \dots + z^{2r} w^1$, is non-degenerate and symmetric. The form Ω, with

$$\Omega(z, w) = -z^1 w^{2r} - \dots - z^r w^{r+1} + z^{r+1} w^r + \dots + z^{2r} w^1,$$

is non-degenerate and skew-symmetric.

Proposition 4.26 *Let* $\mathrm{SO}(\mathbb{C}^{2r}, B)$ *be the Lie group of complex matrices preserving the bilinear form* B *and having determinant* 1. *The Lie algebra* $\mathfrak{so}(\mathbb{C}^{2r}, B)$ *of* $\mathrm{SO}(\mathbb{C}^{2r}, B)$ *consists of all matrices*

$$A = \begin{pmatrix} a & b \\ c & -s_0{}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r, \mathbb{C})$, *and* b, c *are* $r \times r$ *matrices such that*

$${}^t b = -s_0 b s_0, \qquad {}^t c = -s_0 c s_0$$

(*that is*, b *and* c *are skew-symmetric around the skew diagonal*).

Let $\mathrm{Sp}(\mathbb{C}^{2r}, \Omega)$ *be the Lie group of complex matrices preserving the bilinear form* Ω. *The Lie algebra* $\mathfrak{sp}(\mathbb{C}^{2r}, \Omega)$ *of* $\mathrm{Sp}(\mathbb{C}^{2r}, \Omega)$ *consists of all matrices*

$$A = \begin{pmatrix} a & b \\ c & -s_0{}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r, \mathbb{C})$, *and* b, c *are* $r \times r$ *matrices such that* ${}^t b = s_0 b s_0$, ${}^t c = s_0 c s_0$ (*that is*, b *and* c *are symmetric around the skew diagonal*).

Suppose now that $n = 2r + 1$. One then embeds the group $\mathrm{SO}(\mathbb{C}^{2r}, B)$ into the group $\mathrm{SO}(\mathbb{C}^{2r+1}, B)$, for $r \geqslant 2$, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}, \tag{4.2}$$

and one considers the symmetric bilinear form

$$B(z, w) = \sum_{i+j=n+1} z_i w_j, \quad z, w \in \mathbb{C}^n.$$

One can write this form as $B(x, y) = (x, Sy)$, where the $n \times n$ symmetric matrix S has block form

$$\begin{pmatrix} 0 & 0 & s_0 \\ 0 & 1 & 0 \\ s_0 & 0 & 0 \end{pmatrix}.$$

Writing the elements of $M(n, \mathbb{C})$ in the same block form, one has the following description (see [6]) of the Lie algebra of the complex orthogonal group in this case:

Proposition 4.27 *The Lie algebra* $\mathfrak{so}(\mathbb{C}^{2r+1}, B)$ *of* $\mathrm{SO}(\mathbb{C}^{2r+1}, B)$ *consists of all matrices*

$$A = \begin{pmatrix} a & w & b \\ u & 0 & -{}^t w_0 \\ c & -s_0{}^t u & -s_0{}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r, \mathbb{C})$, b, c *are* $r \times r$ *matrices such that*

$${}^t b = -s_0 b s_0, \qquad {}^t c = -s_0 c s_0$$

(*that is*, *b and c are skew-symmetric around the skew diagonal*), w *is an* $r \times 1$ *matrix* (*column vector*), *and* u *is a* $1 \times r$ *matrix* (*row vector*).

Definitions 4.28 Let G be one of the following *classical Lie groups* of rank n:

$$\mathrm{GL}(n, \mathbb{C}), \quad \mathrm{SL}(n+1, \mathbb{C}), \quad \mathrm{Sp}\big(\mathbb{C}^{2n}, \Omega\big), \quad \mathrm{SO}\big(\mathbb{C}^{2n}, B\big), \quad \mathrm{SO}\big(\mathbb{C}^{2n+1}, B\big),$$

and let $\mathfrak{g}$ be its Lie algebra. The subgroup H of diagonal matrices in G is a maximal torus of rank n, and we denote its Lie algebra by $\mathfrak{h}$. Fix a basis for the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ as follows:

(i) Let $G = \mathrm{GL}(n, \mathbb{C})$. Define the linear functional ε_i on $\mathfrak{h}$ by

$$\langle \varepsilon_i, A \rangle = a_i, \quad A = \mathrm{diag}(a_1, \dots, a_n).$$

Then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

(ii) Let $G = \mathrm{SL}(n+1, \mathbb{C})$. Then $\mathfrak{h}$ consists of all diagonal traceless matrices. Define ε_i as in (i) as a linear functional on the space of diagonal matrices for $i = 1, \dots, n+1$. The restriction of ε_i to $\mathfrak{h}$ is then an element of $\mathfrak{h}^*$, again denoted as ε_i. The elements of $\mathfrak{h}^*$ can be written uniquely as

$$\sum_{i=1}^{n+1} \lambda_i \varepsilon_i, \quad \lambda_i \in \mathbb{C}, \ \sum_{i=1}^{n+1} \lambda_i = 0.$$

The functionals

$$\varepsilon_i - \frac{1}{n+1}(\varepsilon_1 + \dots + \varepsilon_{n+1}), \quad i = 1, \dots, n,$$

are a basis for $\mathfrak{h}^*$.

(iii) Let G be $\mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$ or $\mathrm{SO}(\mathbb{C}^{2n}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A \rangle = a_i$ for $A = \mathrm{diag}(a_1, \dots, a_n, -a_n, \dots, -a_1) \in \mathfrak{h}$ and $i = 1, \dots, n$. Then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

(iv) Let $G = \mathrm{SO}(\mathbb{C}^{2n+1}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A \rangle = a_i$ for $A = \mathrm{diag}(a_1, \dots, a_n, 0, -a_n, \dots, -a_1) \in \mathfrak{h}$ and $i = 1, \dots, n$. Then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

We recall the following result (cf. e.g. [11, Lect. 14])).

Theorem 4.29 *Let* $\mathrm{G}_2 \subset \mathrm{SO}(7)$ *be the* 14*-dimensional compact exceptional simple Lie group. Its Lie subalgebra* $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ *consists of the skew-symmetric real* 7×7 *matrices* (a_{ij}) *such that*

$$a_{32} + a_{45} + a_{76} = 0, \qquad a_{13} + a_{64} + a_{75} = 0, \qquad a_{21} + a_{65} + a_{47} = 0,$$
$$a_{14} + a_{36} + a_{27} = 0, \qquad a_{51} + a_{26} + a_{73} = 0, \qquad a_{17} + a_{42} + a_{53} = 0,$$
$$a_{61} + a_{52} + a_{34} = 0.$$

Definitions 4.30 Let (M, Ω) be a symplectic manifold, and let G be a Lie group acting symplectically on M, i.e. $g^*\Omega = \Omega$ for all $g \in G$. Let $\mathfrak{g}$ be the Lie algebra of G. For any $X \in \mathfrak{g}$, denote by $\hat{X}$ the vector field on M generated by the one-parameter subgroup $\exp tX \subset G$:

$$\hat{X}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp tX)p, \qquad p \in M.$$

The symplectic action of G on M is called *Hamiltonian* if there exists a G-equivariant linear map $X \mapsto f_X$ from the Lie algebra $\mathfrak{g}$ to the space $C^\infty M$ such that $\hat{X}$ is the Hamiltonian vector field of f_X for all $X \in \mathfrak{g}$. In other words, for all $X, Y \in \mathfrak{g}$, $g \in G$, $a, b \in \mathbb{R}$,

$$f_{aX+bY} = af_X + bf_Y, \qquad -\mathrm{d}f_X = i_{\hat{X}}\Omega, \qquad \left(g^{-1}\right)^* f_X = f_{\mathrm{Ad}_g X}.$$

It can be proved that in this case

$$\{f_X, f_Y\} = f_{[X,Y]}, \qquad X, Y \in \mathfrak{g},$$

i.e. this G-equivariant linear map $X \mapsto f_X$ is a homomorphism from the Lie algebra $\mathfrak{g}$ to the Lie algebra $C^\infty M$ with respect to the standard Poisson bracket on M induced by Ω. The map

$$\mu\colon M \to \mathfrak{g}^*, \qquad \mu(p)(X) = f_X(p),$$

from M to the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is called the *momentum map*. The momentum map is G-equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$, i.e.

$$\mu(gp)(X) = \mu(p)(\mathrm{Ad}_{g^{-1}} X).$$

Definitions 4.31 The action of a Lie group G on a connected differentiable manifold M is said to be *effective* if $gp = p$ for all $p \in M$ implies that $g = e$, the identity element of G.

The action of a Lie group G on a connected differentiable manifold M is said to be *free* if $gp = p$ for a point $p \in M$ implies that $g = e$.

The action of a Lie group G on a connected differentiable manifold M is said to be *transitive* if for each two points $p, q \in M$, there exists $g \in G$ such that $gp = q$.

The action of a Lie group G on a connected differentiable manifold M is said to be *properly discontinuous* if the two following conditions hold:

(i) Each point $p \in M$ has a neighbourhood U such that $U \cap g(U)$ is empty for all $g \in G \setminus \{e\}$.
(ii) Any two points $p, p' \in M$ that are not equivalent modulo G (i.e. $gp \neq p'$ for every $g \in G$) have neighbourhoods U, U', respectively, such that $U \cap g(U')$ is empty for all $g \in G$.

Conditions (i), (ii) together imply that M/G is a Hausdorff manifold of the same dimension as M.

Definition 4.32 A Lie group G is said to act *simply transitively* on a manifold M if the action is transitive and free.

Theorem 4.33 *Let H be a closed subgroup of a Lie group G. Then the quotient manifold G/H admits a unique structure of smooth manifold in such a way that the natural projection $G \to G/H$ is a submersion and the natural action of G on G/H is smooth.*

Theorem 4.34 *Let $G \times M \to M$, $(s, p) \mapsto sp$, be a transitive action of the Lie group G on the differentiable manifold M on the left. Let $p \in M$, and let H be the isotropy group at p. Define the map*

$$\Phi\colon G/H \to M, \qquad \Phi(sH) = sp.$$

Then Φ is a diffeomorphism.

Proposition 4.35 *Let G/H be a homogeneous space, and let N be the maximal normal subgroup of G contained in H. Notice that N is a closed subgroup. Then $G' = G/N$ acts on G/H with isotropy subgroup $H' = H/N$, and G' acts effectively on $G/H = G'/H'$.*

Definition 4.36 A homogeneous space G/H is said to be *reductive* if there exists an $\mathrm{Ad}(H)$-invariant direct sum complement vector space $\mathfrak{m}$ to the Lie algebra $\mathfrak{h}$ of the isotropy group H.

4.2 Lie Groups and Lie Algebras

Problem 4.37 Prove that the following are Lie groups:

(i) Each finite-dimensional real vector space with its structure of additive group. In particular $\mathbb{R}^n$.

(ii) The set of non-zero complex numbers $\mathbb{C}^*$ with the multiplication of complex numbers.
(iii) $G \times H$, where G, H are Lie groups, with the product $(g, h)(g', h') = (gg', hh')$, $g, g' \in G$, $h, h' \in H$. In general, if G_i, $i = 1, \dots, n$, is a Lie group, then $G_1 \times \cdots \times G_n$ is a Lie group.
(iv) T^n for $n \geqslant 1$ (*toral group*).
(v) $\operatorname{Aut} V$, where V is a vector space of finite dimension over $\mathbb{R}$ or $\mathbb{C}$, with the composition product, and in particular $\mathrm{GL}(n, \mathbb{R}) = \operatorname{Aut}_{\mathbb{R}} \mathbb{R}^n$ and $\mathrm{GL}(n, \mathbb{C}) = \operatorname{Aut}_{\mathbb{C}} \mathbb{C}^n$.
(vi) $K = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$, $n > 1$, with the group structure defined by

$$(x, A)\big(x', A'\big) = \big(x + Ax', AA'\big).$$

The relevant theory is developed, for instance, in Warner [13].

Solution

(i) Let V be a finite-dimensional real vector space. If $\dim V = n$, then V has a natural structure of C^∞ manifold, defined by the global chart (V, φ), $\varphi\colon V \to \mathbb{R}^n$, the coordinate functions being the dual basis to a given basis of V. The structure does not depend on the given basis, as it is easily checked. On the other hand, V has the structure of an additive group with the internal law, and the map $V \times V \to V$, $(v, w) \mapsto v - w$, is C^∞.
(ii) $\mathbb{C}^*$ has a natural structure of a two-dimensional manifold as an open subset of the two-dimensional real vector space $\mathbb{C}$. $\mathbb{C}^*$ has the structure of a multiplicative group, and the map $\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(z, w) \mapsto zw^{-1}$, is C^∞, since if $z = a + b\mathrm{i}$, $w = c + d\mathrm{i}$, one has

$$zw^{-1} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\mathrm{i} \equiv \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right) \in \mathbb{R}^2.$$

(iii) $G \times H$ is a Lie group with the structure of product manifold and the given product, since

$$\big((g, h), \big(g', h'\big)\big) \mapsto (g, h)\big(g', h'\big)^{-1} = \big(gg'^{-1}, hh'^{-1}\big)$$

is C^∞.
(iv) $T^n = S^1 \times \overset{(n)}{\cdots} \times S^1$. Hence T^n is a Lie group as it is a finite product of Lie groups.
(v) $\operatorname{Aut} V$ is an open subset of $\operatorname{End} V$ because

$$\operatorname{Aut} V = \{A \in \operatorname{End} V : \det A \neq 0\}$$

and det is a continuous function. Therefore $\operatorname{Aut} V$ has a structure of C^∞ manifold (as an open submanifold of $\mathbb{R}^{n^2}$, $n = \dim V$). The multiplication in $\operatorname{Aut} V$ is the composition. Taking as its chart the map which associates to an automorphism its matrix in a basis, the product is calculated by multiplication of matrices. The map $\operatorname{Aut} V \times \operatorname{Aut} V \to \operatorname{Aut} V$, $(A, B) \mapsto AB^{-1}$, is C^∞, as the components of AB and B^{-1} are rational functions in the components of A and B. Hence $\operatorname{Aut} V$ is a Lie group. We have as particular cases the sets $\mathrm{GL}(n, \mathbb{R}) = \operatorname{Aut}_{\mathbb{R}} \mathbb{R}^n$ and $\mathrm{GL}(n, \mathbb{C}) = \operatorname{Aut}_{\mathbb{C}} \mathbb{C}^n$.

(vi) $K = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$ $(n > 1)$ has the structure of a product manifold, and with the law $(x, A) \cdot (x', A') = (x + Ax', AA')$, it has the structure of a group. Let us show that K is a Lie group. In fact, the above product is C^∞, and the inverse of (x, A) is (y, B) such that $(y, B) \cdot (x, A) = (0, I)$. Hence the inverse of (x, A) is $(-A^{-1}x, A^{-1})$, so the map $(x, A) \mapsto (x, A)^{-1}$ is C^∞. This is the Lie *group of affine transformations of* $\mathbb{R}^n$ (identify the element (v, A) of K with the affine transformations $x \mapsto v + Ax$ of $\mathbb{R}^n$). The multiplication in K corresponds with the composition of affine transformations of $\mathbb{R}^n$.

Problem 4.38 Consider the product $T^1 \times \mathbb{R}^+$ of the one-dimensional torus by the multiplicative group of strictly positive numbers (that group is called the *group of similarities of the plane*). Let (θ, x) denote local coordinates. Show that the vector field

$$\frac{\partial}{\partial \theta} + x\frac{\partial}{\partial x}$$

is left-invariant.

Solution (i) Let $L_s \colon G \to G$ denote the left translation $L_s s_1 = s s_1$ on a Lie group G. A vector field Y on a Lie group is left-invariant if $L_{s*} Y_e = Y_s$ for all $s \in G$, where e stands for the identity element of G.

In the present case, let $(\alpha, a), (\theta, x)$ be in the coordinate domain with (α, a) arbitrarily fixed and any (θ, x). The left translation is given by $L_{(\alpha,a)}(\theta, x) = (\alpha + \theta, ax)$. Therefore one has

$$L_{(\alpha,a)*} = \begin{pmatrix} \frac{\partial(\alpha+\theta)}{\partial\theta} & \frac{\partial(\alpha+\theta)}{\partial x} \\ \frac{\partial(ax)}{\partial\theta} & \frac{\partial(ax)}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

A vector field Z on $T^1 \times \mathbb{R}^+$ is left-invariant if

$$L_{(\alpha,a)*} Z_{(0,1)} = Z_{(\alpha,a)}. \tag{$\star$}$$

For the vector field

$$X_{(\theta,x)} = \frac{\partial}{\partial\theta} + x\frac{\partial}{\partial x}$$

(see Fig. 4.1), we have

$$L_{(\alpha,a)*} X_{(0,1)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \frac{\partial}{\partial\theta} + a\frac{\partial}{\partial x} = X_{(\alpha,a)}.$$

Since (α, a) is arbitrary, we have that condition $(\star)$ is satisfied at any point, and X is in fact left-invariant on the given coordinate domain. The prolongation to all of G is immediate.

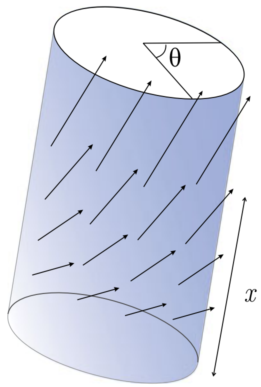


Fig. 4.1 The vector field $\partial/\partial\theta + x\partial/\partial x$ on the group of similarities of the plane

Problem 4.39 Using the coordinate vector fields $\partial/\partial x^i_j$, $1 \leqslant i, j \leqslant n$, on $\mathrm{GL}(n,\mathbb{R})$, prove that the vector field Y on $\mathrm{GL}(n,\mathbb{R})$ whose matrix of components at the identity is $A = (a^i_j)$ and whose matrix of components is equal to BA at the element $B = (b^i_j)$ of $\mathrm{GL}(n,\mathbb{R})$ is a left-invariant vector field.

Solution We have $Y_I = \sum_{i,j=1}^n a^i_j(\partial/\partial x^i_j)_I$, where I denotes the identity element of $\mathrm{GL}(n,\mathbb{R})$. Since $(L_{B*}Y_I)x^i_j = Y_I(x^i_j \circ L_B)$ and

$$\left(x^i_j \circ L_B\right)(C) = x^i_j(BC) = \sum_k b^i_k c^k_j,$$

one has $x^i_j \circ L_B = \sum_k b^i_k x^k_j$. Hence,

$$Y_I\left(x^i_j \circ L_B\right) = \sum_{h,k,l=1}^n a^h_l \frac{\partial}{\partial x^h_l}\bigg|_I \left(b^i_k x^k_j\right) = \sum_{h=1}^n a^h_j b^i_h,$$

that is,

$$L_{B*}Y_I = \sum_{i,j,h=1}^n b^i_h a^h_j \frac{\partial}{\partial x^i_j}\bigg|_B = \sum_{i,j=1}^n (BA)^i_j \frac{\partial}{\partial x^i_j}\bigg|_B = Y_B.$$

Problem 4.40 Show that the following are Lie algebras:

1. The vector space $\mathfrak{X}(M)$ of C^∞ vector fields X on a manifold M, with the bracket of vector fields, satisfying $L_X\mathscr{T} = 0$, where $\mathscr{T}$ denotes a tensor field on M such that either
   (i) $\mathscr{T}$ is the identity endomorphism (the corresponding vector fields are all $X \in \mathfrak{X}(M)$), or
   (ii) $\mathscr{T}$ is a volume element (the corresponding X are called *divergence-free vector fields*).
2. The vector space $\mathbb{R}^3$ with the vector product operation $\times$ of vectors.
3. The space $\mathrm{End}\,V$ of endomorphisms of a vector space V of dimension n, with the operation $[A, B] = AB - BA$.

The relevant theory is developed, for instance, in Warner [13].

Solution

1. (i) Let $a, b \in \mathbb{R}$ and $X, Y \in \mathfrak{X}(M)$. Since

$$[aX_1 + bX_2, Y]f = a[X_1, Y]f + b[X_2, Y]f,$$

$[X, Y]$ is linear in the first variable. As $[X, Y] = -[Y, X]$, linearity on the first variable implies linearity on the second one. So $[X, Y]$ is $\mathbb{R}$-bilinear and anticommutative. The Jacobi identity

$$\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0$$

is satisfied, as it follows by adding

$$\begin{aligned}\big[[X, Y], Z\big]f &= [X, Y](Zf) - Z\big([X, Y]f\big)\\ &= X\big(Y(Zf)\big) - Y\big(X(Zf)\big) - Z\big(X(Yf)\big) + Z\big(Y(Xf)\big),\end{aligned}$$

and the two similar identities obtained by cyclic permutation of X, Y and Z.

(ii) If v is an n-form on an n-dimensional manifold M, then the conditions $L_X v = L_Y v = 0$, $X, Y \in \mathfrak{X}(M)$, imply

$$L_{\lambda X + \mu Y} v = 0, \quad \lambda, \mu \in \mathbb{R},$$

by virtue of the linear properties of the Lie derivative. Hence the set

$$\mathscr{L}(v) = \big\{X \in \mathfrak{X}(M) : L_X v = 0\big\}$$

is a vector space. Moreover, by using the formula $L_{[X,Y]} = [L_X, L_Y]$, we have $L_{[X,Y]}v = L_X(L_Y v) - L_Y(L_X v) = 0$, thus proving that $\mathscr{L}(v)$ is a Lie algebra as well.

2. One defines

$$v \times w = (bf - ce, -af + cd, ae - bd), \quad v = (a, b, c),\; w = (d, e, f).$$

Then we have:

(a) (bilinearity) $(\lambda v + \mu w) \times u = \lambda v \times u + \mu w \times u$, $\lambda, \mu \in \mathbb{R}$, and $u \times (\lambda v + \mu w) = \lambda u \times v + \mu u \times w$, as it is easily seen.

(b) (skew-symmetry) $u \times w + w \times u = 0$. Immediate from the definition of the vector product.

(c) (Jacobi identity) $(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$.

In fact, by using the formula relating the vector product and the scalar product, we obtain:

$$(u \times v) \times w = (wu)v - (wv)u, \qquad (v \times w) \times u = (uv)w - (uw)v,$$
$$(w \times u) \times v = (vw)v - (vu)w.$$

Adding these equalities and taking into account the skew-symmetry of the vector product, we obtain the Jacobi identity.

3. The map

$$\operatorname{End} V \times \operatorname{End} V \to \operatorname{End} V, \qquad (A, B) \mapsto [A, B] = AB - BA,$$

is bilinear, skew-symmetric and satisfies the Jacobi identity, as it is easily seen.

Problem 4.41 Consider the set G of matrices of the form

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R},\ x > 0,$$

with a structure of C^∞ manifold defined by the chart mapping each element of G as above to $(x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^2$.

(i) Is G a Lie subgroup of $\mathrm{GL}(3, \mathbb{R})$?

(ii) Prove that

$$\left\{ X = x\frac{\partial}{\partial x},\ Y = x\frac{\partial}{\partial y},\ Z = x\frac{\partial}{\partial z} \right\}$$

is a basis of left-invariant vector fields.

(iii) Find the structure constants of G with respect to the basis in (ii).

Solution

(i) The product of elements of G

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 & v \\ 0 & u & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} xu & 0 & xv + y \\ 0 & xu & xw + z \\ 0 & 0 & 1 \end{pmatrix} \in G$$

and the inverse of an element

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & 0 & -y/x \\ 0 & 1/x & -z/x \\ 0 & 0 & 1 \end{pmatrix} \in G$$

yield C^∞ maps $G \times G \to G$ and $G \to G$, respectively. Hence G is a Lie group, which is in addition an abstract subgroup of $\mathrm{GL}(3, \mathbb{R})$. The inclusion $G \to \mathrm{GL}(3, \mathbb{R})$ is an immersion, as its rank (that of the map $(x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^2 \mapsto (x, 0, y, 0, x, z, 0, 0, 1) \in \mathbb{R}^9$) is 3, so that G is a submanifold, hence a Lie subgroup, of $\mathrm{GL}(3, \mathbb{R})$.

(ii) Let $(a, b, c) \in G$ be arbitrarily fixed and any (x, y, z) in G. As the left translation by (a, b, c) is

$$L_{(a,b,c)}(x, y, z) = (ax, ay + b, az + c),$$

we have

$$L_{(a,b,c)*} \equiv \operatorname{diag}(a, a, a).$$

Let $e = (1, 0, 0)$ denote the identity element of G. We have

$$X_e = \left.\frac{\partial}{\partial x}\right|_e, \qquad Y_e = \left.\frac{\partial}{\partial y}\right|_e, \qquad Z_e = \left.\frac{\partial}{\partial z}\right|_e.$$

We deduce $L_{(a,b,c)*}X_e = X_{(a,b,c)}$ and similar expressions for Y and Z. Since X, Y, Z are C^∞ left-invariant vector fields that are linearly independent at e, they are a basis of left-invariant vector fields.

(iii) Let $X_1 = X$, $X_2 = Y$, $X_3 = Z$. Then

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = X_3, \qquad [X_2, X_3] = 0,$$

so, with respect to that basis, the non-zero structure constants are

$$c_{12}^2 = -c_{21}^2 = c_{13}^3 = -c_{31}^3 = 1.$$

Problem 4.42 Let

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

(i) Show that H admits a structure of C^∞ manifold with which it is diffeomorphic to $\mathbb{R}^3$.
(ii) Show that H with matrix multiplication is a Lie group (H is called the *Heisenberg group*).
(iii) Show that $B = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\}$ is a basis of the Lie algebra $\mathfrak{h}$ of H.

Solution

(i) The map

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{R}^3 \\ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} & \mapsto & (x, y, z) \end{array}$$

is obviously bijective. Thus $\{(H, \varphi)\}$ is an atlas for H, which defines a C^∞ structure on H such that φ is a diffeomorphism with $\mathbb{R}^3$.

(ii) H is a group with the product of matrices, because if $A, B \in H$, then $AB \in H$, and if

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in H, \quad \text{then} \quad A^{-1} = \begin{pmatrix} 1 & -x & xz-y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \in H.$$

Moreover, the maps

$$\begin{array}{rcl} H \times H & \stackrel{\Phi}{\to} & H \\ (A,B) & \mapsto & AB \end{array} \quad \text{and} \quad \begin{array}{rcl} H & \stackrel{\Psi}{\to} & H \\ A & \mapsto & A^{-1} \end{array}$$

are C^∞. Indeed, $\varphi \circ \Phi \circ (\varphi \times \varphi)^{-1} \colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, given by

$$\big(\varphi \circ \Phi \circ (\varphi \times \varphi)^{-1}\big)\big((x,y,z),(a,b,c)\big) = (a+x, b+xc+y, c+z),$$

is obviously C^∞. Similarly,

$$\varphi \circ \Psi \circ \varphi^{-1} \colon \qquad \begin{array}{rcl} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (-x, xz-y, -z) \end{array}$$

is also C^∞. Thus H is a Lie group.

One can also prove it considering H as the closed subgroup of the general linear group $\mathrm{GL}(3,\mathbb{R})$, defined by the equations

$$x_1^1 = x_2^2 = x_3^3 = 1, \qquad x_1^2 = x_1^3 = x_2^3 = 0,$$

where x_j^i denote the usual coordinates of $\mathrm{GL}(3,\mathbb{R}) \subset M(3,\mathbb{R}) \cong \mathbb{R}^9$. Hence by Cartan's Criterion on Closed Subgroups 4.4, G is a Lie subgroup of $\mathrm{GL}(3,\mathbb{R})$.

(iii) We have that $\dim H = 3$. Thus $\dim \mathfrak{h} = 3$, and so we only have to prove that

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent, which is immediate, and that they are left-invariant, for which we shall write

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

We have to prove that for every $A \in H$, one has

$$(L_A)_{*B}(X_i|_B) = X_i|_{AB}, \quad B \in H, \ i = 1,2,3. \tag{$\star$}$$

Let (a,b,c) be arbitrarily fixed and any (x,y,z) in H. As the left translation by (a,b,c) is $L_{(a,b,c)}(x,y,z) = (x+a, y+az+b, z+c)$, we have

$$L_{(a,b,c)*} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence,

$$(L_A)_{*B}(X_1|_B) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \equiv \frac{\partial}{\partial x}\bigg|_{AB},$$

and similarly we obtain

$$(L_A)_{*B}(X_2|_B) = \frac{\partial}{\partial y}\bigg|_{AB}, \qquad (L_A)_{*B}(X_3|_B) = (x+a)\frac{\partial}{\partial y}\bigg|_{AB} + \frac{\partial}{\partial z}\bigg|_{AB},$$

so condition ($\star$) is satisfied.

Problem 4.43 Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on a Lie group G. For every $X \in \mathfrak{g}$, let $X^\star$ be the infinitesimal generator of the one-parameter group

$$\Phi : \mathbb{R} \times G \to G, \qquad \Phi(t, x) = \exp(tX)x, \quad t \in \mathbb{R},\ x \in G.$$

Prove:

(i) The vector field $X^\star$ is right-invariant.
(ii) $L_{X^\star}\omega = 0 \quad \forall \omega \in \mathfrak{g}^*$.

Solution

(i) The vector field $X^\star$ is said to be right-invariant if $R_g \cdot X^\star = X^\star$ for all elements $g \in G$, which is equivalent to saying that every Φ_t commutes with every R_g (see, e.g. [9, Vol. I, Chap. I, Corollary 1.8]). In fact, one has

$$(\Phi_t \circ R_g)(x) = (R_g \circ \Phi_t)(x) = \exp(tX)xg, \quad g, x \in G,\ t \in \mathbb{R}.$$

(ii) Moreover (see formula (2.1)),

$$(L_{X^\star}\omega)_x = \lim_{t\to 0} \frac{\omega_x - \Phi^*_{-t}(\omega_{\Phi_t(x)})}{t},$$

but

$$\Phi^*_{-t}(\omega_{\Phi_t(x)}) = \omega_{\Phi_t(x)} \circ (L_{\exp(-tX)})_* = \omega_x,$$

as ω is left-invariant.

Hence $L_{X^\star}\omega = 0$.

Problem 4.44 Find the left- and right-invariant measures on:

(i) The *Euclidean group* $E(2)$ of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ x & \cos\theta & \sin\theta \\ y & -\sin\theta & \cos\theta \end{pmatrix}.$$

(ii) The group of matrices of the form $\left(\begin{smallmatrix} x & z \\ 0 & y \end{smallmatrix}\right)$, $x, y > 0$.
(iii) The Heisenberg group (see Problem 4.42).
(iv) The real general linear group $\mathrm{GL}(2, \mathbb{R})$.

Remark Given a matrix of functions, $A = (a^i_j)$, we shall denote by $\mathrm{d}A$ the matrix $(\mathrm{d}a^i_j)$.

The relevant theory is developed, for instance, in Sattinger and Weaver [12].

Solution Let A be a generic element of any of the groups above. We have [12, pp. 90–91] that one basis of left- (resp. right-) invariant 1-forms on G is given by a set of different elements of the matrix $A^{-1}\mathrm{d}A$ (resp. $(\mathrm{d}A)A^{-1}$). Then a left- (resp. right-) invariant measure is given by the wedge product of the given basis of left- (resp. right-) invariant 1-forms. In the present cases we obtain:

(i)

$$A^{-1}\mathrm{d}A = \begin{pmatrix} 0 & 0 & 0 \\ \cos\theta\,\mathrm{d}x - \sin\theta\,\mathrm{d}y & 0 & \mathrm{d}\theta \\ \sin\theta\,\mathrm{d}x + \cos\theta\,\mathrm{d}y & -\mathrm{d}\theta & 0 \end{pmatrix},$$

$$(\mathrm{d}A)A^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{d}x - y\,\mathrm{d}\theta & 0 & \mathrm{d}\theta \\ \mathrm{d}y + x\,\mathrm{d}\theta & -\mathrm{d}\theta & 0 \end{pmatrix},$$

and hence the left- and right-invariant measures ω_L and ω_R are, up to a constant factor,

$$\omega_L = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\theta = \omega_R.$$

(ii)

$$A^{-1}\mathrm{d}A = \begin{pmatrix} \frac{\mathrm{d}x}{x} & \frac{1}{x}(\mathrm{d}z - z\frac{\mathrm{d}y}{y}) \\ 0 & \frac{\mathrm{d}y}{y} \end{pmatrix}, \qquad (\mathrm{d}A)A^{-1} = \begin{pmatrix} \frac{\mathrm{d}x}{x} & \frac{1}{y}(\mathrm{d}z - z\frac{\mathrm{d}x}{x}) \\ 0 & \frac{\mathrm{d}y}{y} \end{pmatrix},$$

hence,

$$\omega_L = \frac{1}{x^2 y}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \qquad \omega_R = \frac{1}{x y^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

(iii) We have

$$A^{-1}\mathrm{d}A = \begin{pmatrix} 0 & \mathrm{d}x & \mathrm{d}y - x\,\mathrm{d}z \\ 0 & 0 & \mathrm{d}z \\ 0 & 0 & 0 \end{pmatrix}, \qquad (\mathrm{d}A)A^{-1} = \begin{pmatrix} 0 & \mathrm{d}x & \mathrm{d}y - z\,\mathrm{d}x \\ 0 & 0 & \mathrm{d}z \\ 0 & 0 & 0 \end{pmatrix},$$

hence,

$$\omega_L = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \omega_R.$$

(iv) Given $A = (a_{ij}) \in \mathrm{GL}(2,\mathbb{R})$, $i, j = 1, 2$, let $A^{-1} = (\alpha^{ij})$ be its inverse. Then a basis of left-invariant 1-forms is given by the components of $A^{-1}\mathrm{d}A$. The left-invariant measure ω_L on $\mathrm{GL}(2,\mathbb{R})$ is given by the wedge product of such components:

$$\begin{aligned}\omega_L &= \left(\alpha^{11}\,\mathrm{d}a_{11} + \alpha^{12}\,\mathrm{d}a_{21}\right) \wedge \left(\alpha^{11}\,\mathrm{d}a_{12} + \alpha^{12}\,\mathrm{d}a_{22}\right)\\ &\quad \wedge \left(\alpha^{21}\,\mathrm{d}a_{11} + \alpha^{22}\,\mathrm{d}a_{21}\right) \wedge \left(\alpha^{21}\,\mathrm{d}a_{12} + \alpha^{22}\,\mathrm{d}a_{22}\right)\\ &= \left(\alpha^{11}\alpha^{22} - \alpha^{12}\alpha^{21}\right)^2 \mathrm{d}a_{11} \wedge \mathrm{d}a_{12} \wedge \mathrm{d}a_{21} \wedge \mathrm{d}a_{22}\\ &= \frac{1}{(\det A)^2}\mathrm{d}a_{11} \wedge \mathrm{d}a_{12} \wedge \mathrm{d}a_{21} \wedge \mathrm{d}a_{22}.\end{aligned}$$

One has $\omega_L = \omega_R$, as the computation of the components of $(\mathrm{d}A)A^{-1}$ shows.

Problem 4.45 Let A be a finite-dimensional $\mathbb{R}$-algebra (not necessarily commutative). Set $n = \dim_{\mathbb{R}} A$. Let $\mathrm{Aut}_{\mathbb{R}} A \cong \mathrm{GL}(n,\mathbb{R})$ be the group of all $\mathbb{R}$-linear automorphisms of A, and let $G(A)$ be the group of $\mathbb{R}$-algebra automorphisms of A. Let $\mathrm{Der}\, A$ be the set of all $\mathbb{R}$-linear maps $X\colon A \to A$ such that

$$X(a \cdot b) = X(a) \cdot b + a \cdot X(b), \quad a, b \in A.$$

Prove:

(i) $\mathrm{Der}\, A$ is a Lie algebra with the bracket

$$[X, Y](a) = X\big(Y(a)\big) - Y\big(X(a)\big).$$

(ii) $G(A)$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ and hence is a Lie group.
(iii) $\dim G(A) \leqslant (n-1)^2$.
(iv) The Lie algebra of $G(A)$ is isomorphic to $(\mathrm{Der}\, A, [\,,\,])$.

Solution

(i) Certainly, $\mathrm{Der}\, A$ is an $\mathbb{R}$-vector space. Further, the bracket of two derivations is another derivation, as

$$\begin{aligned}[X, Y](a \cdot b) &= X\big(Y(a \cdot b)\big) - Y\big(X(a \cdot b)\big)\\ &= X\big\{Y(a) \cdot b + a \cdot Y(b)\big\} - Y\big\{X(a) \cdot b + a \cdot X(b)\big\}\\ &= \big\{X\big(Y(a)\big) \cdot b + Y(a) \cdot X(b) + X(a) \cdot Y(b) + a \cdot X\big(Y(b)\big)\big\}\\ &\quad - \big\{Y\big(X(a)\big) \cdot b + X(a) \cdot Y(b) + Y(a) \cdot X(b) + a \cdot Y\big(X(b)\big)\big\}\\ &= \big\{X\big(Y(a)\big) - Y\big(X(a)\big)\big\} \cdot b + a \cdot \big\{X\big(Y(b)\big) - Y\big(X(b)\big)\big\}\\ &= [X, Y](a) \cdot b + a \cdot [X, Y](b).\end{aligned}$$

Accordingly, $\operatorname{Der} A$ is endowed with a skew-symmetric bilinear map

$$[\,,\,]\colon \operatorname{Der} A \times \operatorname{Der} A \to \operatorname{Der} A,$$

and the Jacobi identity follows from the following calculation:

$$\begin{aligned}
&\big(\big[[X,Y],Z\big]+\big[[Y,Z],X\big]+\big[[Z,X],Y\big](a)\big)\\
&\quad=\big([X,Y]\big(Z(a)\big)-Z\big([X,Y](a)\big)\big)+\big([Y,Z]\big(X(a)\big)-X\big([Y,Z](a)\big)\big)\\
&\qquad+\big([Z,X]\big(Y(a)\big)-Y\big([Z,X](a)\big)\big)\\
&\quad=X\big(Y\big(Z(a)\big)\big)-Y\big(X\big(Z(a)\big)\big)-Z\big(X\big(Y(a)\big)\big)+Z\big(Y\big(X(a)\big)\big)\\
&\qquad+Y\big(Z\big(X(a)\big)\big)-Z\big(Y\big(X(a)\big)\big)-X\big(Y\big(Z(a)\big)\big)+X\big(Z\big(Y(a)\big)\big)\\
&\qquad+Z\big(X\big(Y(a)\big)\big)-X\big(Z\big(Y(a)\big)\big)-Y\big(Z\big(X(a)\big)\big)+Y\big(X\big(Z(a)\big)\big)=0.
\end{aligned}$$

(ii) For every pair $a,b\in A$, let $\Phi_{a,b}\colon \operatorname{Aut}_{\mathbb{R}} A\to A$ be the map given by $\Phi_{a,b}(f)=f(a\cdot b)-f(a)\cdot f(b)$. Then we have

$$G(A)=\bigcap_{a,b\in A}\Phi_{a,b}^{-1}(0).$$

As each $\Phi_{a,b}$ is a continuous map, we conclude that $G(A)$ is a closed subset in $\operatorname{Aut}_{\mathbb{R}} A$. Furthermore, $G(A)$ is an abstract subgroup as if $f,g\in G(A)$, then

$$\begin{aligned}
(f\circ g)(a\cdot b)&=f\big(g(a\cdot b)\big)=f\big(g(a)\cdot g(b)\big)=f\big(g(a)\big)\cdot f\big(g(b)\big)\\
&=(f\circ g)(a)\cdot(f\circ g)(b).
\end{aligned}$$

Hence $f\circ g\in G(A)$. Similarly, $f^{-1}\in G(A)$ since

$$f\big(f^{-1}(a)\cdot f^{-1}(b)\big)=f\big(f^{-1}(a)\big)\cdot f\big(f^{-1}(b)\big)=a\cdot b=f\big(f^{-1}(a\cdot b)\big),$$

and since f is injective, we conclude that $f^{-1}(a)\cdot f^{-1}(b)=f^{-1}(a\cdot b)$ for all $a,b\in A$.

(iii) If $f\in G(A)$, then $f(1)=1$. Hence each $f\in G(A)$ induces an automorphism $\bar f\in \operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R})$ by setting $\bar f(a \mod \mathbb{R})=f(a) \mod \mathbb{R}$, and the map

$$h\colon G(A)\to \operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R}),\qquad f\mapsto \bar f,$$

is clearly a group homomorphism. We claim that h is injective. In fact, $f\in\ker h$ if and only if $\bar f(a) \mod \mathbb{R}=a \mod \mathbb{R}$ for all $a\in A$, and this condition means that $f(a)-a\in\mathbb{R}$ for all $a\in A$. Hence we can write $f(a)=a+\omega(a)$, where $\omega\colon A \to \mathbb{R}$ is a linear form such that $\omega(1) = 0$. By imposing $f(a \cdot b) = f(a) \cdot f(b)$ we obtain

$$\omega(a \cdot b) = \omega(b)a + \omega(a)b + \omega(a)\omega(b).$$

Hence,

$$\omega(b)a + \omega(a)b \in \mathbb{R}, \quad a, b \in A. \tag{$\star$}$$

If $\omega \neq 0$, then there exists $a \in A$ such that $\omega(a) = 1$, and from ($\star$) it follows that $b \in \mathbb{R} + \mathbb{R}a$ for every $b \in A$. Hence $\dim A = 2$, and then either $A \cong \mathbb{R}[\varepsilon]$ or $A \cong \mathbb{R}[\mathrm{i}]$ or $A \cong \mathbb{R}[\mathrm{j}]$, with $\varepsilon^2 = 0$, $\mathrm{i}^2 = -1$, $\mathrm{j}^2 = 1$, thus leading us to a contradiction, as in these cases $\ker h$ is the identity. Accordingly, $G(A)$ is a subgroup of $\mathrm{Aut}_{\mathbb{R}}(A/\mathbb{R})$, so that $\dim G(A) \leqslant \dim \mathrm{Aut}_{\mathbb{R}}(A/\mathbb{R}) = (n-1)^2$.

(iv) Let $\mathfrak{g}(A)$ be the Lie algebra of $G(A)$, which is a Lie subalgebra of $\mathrm{End}_{\mathbb{R}} A = \mathrm{Lie}(\mathrm{Aut}_{\mathbb{R}} A)$. We know that an element $X \in \mathrm{End}_{\mathbb{R}} A$ belongs to $\mathfrak{g}(A)$ if and only if for every $t \in \mathbb{R}$, we have $\exp(tX) \in G(A)$, or equivalently,

$$\exp(tX)(a \cdot b) = \exp(tX)(a) \cdot \exp(tX)(b).$$

Differentiating this equation at $t = 0$, we conclude that X is a derivation of A.

Conversely, if X is a derivation, then by recurrence on k it is readily checked that

$$X^k(a \cdot b) = \sum_{h=0}^{k} \binom{k}{h} X^h(a) X^{k-h}(b).$$

Hence,

$$\begin{aligned}\exp(tX)(a \cdot b) &= \sum_{k=0}^{\infty} t^k \frac{X^k(a \cdot b)}{k!} = \sum_{k=0}^{\infty} \sum_{h=0}^{k} t^k \frac{1}{(k-h)!h!} X^h(a) X^{k-h}(b) \\ &= \exp(tX)(a) \cdot \exp(tX)(b),\end{aligned}$$

thus proving that the Lie algebra of $G(A)$ is isomorphic to $(\mathrm{Der}\, A, [\,,\,])$.

Problem 4.46 Prove that the Lie algebra $\mathfrak{so}(3)$ does not admit any two-dimensional Lie subalgebra.

Solution Let $\{e_1, e_2, e_3\}$ be the standard basis; that is,

$$[e_1 e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2.$$

Assume that $\mathfrak{g}$ is a two-dimensional Lie subalgebra. Let $\{v = \lambda^i e_i, w = \mu^i e_i\}$ be a basis of $\mathfrak{g}$. As the rank of the 3×2-matrix

$$\begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \\ \lambda^3 & \mu^3 \end{pmatrix}$$

is 2, we can assume that

$$\det \begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \end{pmatrix} \neq 0.$$

By making a change of basis in $\mathfrak{g}$ we can thus suppose that

$$\begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$v = e_1 + \lambda^3 e_3, \qquad w = e_2 + \mu^3 e_3. \tag{$\star$}$$

As $\mathfrak{g}$ is a Lie subalgebra, we have $[v, w] = \alpha v + \beta w$. By using ($\star$) we obtain $\lambda^3 = -\alpha$, $\mu^3 = -\beta$, $\alpha\lambda^3 + \beta\mu^3 = 1$, and substituting the first two relations into the third one, we obtain $\alpha^2 + \beta^2 + 1 = 0$, thus leading us to a contradiction.

4.3 Homomorphisms of Lie Groups and Lie Algebras

Problem 4.47 Consider the Heisenberg group (see Problem 4.42) and the map $f : H \to \mathbb{R}$, $A \mapsto f(A) = x + y + z$.

(i) Is f differentiable?
(ii) Is it a homomorphism of Lie groups?

Solution Let

$$\psi : H \to \mathbb{R}^3, \qquad \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z),$$

be the usual global chart of H. The map

$$f \circ \psi^{-1} : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \mapsto x + y + z,$$

is C^∞, and thus f is a C^∞ map. The additive group of real numbers $(\mathbb{R}, +)$ is a Lie group. Given

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

it is easy to see that $f(AB) \neq f(A) + f(B)$, so that f is not a homomorphism of Lie groups.

Problem 4.48 Prove that one has:

(i) An isomorphism of Lie groups $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1$.
(ii) A homeomorphism $\mathrm{O}(n) \cong \mathrm{SO}(n) \times \{-1, +1\}$.

Hint (to (i)) Consider the real representation of the general linear group $\mathrm{GL}(1, \mathbb{C})$:

$$\rho\colon \mathrm{GL}(1,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{R})$$
$$a + b\mathrm{i} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Solution

(i)

$$\begin{aligned}
\mathrm{SO}(2) &= \left\{A \in \mathrm{GL}(2,\mathbb{R}) : {}^t\!AA = I,\ \det A = 1\right\} \\
&= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2 + c^2 = b^2 + d^2 = 1,\ ab + cd = 0,\ ad - bc = 1\right\} \\
&\cong \left\{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} :\ \alpha \in \mathbb{R}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{U}(1) &= \left\{A \in \mathrm{GL}(1,\mathbb{C}) : {}^t\!\bar{A}A = 1\right\} = \left\{z \in \mathbb{C} \setminus \{0\} : \bar{z}z = 1\right\} \\
&= \{z \in \mathbb{C} : z = \cos\alpha + \mathrm{i}\sin\alpha\} \cong \left\{(\cos\alpha, \sin\alpha) : \alpha \in \mathbb{R}\right\} = S^1.
\end{aligned}$$

Let ρ be the real representation of $\mathrm{GL}(1,\mathbb{C})$. If $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have $a = 1$, $b = 0$; so ρ is injective. We have

$$\rho\big(\mathrm{U}(1)\big) = \rho\big(\{\cos\alpha + \mathrm{i}\sin\alpha\}\big) = \left\{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\right\} \cong \mathrm{SO}(2).$$

Since ρ is injective, one obtains $\mathrm{U}(1) \cong \mathrm{SO}(2)$.

(ii)

$$\mathrm{O}(n) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : {}^t\!AA = I\right\}.$$

Hence, if $A \in \mathrm{O}(n)$, then $\det {}^t\!AA = 1$. Consider the exact sequence

$$1 \to \mathrm{SO}(n) \overset{j}{\hookrightarrow} O(n) \overset{\det}{\to} \{-1, +1\} \to 1,$$

where j denotes the inclusion map of $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}$ in $\mathrm{O}(n)$. The map

$$\sigma : \{-1, +1\} \to \mathrm{O}(n)$$
$$1 \mapsto I$$
$$-1 \mapsto \mathrm{diag}\{-1, 1, \dots, 1\}$$

is a section of det, and hence we have a homeomorphism

$$\mathrm{O}(n) \cong \mathrm{SO}(n) \times \{-1, +1\}, \qquad A \mapsto \big(\sigma(\det A)A, \det A\big).$$

Problem 4.49 Let $\psi\colon G \to G$ be the diffeomorphism of the Lie group G defined by $\psi(a) = a^{-1}$, $a \in G$. Prove that ω is a left-invariant form if and only if $\psi^*\omega$ is right-invariant.

Solution We have

$$\psi(R_b x) = \psi(xb) = (xb)^{-1} = b^{-1}x^{-1}$$

and

$$L_{b^{-1}}\psi(x) = L_{b^{-1}}\big(x^{-1}\big) = b^{-1}x^{-1}.$$

Thus $\psi \circ R_b = L_{b^{-1}} \circ \psi$, and hence $(\psi \circ R_b)^* = (L_{b^{-1}} \circ \psi)^*$, that is, $R_b^* \circ \psi^* = \psi^* \circ L_{b^{-1}}^*$. If ω is left-invariant, we have

$$\big(R_b^* \circ \psi^*\big)(\omega) = \big(\psi^* \circ L_{b^{-1}}^*\big)(\omega) = \psi^*\omega,$$

and thus $\psi^*\omega$ is right-invariant. Conversely, if $\psi^*\omega$ is right-invariant, $\psi^*\omega = R_b^*\psi^*\omega = \psi^* L_{b^{-1}}^*\omega$, and thus $\omega = L_{b^{-1}}^*\omega$, because ψ^* is an isomorphism, and ω is left-invariant.

Problem 4.50 Let G be a compact, connected Lie group oriented by a left-invariant volume form v. Prove that for every continuous function f on G and every $s \in G$, we have

$$\int_G f v = \int_G (f \circ R_s) v,$$

where $R_s\colon G \to G$ denotes the right translation by s; that is, the left-invariant integral $f \mapsto \int_G f v$ is also right-invariant.

Solution For every $s \in G$, there exists a unique scalar $\varphi(s) \in \mathbb{R}^*$ such that $R_s^* v = \varphi(s)v$. The map $\varphi\colon G \to \mathbb{R}^*$ is clearly differentiable, and since $\varphi(e) = 1$ (where e denotes the identity element of G) and G is connected, we have $\varphi(G) \subseteq \mathbb{R}^+$; hence R_s is orientation-preserving. By applying the formula of change of variables to the diffeomorphism $R_s\colon G \to G$ we obtain

$$\int_G f v = \int_G R_s^*(f v) = \int_G (f \circ R_s) R_s^* v = \int_G (f \circ R_s)\varphi(s) v.$$

Hence,

$$\int_G f v = \varphi(s) \int_G (f \circ R_s) v.$$

Letting $f = 1$ and taking into account that $\int_G v \neq 0$, we conclude that $\varphi(s) = 1$ for all $s \in G$, and consequently,

$$\int_G f v = \int_G (f \circ R_s) v.$$

We also remark that $\varphi(s) = 1$ for all $s \in G$ implies that v is right-invariant.

Problem 4.51 Let G be a compact, connected Lie group oriented by a left-invariant volume form v and consider the map

$$\psi : G \to G, \qquad \psi(a) = a^{-1}, \quad a \in G.$$

Prove that for every continuous function f on G, we have

$$\int_G f v = \int_G (f \circ \psi) v.$$

Solution For every $s \in G$, we have from Problem 4.49:

$$R_s^*\big(\psi^* v\big) = (\psi \circ R_s)^* v = (L_{s^{-1}} \circ \psi)^* v = \psi^*\big(L_{s^{-1}}^* v\big) = \psi^* v.$$

Hence $\psi^* v$ is right-invariant, and since v is also right-invariant (see Problem 4.50), there exists $\varepsilon \in \mathbb{R}^*$ such that $\psi^* v = \varepsilon v$. Moreover, $\varepsilon^2 = 1$ as ψ is an involution. By applying the formula of change of variables to the diffeomorphism ψ (which may be orientation-reversing) we obtain

$$\int_G f v = \varepsilon \int_G \psi^*(f v) = \varepsilon \int_G (f \circ \psi)\psi^* v = \varepsilon \int_G (f \circ \psi)\varepsilon v = \int_G (f \circ \psi) v.$$

Problem 4.52 Let λ be an irrational real number, and let φ be the map

$$\varphi : \mathbb{R} \to T^2 = S^1 \times S^1, \qquad \varphi(t) = \big(\mathrm{e}^{2\pi \mathrm{i} t}, \mathrm{e}^{2\pi \mathrm{i}\lambda t}\big).$$

(i) Prove that it is an injective homomorphism of Lie groups.
(ii) Prove that the image of φ is dense in the torus (see Fig. 4.2).

Solution

(i) That φ is a homomorphism of Lie groups is immediate. We have $\varphi(t_1) = \varphi(t_2)$ if and only if $(\mathrm{e}^{2\pi \mathrm{i} t_1}, \mathrm{e}^{2\pi \mathrm{i}\lambda t_1}) = (\mathrm{e}^{2\pi \mathrm{i} t_2}, \mathrm{e}^{2\pi \mathrm{i}\lambda t_2})$ or equivalently if $t_1 - t_2$ and $\lambda(t_1 - t_2)$ are integers, which happens only if $t_1 = t_2$. Hence φ is injective.
(ii) It suffices to show that the subgroup $\mathbb{Z} + \lambda\mathbb{Z}$ is dense in $\mathbb{R}$, since if this happens, given the real numbers t_1, t_2, there exists a sequence $m_j + \lambda n_j$ such that

$$t_2 - \lambda t_1 = \lim_{j \to \infty} (m_j + \lambda n_j),$$

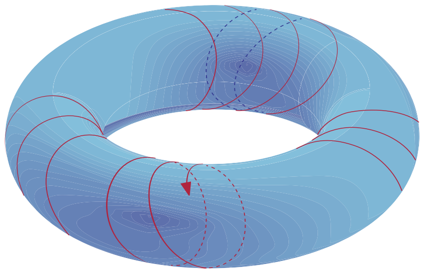


Fig. 4.2 A geodesic of irrational slope is dense in the 2-torus

that is,

$$t_2 = \lim_{j\to\infty} \bigl(m_j + \lambda(n_j + t_1)\bigr).$$

Hence,

$$\varphi(n_j + t_1) = \bigl(\mathrm{e}^{2\pi \mathrm{i} t_1}, \mathrm{e}^{2\pi \mathrm{i}\lambda(n_j+t_1)}\bigr) = \bigl(\mathrm{e}^{2\pi \mathrm{i} t_1}, \mathrm{e}^{2\pi \mathrm{i}(m_j+\lambda(n_j+t_1))}\bigr),$$

and thus,

$$\lim_{j\to\infty} \varphi(n_j + t_1) = \bigl(\mathrm{e}^{2\pi \mathrm{i} t_1}, \mathrm{e}^{2\pi \mathrm{i} t_2}\bigr).$$

Now, to prove that the subgroup $A = \mathbb{Z} + \lambda\mathbb{Z}$ is dense in $\mathbb{R}$, it suffices to see that the origin is an accumulation point in A, since in this case, given $x > 0$ and $0 < \varepsilon < x$, there exists $a \in A$ such that $0 < a < \varepsilon$, and if N denotes the greatest integer less than or equal to $(x - \varepsilon)/a$, then $Na \leqslant x - \varepsilon < (N + 1)a$, which implies $(N + 1)a < x + \varepsilon$, as in the contrary case we would have

$$x + \varepsilon \leqslant (N + 1)a \leqslant x - \varepsilon + a,$$

that is, $2\varepsilon \leqslant a < \varepsilon$, so getting a contradiction. Thus we have

$$x - \varepsilon < (N + 1)a < x + \varepsilon,$$

that is, $|(N + 1)a - x| < \varepsilon$.

If the origin is not an accumulation point in A, it is an isolated point, and then every point in A is isolated as A is a subgroup. Hence A is a closed discrete subset of $\mathbb{R}$. In fact, if $\lim_{k\to\infty} x_k = x$, $x_k \in A$, then for k large enough, $x_k - x_{k+1}$ belongs to an arbitrarily small neighbourhood of the origin. As $x_k - x_{k+1} \in A$ and the origin is isolated, we conclude $x_k = x_{k+1}$. Hence $x \in A$. Accordingly, $\mu = \inf\{x \in A : x > 0\}$ is a positive element in A. We will prove that A is generated by μ, that is, that $\mathbb{Z} + \lambda\mathbb{Z} = \mu\mathbb{Z}$. This will lead us to a contradiction, as λ is irrational.

Let $x \in A$ be a positive element. Let n denote the greatest integer less than or equal to x/μ, so that $n \leqslant x/\mu < n + 1$. Hence $0 \leqslant x - n\mu < \mu$. As $x - n\mu \in A$, from the very definition of μ we conclude that $x - n\mu = 0$.

4.4 Lie Subgroups and Lie Subalgebras

Problem 4.53 Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers.

(i) Prove that the map

$$j\colon \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{R}), \qquad x + \mathrm{i}y \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

is a faithful representation of the Lie group $\mathbb{C}^*$ (faithful means that j is injective).

(ii) Find the Lie subalgebra $\mathrm{Lie}(j(\mathbb{C}^*))$ of $\mathfrak{gl}(2,\mathbb{R})$.

Solution

(i) Since $\mathbb{C}^* \cong \mathrm{GL}(1,\mathbb{C})$, this was proved in Problem 4.48.

(ii)

$$\mathrm{Lie}\big(j\big(\mathbb{C}^*\big)\big) = j_*\big(T_1\mathbb{C}^*\big) \cong \left\{ \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix},\ \lambda, \mu \in \mathbb{R} \right\}.$$

Problem 4.54 Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Consider the distribution $\mathscr{D}(s) = \{X_s : X \in \mathfrak{h}\}$, $s \in G$.

(i) Show that $\mathscr{D}$ is a C^∞ distribution of the same dimension as $\mathfrak{h}$. Is it involutive?

(ii) Consider the two-dimensional C^∞ distributions

$$\mathscr{D}_1 = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle, \qquad \mathscr{D}_2 = \left\langle \frac{\partial}{\partial x},\ x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle, \qquad \mathscr{D}_3 = \left\langle \frac{\partial}{\partial y},\ x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle$$

on the Heisenberg group (see Problem 4.42). Are they involutive?

(iii) Let $\mathscr{I}(\mathscr{D}_i)$ be the differential ideal corresponding to $\mathscr{D}_i$, $i = 1, 2, 3$. If $\alpha = \mathrm{d}x \wedge \mathrm{d}z$, $\beta = \mathrm{d}x + \mathrm{d}z$. Do we have $\alpha, \beta \in \mathscr{I}(\mathscr{D}_1)$? And $\alpha, \beta \in \mathscr{I}(\mathscr{D}_3)$?

Solution

(i) The space $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$. Let $\{X_1, \dots, X_k\}$ be left-invariant vector fields which are a basis of $\mathfrak{h}$. Then,

$$\mathscr{D}(s) = \big\langle X_1|_s, \dots, X_k|_s \big\rangle, \quad s \in G,$$

is a vector subspace of T_sG of dimension k. Hence $\mathscr{D}$ is a C^∞ k-dimensional distribution on G, because it is globally spanned by $X_1, \dots, X_k$, which are left-invariant vector fields and hence C^∞. Moreover, $\mathscr{D}$ is involutive. In fact, $X_1, \dots, X_k$ span $\mathscr{D}$, and $\mathfrak{h}$ is a subalgebra, so $[X_i, X_j] \in \mathfrak{h}$.

(ii) The Lie algebra $\mathfrak{h}$ of H is spanned (see Problem 4.42) by

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Since

$$\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]=0,\qquad \left[\frac{\partial}{\partial x},x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]=\frac{\partial}{\partial y},\qquad \left[\frac{\partial}{\partial y},x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]=0,$$

it follows that $\mathscr{D}_1$ and $\mathscr{D}_3$ are involutive but $\mathscr{D}_2$ is not.

(iii) $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$ is a basis of the $(C^\infty H)$-module $\mathfrak{X}(H)$, with dual basis $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$, and thus $\mathscr{I}(\mathscr{D}_1) = \langle \mathrm{d}z\rangle$. Hence $\alpha \in \mathscr{I}(\mathscr{D}_1)$, but $\beta \notin \mathscr{I}(\mathscr{D}_1)$. Also, $\{\partial/\partial x, \partial/\partial y, x\partial/\partial y + \partial/\partial z\}$ is a basis of $\mathfrak{X}(H)$, with dual basis $\{\theta^1,\theta^2,\theta^3\}$, and we have $\mathscr{I}(\mathscr{D}_3) = \langle\theta^1\rangle = \langle \mathrm{d}x\rangle$. Hence $\alpha \in \mathscr{I}(\mathscr{D}_3)$, and $\beta \notin \mathscr{I}(\mathscr{D}_3)$.

Problem 4.55 Consider the set G of matrices of the form

$$g=\begin{pmatrix} x & y\\ 0 & 1\end{pmatrix},\qquad x,y\in\mathbb{R},\ x\neq 0.$$

(i) Show that G is a Lie subgroup of $\mathrm{GL}(2,\mathbb{R})$.
(ii) Show that the elements of $\omega = g^{-1}\mathrm{d}g$ are left-invariant 1-forms.
(iii) Since $\mathfrak{g}\cong T_eG=\mathbb{R}^2$, we have $\dim\mathfrak{g}=2$, and we can choose $\{\omega_1=\mathrm{d}x/x, \omega_2 = \mathrm{d}y/x\}$ as a basis of the space of left-invariant 1-forms. Compute the structure constants of G with respect to this basis.
(iv) Prove that ω satisfies the relation $\mathrm{d}\omega+\omega\wedge\omega=0$.

Remark Here $\mathrm{d}\omega$ denotes the matrix $(\mathrm{d}\omega)^i_j = (\mathrm{d}\omega^i_j)$, and $\omega\wedge\omega$ denotes the wedge product of matrices, that is, with entries $(\omega\wedge\omega)^i_k=\sum_k \omega^i_k\wedge\omega^k_j$.

Solution

(i) Since

$$\begin{pmatrix} x & y\\ 0 & 1\end{pmatrix}\begin{pmatrix} x' & y'\\ 0 & 1\end{pmatrix}=\begin{pmatrix} xx' & xy'+y\\ 0 & 1\end{pmatrix}\in G,$$
$$\begin{pmatrix} x & y\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix} 1/x & -y/x\\ 0 & 1\end{pmatrix}\in G,$$

G is an abstract subgroup of $\mathrm{GL}(2,\mathbb{R})$, and as both the product and the inverse are C^∞ maps, G is a Lie group.

Moreover, G is a closed subgroup of $\mathrm{GL}(2,\mathbb{R})$, defined by the equations $x^2_2 = x^3_3 - 1 = 0$, x^i_j being the usual coordinates of $\mathrm{GL}(2,\mathbb{R})\subset M(2,\mathbb{R})\cong\mathbb{R}^4$. Hence G is closed in $\mathrm{GL}(2,\mathbb{R})$, and accordingly G is a Lie subgroup of $\mathrm{GL}(2,\mathbb{R})$.

(ii) If $g=\left(\begin{smallmatrix} x & y\\ 0 & 1\end{smallmatrix}\right)$, one has $\omega=g^{-1}\mathrm{d}g=\frac{1}{x}\left(\begin{smallmatrix} \mathrm{d}x & \mathrm{d}y\\ 0 & 0\end{smallmatrix}\right)$. We must prove $L^*_g\omega_g=\omega_e$. Let $s=\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right)$ be arbitrarily fixed. Proceeding similarly to Problem 4.38, we obtain

$L_{s*} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Thus,

$$L_g^* \omega_g = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1/x \\ 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 \\ 1/x \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\equiv \{\mathrm{d}x, \mathrm{d}y\} \equiv \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \\ 0 & 0 \end{pmatrix} = \omega_e$$

for $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(iii) As

$$\mathrm{d}\left(\frac{\mathrm{d}x}{x}\right) = 0, \qquad \mathrm{d}\left(\frac{\mathrm{d}y}{x}\right) = -\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{x},$$

from the Maurer–Cartan equations $\mathrm{d}\omega_i = -\sum_{j<k} c^i_{jk} \omega^j \wedge \omega^k$ we deduce

$$c^2_{12} = -c^2_{21} = 1.$$

(iv)

$$\mathrm{d}\omega = -\frac{1}{x^2} \begin{pmatrix} 0 & \mathrm{d}x \wedge \mathrm{d}y \\ 0 & 0 \end{pmatrix} = -\omega \wedge \omega.$$

Problem 4.56 Let S be the set of matrices of the form

$$M(u, v, w) = \begin{pmatrix} \cos w & \sin w & 0 & u \\ -\sin w & \cos w & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u, v, w \in \mathbb{R}.$$

(i) Prove that S is a Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$.

(ii) Let $j : \mathbb{R}^3 \to \mathrm{GL}(4, \mathbb{R})$, $(u, v, w) \mapsto M(u, v, w)$. Compute

$$j_* \frac{\partial}{\partial u}, \qquad j_* \frac{\partial}{\partial v}, \qquad j_* \frac{\partial}{\partial w}$$

and show that j is an immersion.

(iii) Prove that the tangent space to S at the identity element $e \in S$ admits the basis

$$\left\{ X_1 = \left.\frac{\partial}{\partial x^1_4}\right|_e, \ X_2 = \left.\frac{\partial}{\partial x^2_4}\right|_e, \ X_3 = \left.\frac{\partial}{\partial x^1_2}\right|_e - \left.\frac{\partial}{\partial x^2_1}\right|_e + \left.\frac{\partial}{\partial x^3_4}\right|_e \right\}.$$

Solution

(i) For all $M(u, v, w) \in S$, one has $\det M(u, v, w) = 1$, so $S \subset \mathrm{GL}(4, \mathbb{R})$. Moreover, the product of two elements of S and also the inverse of any element belong to S, as it follows by direct computation, so that S is a subgroup of

$\mathrm{GL}(4,\mathbb{R})$. Further, S can be considered as the closed subgroup of $\mathrm{GL}(4,\mathbb{R})$ determined by the equations

$$x_1^1 = x_2^2 = \cos x_4^3, \qquad x_2^1 = -x_1^2 = \sin x_4^3, \qquad x_3^3 = x_4^4 = 1,$$
$$x_3^1 = x_3^2 = x_1^3 = x_2^3 = x_1^4 = x_2^4 = x_3^4 = 0,$$

x_j^i being the usual coordinates of $\mathrm{GL}(4,\mathbb{R}) \subset M(4,\mathbb{R}) \cong \mathbb{R}^{16}$. Hence by Cartan's Criterion on Closed Subgroups, S is a Lie subgroup of $\mathrm{GL}(4,\mathbb{R})$.

(ii) We have

$$j_*\frac{\partial}{\partial u} = \frac{\partial}{\partial x_4^1}, \qquad j_*\frac{\partial}{\partial v} = \frac{\partial}{\partial x_4^2},$$
$$j_*\frac{\partial}{\partial w} = -\sin w\frac{\partial}{\partial x_1^1} + \cos w\frac{\partial}{\partial x_2^1} - \cos w\frac{\partial}{\partial x_1^2} - \sin w\frac{\partial}{\partial x_2^2} + \frac{\partial}{\partial x_4^3}.$$

Therefore j is an immersion.

(iii) The identity element of S, $e = I$, corresponds to $u = v = w = 2k\pi$. By (ii), T_eS admits the basis in the statement.

Problem 4.57 Let $G = \{\left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right) : a, b \in \mathbb{R}, a > 0\}$.

(i) Prove that G admits a Lie group structure.
(ii) Is G a Lie subgroup of $\mathrm{GL}(2,\mathbb{R})$?
(iii) Let μ be the map defined by

$$G \to \mathrm{GL}(2,\mathbb{R}), \qquad \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Is it differentiable? Is it a homomorphism of Lie groups? Is it an immersion? (cf. Problem 4.55).

Solution

(i) The map

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & U = \{(a,b) \in \mathbb{R}^2 : a > 0\} \\ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} & \mapsto & (a,b) \end{array}$$

is obviously bijective. Since U is open in $\mathbb{R}^2$, it is a two-dimensional C^∞ manifold, and thus there exists a unique differentiable structure on G such that $\dim G = 2$ and φ is a diffeomorphism.

G is a group with the product of matrices, since given

$$A = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix},$$

we have

$$AB = \begin{pmatrix} aa' & 0 \\ ba' + b' & 1 \end{pmatrix} \in G, \qquad A^{-1} = \begin{pmatrix} 1/a & 0 \\ -b/a & 1 \end{pmatrix} \in G.$$

Therefore G is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.

The operations

$$\begin{array}{rcl} G \times G & \xrightarrow{\Phi} & G \\ (A, B) & \mapsto & AB \end{array} \quad \text{and} \quad \begin{array}{rcl} G & \xrightarrow{\Psi} & G \\ A & \mapsto & A^{-1} \end{array}$$

are C^∞. In fact,

$$\left(\varphi \circ \Phi \circ (\varphi \times \varphi)^{-1}\right)\left((a, b), \left(a', b'\right)\right) = \left(aa', ba' + b'\right), \quad a, a' > 0,$$

$$\left(\varphi \circ \Psi \circ \varphi^{-1}\right)(a, b) = (1/a, -b/a), \quad a > 0,$$

are C^∞.

(ii) G is the closed submanifold of the open subset $x_1^1 > 0$ in $\mathrm{GL}(2, \mathbb{R})$ given by the equations $x_2^1 = 0$, $x_2^2 - 1 = 0$, x_j^i being the usual coordinates of $\mathrm{GL}(2, \mathbb{R}) \subset M(2, \mathbb{R}) \cong \mathbb{R}^4$. Thus G is a Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$.

Another way to prove that G is a Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$ is to observe that G is closed in $\mathrm{GL}(2, \mathbb{R})$, as if the sequence

$$\begin{pmatrix} a_n & 0 \\ b_n & 1 \end{pmatrix}$$

goes to

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

as $n \to \infty$, then it implies that $a_{11} \geqslant 0$, $a_{12} = 0$, $a_{22} = 1$; hence $a_{11} > 0$, and we can apply Cartan's Criterion on Closed Subgroups.

(iii) μ can be written in local coordinates as

$$\left(\psi \circ j \circ \mu \circ \varphi^{-1}\right)(a, b) = (a, b, 0, 1), \quad (a, b) \in U,$$

where ψ stands for the coordinate map of a local coordinate system on $\mathrm{GL}(2, \mathbb{R})$, and j denotes the inclusion map $j \colon \mu(G) \to \mathrm{GL}(2, \mathbb{R})$. As $\psi \circ j \circ \mu \circ \varphi^{-1}$ is a C^∞ map, μ is C^∞. On the other hand,

$$\mu\left(\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix}\right) = \mu\begin{pmatrix} aa' & 0 \\ ba' + b' & 1 \end{pmatrix} = \begin{pmatrix} aa' & ba' + b \\ 0 & 1 \end{pmatrix}$$

and

$$\mu\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\mu\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix},$$

and hence μ is not even a homomorphism of groups.

Finally, we have

$$\operatorname{rank} \mu_{\left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right)} = \operatorname{rank}\big(\psi \circ \mu \circ \varphi^{-1}\big)_{(a,b)} = 2.$$

Hence μ is an immersion.

Problem 4.58

(i) Determine all the two-dimensional Lie algebras. In fact, prove that there is a unique non-Abelian two-dimensional Lie algebra.

(ii) Prove that the map ρ of the non-Abelian two-dimensional Lie Algebra $\mathfrak{g}$ to $\operatorname{End}\mathfrak{g}$ given by $e \mapsto [e, \cdot]$ (that is, the adjoint representation) is a faithful representation of $\mathfrak{g}$.

(iii) Give a basis of left-invariant vector fields on the image of ρ and their bracket.

(iv) Let $\mathscr{D}$ be the distribution on $\operatorname{Aut}\mathfrak{g}$ spanned by the left-invariant vector fields on the image of ρ. Find a coordinate system (u, v, w, z) on $\operatorname{Aut}\mathfrak{g}$ such that $\partial/\partial z$, $\partial/\partial u$ span $\mathscr{D}$ locally.

(v) Prove that the subgroup $G_0 \subset \mathrm{GL}(2, \mathbb{R})$ determined by the subalgebra $\mathfrak{g}$ is the identity component ($\beta > 0$) of the group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} : \beta \neq 0 \right\}.$$

(vi) Prove that G can be viewed as the *group* $\operatorname{Aff}(\mathbb{R})$ *of affine transformations of the real line* $\mathbb{R}$. That is, the group of transformations

$$t' = \beta t + \alpha, \quad \beta \neq 0,$$

where $t = y/x$, $t' = y'/x'$ are the affine coordinates.

Solution

(i) Let $\mathfrak{g}$ be a two-dimensional Lie algebra with basis $\{e_1, e_2\}$. The Lie algebra structure is completely determined, up to isomorphism, knowing the constants a and b in the only bracket

$$[e_1, e_2] = a e_1 + b e_2.$$

If $a = b = 0$, the Lie algebra is Abelian, that is, $[e, e'] = 0$ for all $e, e' \in \mathfrak{g}$.

Otherwise, permuting e_1 and e_2 if necessary, we can suppose that $b \neq 0$, and so $\{e_1' = (1/b)e_1, e_2' = (a/b)e_1 + e_2\}$ is a basis of $\mathfrak{g}$, and one has

$$\left[e_1', e_2'\right] = e_2'.$$

Hence there exist, up to isomorphism, only two two-dimensional Lie algebras.

(ii) Let $\mathfrak{g} = \langle e'_1, e'_2 \rangle$ be the two-dimensional non-Abelian Lie algebra. That the map ρ is a representation follows from the Jacobi identity. The representation is faithful (that is, the homomorphism is injective) as we have

$$\rho\big(ae'_1 + be'_2\big) = \begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix}$$

in the basis $\{e'_1, e'_2\}$.

(iii) Fixing that basis, $\operatorname{End}\mathfrak{g}$ can be identified to the space of 2×2 square matrices, which is the Lie algebra of $GL(2, \mathbb{R})$.

Let E^i_j be the $n \times n$-matrix with zero entries except the (i, j)th one, which is 1. The left-invariant vector field X^i_j associated to E^i_j generates the one-parameter group $(\varphi^i_j)_t$ given by

$$\big(\varphi^i_j\big)_t X = X \cdot \exp\big(tE^i_j\big), \quad X \equiv \big(x^k_l\big).$$

Now,

$$\big(E^i_j\big)^2 = \begin{cases} 0 & \text{if } i \neq j, \\ E^i_j & \text{if } i = j. \end{cases}$$

Hence,

$$\exp\big(tE^i_j\big) = \begin{cases} I + tE^i_j & \text{if } i \neq j, \\ I + (\mathrm{e}^t - 1)E^i_j & \text{if } i = j, \end{cases}$$

where I denotes the identity matrix.

As a computation shows,

$$X^i_j = \sum_k x^k_i \frac{\partial}{\partial x^k_j}.$$

So, in the present case we have

$$X^2_1 = x^1_2 \frac{\partial}{\partial x^1_1} + x^2_2 \frac{\partial}{\partial x^2_1}, \qquad X^2_2 = x^1_2 \frac{\partial}{\partial x^1_2} + x^2_2 \frac{\partial}{\partial x^2_2},$$

and

$$\big[X^2_1, X^2_2\big] = -X^2_1.$$

(iv) Let us reduce X^2_1 to canonical form. The functions

$$u = \frac{x^2_1}{x^2_2}, \qquad v = x^1_1 - \frac{x^1_2 x^2_1}{x^2_2}, \qquad x^1_2, \qquad x^2_2$$

are coordinate functions on the neighbourhood defined by $x_2^2 \neq 0$ of the identity element I. In fact,

$$\frac{\partial(u, v, x_2^1, x_2^2)}{\partial(x_1^1, x_2^1, x_1^2, x_2^2)} = \frac{1}{x_2^2}.$$

In the new system we have

$$X_1^2 = \frac{\partial}{\partial u}, \qquad X_2^2 = -u\frac{\partial}{\partial u} + x_2^1\frac{\partial}{\partial x_2^1} + x_2^2\frac{\partial}{\partial x_2^2}.$$

Now, taking $w = x_2^1/x_2^2$, the functions (u, v, w, x_2^2) are coordinate functions on the neighbourhood given by $x_2^2 \neq 0$, since

$$\frac{\partial(u, v, w, x_2^2)}{\partial(u, v, x_2^1, x_2^2)} = \frac{1}{x_2^2}.$$

In this system we have

$$X_1^2 = \frac{\partial}{\partial u}, \qquad X_2^2 = -u\frac{\partial}{\partial u} + x_2^2\frac{\partial}{\partial x_2^2}.$$

Finally, defining $z = \log x_2^2$ in the neighbourhood $x_2^2 > 0$ of the identity element, we obtain coordinate functions (u, v, w, z) in which

$$X_1^2 = \frac{\partial}{\partial u}, \qquad X_2^2 = -u\frac{\partial}{\partial u} + \frac{\partial}{\partial z}.$$

Thus, the involutive distribution $\mathscr{D}$ corresponding to the subalgebra $\mathfrak{g}$ is spanned by $\partial/\partial u$, $\partial/\partial z$, that is,

$$\mathscr{D} = \langle X_1^2, X_2^2 \rangle = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial z} \right\rangle.$$

(v) By the above results, the integral submanifolds of $\mathscr{D}$ are defined by

$$v = x_1^1 - \frac{x_2^1 x_1^2}{x_1^2} = A, \qquad w = \frac{x_2^1}{x_2^2} = B,$$

where A, B denote arbitrary constants. In particular, the integral submanifold passing through the identity element I is obtained for $A = 1$, $B = 0$, that is, it is defined by

$$x_1^1 = 1, \qquad x_2^1 = 0.$$

Consequently, the subgroup G_0 of $\operatorname{Aut}\mathfrak{g}$ defined by the subalgebra $\mathfrak{g}$ is the (identity component of the) one in the statement.

(vi) The group G represents the transformations

$$x' = x, \qquad y' = \alpha x + \beta y.$$

The subgroup G admits a simple geometrical interpretation as the group of affine transformations of the real line $\mathbb{R}$ (see Problem 4.37). In fact, dividing we obtain

$$t' = \beta t + \alpha,$$

where $t = y/x$, $t' = y'/x'$ are the affine coordinates.

The group G has two components, defined by $\beta > 0$ and $\beta < 0$. The component of the identity element, which is the subgroup defined from $\mathfrak{g}$, is the first one.

4.5 The Exponential Map

Problem 4.59 Prove that, up to isomorphisms, the only one-dimensional connected Lie groups are S^1 and $\mathbb{R}$.

Solution The Lie algebra $\mathfrak{g}$ of such a Lie group G is a real vector space of dimension 1, hence isomorphic to $\mathbb{R}$. The exponential map is a homomorphism of Lie groups if the Lie algebra is Abelian, as in the present case. Consequently, here we have that exp is surjective since G is connected:

$$\exp \colon \mathbb{R} \to G, \qquad X \mapsto \exp X = \exp_X(1).$$

As ker(exp) is a closed subgroup of $\mathbb{R}$, then either $\ker(\exp) = 0$ or $\ker(\exp) = a\mathbb{Z}$, $a \in \mathbb{R}$. Hence, either

$$G = \mathbb{R}/\ker(\exp) = \mathbb{R} \quad \text{or} \quad G = \mathbb{R}/a\mathbb{Z} \cong S^1.$$

Problem 4.60 Prove that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not of the form e^A for any $A \in \mathfrak{gl}(2, \mathbb{R})$.

Solution Suppose that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{e}^A$. Then, since $\mathrm{e}^A = \mathrm{e}^{A/2+A/2} = \mathrm{e}^{A/2}\mathrm{e}^{A/2}$, it would be $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = (\mathrm{e}^{A/2})^2$. That is, the matrix would have square root, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$; but a calculation shows that there is no real solution.

Remark Interestingly enough, $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *does* lie in the image of exp, as $\exp\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} = -I$. On the other hand, the square roots of $-I$ in $\mathrm{GL}(2, \mathbb{R})$ are $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Problem 4.61 Let X be an element of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the real special linear group $\mathrm{SL}(2,\mathbb{R})$. Calculate $\exp X$.

The relevant theory is developed, for instance, in Helgason [7].

Solution Since

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{X \in M(2,\mathbb{R}) : \operatorname{tr} X = 0\right\}$$

if $X \in \mathfrak{sl}(2,\mathbb{R})$, it is of the form $X = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, and

$$\exp X = \sum_{n \geqslant 0} \frac{1}{n!} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}^n .$$

It is immediate that $X^2 = (a^2 + bc)I = -(\det X)I$, and hence,

$$\begin{aligned}
\exp X &= \left(\sum_{n \geqslant 0} \frac{(-\det X)^n}{(2n)!}\right) I + \left(\sum_{n \geqslant 0} \frac{(-\det X)^n)}{(2n+1)!}\right) X \\
&= \left(1 - \frac{\det X}{2!} + \frac{(\det X)^2}{4!} - \frac{(\det X)^3}{6!} + \cdots\right) I \\
&\quad + \left(1 - \frac{\det X}{3!} + \frac{(\det X)^2}{5!} - \frac{(\det X)^3}{7!} + \cdots\right) X.
\end{aligned}$$

We have to consider three cases:

(i) $\det X < 0$. Then

$$\begin{aligned}
\exp X &= \left(1 + \frac{|\det X|}{2!} + \frac{|\det X|^2}{4!} + \frac{|\det X|^3}{6!} + \cdots\right) I \\
&\quad + \left(1 + \frac{|\det X|}{3!} + \frac{|\det X|^2}{5!} + \frac{|\det X|^3}{7!} + \cdots\right) X \\
&= (\cosh\sqrt{-\det X})I + \left(\frac{\sinh\sqrt{-\det X}}{\sqrt{-\det X}}\right) X.
\end{aligned}$$

(ii) $\det X = 0$. Hence $\exp X = I + X$.

(iii) $\det X > 0$. Then $\exp X = (\cos\sqrt{\det X})I + (\frac{\sin\sqrt{\det X}}{\sqrt{\det X}})X$.

Problem 4.62 With the same definitions as in Problem 4.53:

(i) Prove that exp is a local diffeomorphism from $\mathrm{Lie}(j(\mathbb{C}^*))$ into $j(\mathbb{C}^*)$.

(ii) Which are the one-parameter subgroups of $j(\mathbb{C}^*)$?

Solution

(i)

$$\exp\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix} = I + \begin{pmatrix}0 & -t\\ t & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-t^2 & 0\\ 0 & -t^2\end{pmatrix} + \frac{1}{3!}\begin{pmatrix}0 & t^3\\ -t^3 & 0\end{pmatrix} + \cdots$$
$$= \begin{pmatrix}1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots & -(t - \frac{t^3}{3!} + \cdots)\\ t - \frac{t^3}{3!} + \cdots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\end{pmatrix} = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix},$$

and since $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix}0 & -t\\ t & 0\end{smallmatrix}\right)$, we have

$$\begin{aligned}\exp\colon \operatorname{Lie}\big(j(\mathbb{C}^*)\big) &\to j(\mathbb{C}^*)\\ \begin{pmatrix}\lambda & -t\\ t & \lambda\end{pmatrix} &\mapsto \exp\lambda \exp\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix} = e^{\lambda}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}.\end{aligned}$$

Hence $\exp\colon \operatorname{Lie}(j(\mathbb{C}^*)) \to j(\mathbb{C}^*)$ is a local diffeomorphism.

(ii) A one-parameter subgroup of $j(\mathbb{C}^*)$ is a homomorphism ρ from the additive group $\mathbb{R}$, considered as a Lie group, into $j(\mathbb{C}^*)$. As there exists a bijective correspondence between one-parameter subgroups and left-invariant vector fields, that is, elements of the Lie algebra, the one-parameter subgroups of $j(\mathbb{C}^*)$ are the maps

$$\begin{aligned}\rho\colon \mathbb{R} &\longrightarrow j(\mathbb{C}^*)\\ t &\longmapsto \exp t\begin{pmatrix}a & -b\\ b & a\end{pmatrix} = e^{at}\begin{pmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{pmatrix}, \quad a, b \in \mathbb{R}.\end{aligned}$$

In fact, it is immediate that $\rho(t)\rho(t') = \rho(t + t')$.

Problem 4.63 Let $x = \left(\begin{smallmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{smallmatrix}\right) \in \mathrm{SO}(2)$. Verify the formula

$$x = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$$

with $X = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) \in \mathfrak{so}(2)$, which justifies (as many other cases) the notation $x = \exp tX$.

Solution One has $\det tX = t^2 > 0$, so the results of Problem 4.61 apply. On the other hand, it is immediate that the powers of X with integer exponents from 1 on are cyclically equal to $X, -I, -X, I$. Hence,

$$\exp tX = (\cos t)I + \frac{\sin t}{t}tX = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right)I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)X$$

$$= I + tX + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-X) + \frac{t^4}{4!}I + \frac{t^5}{5!}X + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}X^n.$$

As $x = (\cos t)I + \frac{\sin t}{t}tX$, we are done.

Problem 4.64 Let H be the Heisenberg group (see Problem 4.42).

(i) Determine its Lie algebra $\mathfrak{h}$.
(ii) Prove that the exponential map is a diffeomorphism from $\mathfrak{h}$ onto H.

Solution

(i) The Lie algebra $\mathfrak{h}$ of H can be identified to the tangent space at the identity element $e \in H$, that is,

$$\mathfrak{h} \equiv \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M(n, \mathbb{R}) \right\},$$

considered as a Lie subalgebra of $\operatorname{End} \mathbb{R}^3$.

(ii) We have $\exp M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$. Since

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

one has

$$\exp \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} 1 & a & b + \frac{1}{2}ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly exp is a diffeomorphism of $\mathfrak{h}$ onto H.

Problem 4.65 Find the matrices $X \in \mathfrak{gl}(n, \mathbb{R}) = M(n, \mathbb{R})$ such that $\exp tX = e^{tX}$ is a one-parameter subgroup of

$$\mathrm{SL}(n, \mathbb{R}) = \left\{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\right\}.$$

Solution Applying the formula $\det \mathrm{e}^X = \mathrm{e}^{\operatorname{tr} X}$, we have that if $\det \mathrm{e}^{tX} = 1$, then $\operatorname{tr}(tX) = 0$, that is, $\operatorname{tr} X = 0$, and conversely.

Problem 4.66 Consider the next subgroups of the general linear group $\mathrm{GL}(n, \mathbb{C})$:

(a) $\mathrm{U}(n) = \{A \in M(n, \mathbb{C}) : {}^t\bar{A}A = I\}$, *unitary group* (the t means "transpose", and the bar indicates complex conjugation).
(b) $\mathrm{SL}(n, \mathbb{C}) = \{A \in M(n, \mathbb{C}) : \det A = 1\}$, *special linear group*.
(c) $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \det A = 1\}$, *special unitary group*.
(d) $\mathrm{O}(n, \mathbb{C}) = \{A \in M(n, \mathbb{C}) : {}^tAA = I\}$, *complex orthogonal group*.
(e) $\mathrm{SO}(n, \mathbb{C}) = \{A \in \mathrm{O}(n, \mathbb{C}) : \det A = 1\}$, *complex special orthogonal group*.
(f)
$$\mathrm{O}(n) = \mathrm{U}(n) \cap \mathrm{GL}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{R}) = \left\{A \in \mathrm{GL}(n, \mathbb{R}) : {}^tAA = I\right\},$$
orthogonal group.
(g) $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}$, *special orthogonal group*.
(h)
$$\mathrm{SL}(n, \mathbb{R}) = \mathrm{SL}(n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{R}) = \left\{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\right\},$$
real special linear group.

Then:

1. Prove that we have a diffeomorphism $\mathrm{U}(n) \cong \mathrm{SU}(n) \times S^1$.
2. Compute the dimensions of each of the groups described above.

The relevant theory is developed, for instance, in Chevalley [4].

Solution

1. Consider the exact sequence
$$1 \to \mathrm{SU}(n) \to \mathrm{U}(n) \overset{\det}{\to} S^1 \to 1,$$
and let $\sigma\colon S^1 \to \mathrm{U}(n)$ be the section of det given by $\sigma(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)$. The map $f\colon \mathrm{SU}(n) \times S^1 \to \mathrm{U}(n)$ given by $f(A, u) = A\sigma(u)$ is clearly differentiable. We will show that f is one-to-one by calculating its inverse. If $B = A\sigma(u)$, then $\det B = \det \sigma(u) = u$, and thus $A = B(\sigma(\det B))^{-1}$. Hence $f^{-1}(B) = (B(\sigma(\det B))^{-1}, \det B)$.
2. Let V and W be neighbourhoods of 0 and I in $M(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$, respectively, such that the exponential map establishes a diffeomorphism between them. Moreover, we can suppose (taking smaller neighbourhoods if necessary) that $A \in V$ implies $\bar{A}$, $-A$, ${}^tA \in V$ or $|\operatorname{tr} A| < 2\pi$.
   (a) Suppose that $A \in V$ is such that $B = \mathrm{e}^A \in W \cap \mathrm{U}(n)$. Then we have $B^{-1} = {}^t\bar{B}$, that is, $\mathrm{e}^{-A} = \mathrm{e}^{{}^t\bar{A}}$. Hence $A + {}^t\bar{A} = 0$, or equivalently ${}^tA + \bar{A} = 0$. Therefore, A is a skew-hermitian matrix. Conversely, if A is a skew-hermitian matrix belonging to V, then $\mathrm{e}^A \in W \cap \mathrm{U}(n)$. Since the space of $n \times n$ skew-hermitian matrices has dimension n^2, it follows that $\dim_{\mathbb{R}} \mathrm{U}(n) = n^2$.

(b) If $A \in V$ is such that $\mathrm{e}^A \in W \cap \mathrm{SL}(n, \mathbb{C})$, then $\det \mathrm{e}^A = 1 = \mathrm{e}^{\operatorname{tr} A}$. Hence $\operatorname{tr} A = 2\pi i k$, but $|\operatorname{tr} A| < 2\pi$, therefore $k = 0$, that is, $\operatorname{tr} A = 0$, so $\dim_{\mathbb{R}} \mathrm{SL}(n, \mathbb{C}) = 2(n^2 - 1)$.

(c) For the unitary special group, we can proceed as in (a) or (b). Alternatively, considering the diffeomorphism above $\mathrm{U}(n) \cong \mathrm{SU}(n) \times S^1$, we obtain $\dim_{\mathbb{R}} \mathrm{SU}(n) = n^2 - 1$.

(d) Given $A \in V$, reasoning as in (a) above, except that one must drop the bars denoting complex conjugation in the corresponding matrices, we obtain that $\mathrm{e}^A \in W \cap \mathrm{O}(n, \mathbb{C})$ if and only if A is skew-symmetric; that is, $A + {}^tA = 0$. Hence, $\dim_{\mathbb{R}} \mathrm{O}(n, \mathbb{C}) = n(n-1)$.

(e) $\dim_{\mathbb{R}} \mathrm{SO}(n, \mathbb{C}) = n(n-1)$ because $\mathrm{SO}(n, \mathbb{C})$ is open in $\mathrm{O}(n, \mathbb{C})$, since one has $\mathrm{SO}(n, \mathbb{C}) = \ker \det$, where $\det \colon \mathrm{O}(n, \mathbb{C}) \to \{+1, -1\}$, and the last space is discrete.

(f) and (g): Proceeding as in (d) but with open subsets $V \subset M(n, \mathbb{R})$ and $W \subset \mathrm{GL}(n, \mathbb{R})$, we have $\dim_{\mathbb{R}} \mathrm{O}(n) = n(n-1)/2$. Proceeding as in (e), we deduce $\dim_{\mathbb{R}} \mathrm{SO}(n) = n(n-1)/2$.

(h) Obviously $\dim \mathrm{SL}(n, \mathbb{R}) = n^2 - 1$.

Problem 4.67 Let G be an Abelian Lie group. Prove that $[X, Y] = 0$ for any left-invariant vector fields X and Y.

Solution The local flow generated by a left-invariant vector field X is given by $\varphi_t(x) = x \exp tX$. Moreover we know that $[X, Y]$ is the Lie derivative of Y with respect to X; hence,

$$[X, Y]_x = \lim_{t \to 0} \frac{1}{t}\big(Y_x - \varphi_{t*}(Y_{\varphi_{-t}(x)})\big).$$

Accordingly, $[X, Y] = 0$ if $\varphi_{t*}(Y_{\varphi_{-t}(x)}) = Y_x$, that is, if Y is invariant by φ_t; and this is equivalent to saying that φ_t and ψ_s commute, where $\psi_s(x) = x \exp sY$ denotes the local flow of Y. As G is Abelian, we have

$$(\varphi_t \circ \psi_s)(x) = x \exp sY \exp tX = x \exp tX \exp sY = (\psi_s \circ \varphi_t)(x).$$

Problem 4.68 Consider, for each $\lambda \in \mathbb{R} \setminus \{0\}$, the Lie algebra

$$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle$$

with the only non-zero bracket

$$[e_1, e_2] = \lambda e_3.$$

Find the corresponding simply connected Lie group G.

Solution The simply connected Lie group corresponding to

$$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle$$

is $G \equiv \mathbb{R}^3$. To determine the operation in G, we consider that

$$\exp\colon \mathfrak{g} \to G \equiv \mathbb{R}^3$$

is a diffeomorphism, and we take the global coordinates on G defined by that diffeomorphism,

$$(x_1, x_2, x_3) = \exp(x_1 e_1 + x_2 e_2 + x_3 e_3), \qquad [e_1, e_2] = \lambda e_3.$$

Then, since G is nilpotent, we have that the Campbell–Baker–Hausdorff formula (see Zachos [14] or (7.5) for the first five summands) reduces to

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] \right),$$

and we accordingly define the group operation on G with respect to these coordinates,

$$\begin{aligned}
(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) &= \exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] \right) \\
&= \exp\Big((x_1 + y_1)e_1 + (x_2 + y_2)e_2 + (x_3 + y_3)e_3 \\
&\quad + \frac{1}{2}[x_1 e_1 + x_2 e_2 + x_3 e_3, y_1 e_1 + y_2 e_2 + y_3 e_3] \Big) \\
&= \exp\Big((x_1 + y_1)e_1 + (x_2 + y_2)e_2 + (x_3 + y_3)e_3 \\
&\quad + \frac{\lambda}{2}(x_1 y_2 - x_2 y_1)e_3 \Big) \\
&= \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{\lambda}{2}(x_1 y_2 - x_2 y_1) \right).
\end{aligned}$$

That is, the product in $G \equiv \mathbb{R}^3$ is given by

$$\begin{aligned}
\mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
\big((t_1, t_2, t_3), (s_1, s_2, s_3)\big) &\longmapsto \left(t_1 + s_1,\ t_2 + s_2,\ t_3 + s_3 + \frac{\lambda}{2}(t_1 s_2 - t_2 s_1) \right).
\end{aligned}$$

Problem 4.69 Consider the Lie algebra

$$\mathfrak{g} = \langle P, X, Y, Q \rangle$$

having the non-zero brackets

$$[X, Y] = P, \qquad [Q, X] = Y, \qquad [Q, Y] = -X.$$

Then:

(i) Prove that its centre is $\langle P\rangle$.
(ii) Find the corresponding simply connected Lie group.

Hint Given the semi-direct product Lie algebra

$$\mathfrak{g} \rtimes_\varphi \mathfrak{h}$$

of the two Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ with respect to a given map $\varphi\colon \mathfrak{h} \to \operatorname{Der}\mathfrak{g}$, in order to find the corresponding simply connected semi-direct product Lie group

$$G \rtimes_\Psi H$$

(see Proposition 4.3 above):

(i) One first considers the map

$$\varphi\colon \mathfrak{h} \to \operatorname{Der}\mathfrak{g} \subset \operatorname{End}\mathfrak{g}$$

to determine $\Phi\colon H \to \operatorname{Aut}\mathfrak{g}$, where $\varphi = \Phi_*$, such that the diagram

$$\begin{array}{ccc} H & \overset{\Phi}{\dashrightarrow} & \operatorname{Aut}\mathfrak{g} \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{h} & \xrightarrow{\varphi=\Phi_*} & \operatorname{Der}\mathfrak{g}\subset\operatorname{End}\mathfrak{g} \end{array} \tag{$\star$}$$

is commutative, i.e. one should have

$$\Phi(\exp A) = \exp\bigl(\varphi(A)\bigr), \qquad A \in \mathfrak{h}.$$

(ii) In turn, this permits one to obtain the map

$$\begin{array}{rcl} \Psi\colon \quad H & \to & \operatorname{Aut} G \\ \exp A \mapsto \Psi(\exp A)\colon & & G \to G \\ & & \exp X \mapsto \exp\bigl\{\bigl(\Phi(\exp A)\bigr)(X)\bigr\} \\ & & \qquad = \exp\bigl\{\bigl(\exp\bigl(\varphi(A)\bigr)\bigr)(X)\bigr\} \end{array}$$

for all $A \in \mathfrak{h}$, $X \in \mathfrak{g}$.

(iii) Finally, denoting (only for the sake of simplicity) $\Psi(h)$ by Ψ_h, the operation in the semi-direct product of Lie groups is given by

$$(G \rtimes_\Psi H) \times (G \rtimes_\Psi H) \longrightarrow G \rtimes_\Psi H$$

$$\bigl((g,h),\bigl(g',h'\bigr)\bigr) \longmapsto \bigl(g\cdot\Psi_h\bigl(g'\bigr), hh'\bigr).$$

Hint (to (ii)) The simply connected Lie group with Lie algebra $\mathfrak{g} = \langle P, X, Y\rangle$ having the only non-zero bracket

$$[X, Y] = P$$

is the usual (i.e. three-dimensional) Heisenberg group H. This can be viewed as either $H \equiv \mathbb{R}^3$ with the operation

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \big((p,x,y),(p',x',y')\big) &\longmapsto \left(p+p'+\frac{1}{2}\big(xy'-yx'\big), x+x', y+y'\right) \end{aligned}$$

or $H = \mathbb{R} \times \mathbb{C}$ with the operation

$$\begin{aligned} (\mathbb{R}\times\mathbb{C}) \times (\mathbb{R}\times\mathbb{C}) &\longrightarrow \mathbb{R}\times\mathbb{C} \\ \big((p,z),(p',z')\big) &\longmapsto \left(p+p'+\frac{1}{2}\operatorname{Im}\big(\bar{z}z'\big), z+z'\right). \end{aligned}$$

Furthermore, with this operation, the exponential map is the identity map on $\mathbb{R}^3$, that is,

$$\begin{aligned} \exp\colon \qquad \mathfrak{h} \equiv \mathbb{R}^3 &\longrightarrow H \equiv \mathbb{R}^3 \\ pP + xX + yY &\longmapsto (p,x,y). \end{aligned}$$

Remark With the operation above, $\mathbb{R}^3$ is isomorphic to the matrix group H in Problem 4.42. In fact, the map

$$\begin{aligned} f\colon H \equiv \mathbb{R}^3 &\longrightarrow H \subset \mathrm{GL}(3,\mathbb{R}) \\ (p,x,y) &\longmapsto f(p,x,y) = \begin{pmatrix} 1 & x & p - \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

is a group isomorphism.

Solution

(i) It follows from

$$\begin{aligned} [pP + xX + yY + qQ, P] &= 0, \\ [pP + xX + yY + qQ, X] &= -yP + qY, \\ [pP + xX + yY + qQ, Y] &= xP - qX, \\ [pP + xX + yY + qQ, Q] &= -xY + yX. \end{aligned}$$

(ii) Consider the two Lie algebras $\mathfrak{g} = \langle P, X, Y \rangle$, with the only non-zero bracket $[X,Y] = P$, and $\mathfrak{h} = \langle Q \rangle$; and the semi-direct product $\mathfrak{g} \rtimes_\varphi \mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$ with respect to the map

$$\begin{aligned} \varphi\colon \mathfrak{h} &\longrightarrow \operatorname{Der}\mathfrak{g} \\ Q &\longmapsto \varphi(Q) \end{aligned}$$

such that

$$\big(\varphi(Q)\big)(P) = \operatorname{ad}_Q P = 0, \qquad \big(\varphi(Q)\big)(X) = \operatorname{ad}_Q X = Y,$$
$$\big(\varphi(Q)\big)(Y) = \operatorname{ad}_Q Y = -X.$$

We have $\varphi(Q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$\varphi(tQ) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix}.$$

The simply connected Lie group H with Lie algebra $\mathfrak{h}$ is $H \equiv \mathbb{R}$. Moreover, the commutative diagram in the hint above is given in the present case by

$$\begin{array}{ccc} H \equiv \mathbb{R} & \overset{\Phi}{\dashrightarrow} & \operatorname{Aut}\mathfrak{g} = \operatorname{Aut}\big(\langle P, X, Y\rangle\big) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{h} \equiv \mathbb{R} & \xrightarrow{\varphi=\Phi_*} & \operatorname{End}\big(\langle P, X, Y\rangle\big) \end{array}$$

with $\exp tQ = t$. Denoting $\Phi(\exp tQ)$ simply by Φ_t, one should have

$$\Phi_t = \exp\big(\varphi(tQ)\big) = \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} = \mathrm{e}^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

Hence we have

$$\begin{aligned} \Psi: \qquad H = \mathbb{R} &\longrightarrow \operatorname{Aut} G \\ \exp tQ \equiv t &\longmapsto \quad \Psi_t \end{aligned}$$

as follows. Let $(p, x, y) \in H$ be given by

$$(p, x, y) = \exp(pP + xX + yY).$$

Then

$$\begin{aligned} \Psi_t(p, x, y) &= \exp\big\{\big(\Phi(\exp tQ)\big)(pP + xX + yY)\big\} \\ &= \exp\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \cdot (pP + xX + yY)\right\} \\ &= \exp\big(pP + (x\cos t - y\sin t)X + (x\sin t + y\cos t)Y\big) \\ &= (p, x\cos t - y\sin t, x\sin t + y\cos t) \in \mathbb{R}^3, \end{aligned}$$

or well

$$\Psi_t(p, x, y) = \big(p, \mathrm{e}^{t\mathrm{i}} z\big) \in \mathbb{R} \times \mathbb{C}.$$

Hence the operation in the semi-direct product group

$$G \rtimes_{\Psi} H \equiv \mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{C} \times \mathbb{R}$$

is given by

$$\begin{aligned}
&(p, x, y, q) \cdot \left(p', x', y', q'\right) \\
&\quad = \left((p, x, y) \cdot \Psi_q\left(p, x', y'\right), q + q'\right) \\
&\quad = \left((p, x, y) \cdot \left(p', x' \cos q - y' \sin q, x' \sin q + y' \cos q\right), q + q'\right) \\
&\quad = \left(p + p' + \frac{1}{2}\left(x\left(x' \sin q + y' \cos q\right) - \left(x' \cos q - y' \sin q\right) y\right),\right. \\
&\qquad \left. x + x' \cos q - y' \sin q, y + x' \sin q + y' \cos q, q + q'\right) \\
&\quad = \left(p + p' + \frac{1}{2}\left((x \sin q - y \cos q) x' + (x \cos q + y \sin q) y'\right),\right. \\
&\qquad \left. x + x' \cos q - y' \sin q, y + x' \sin q + y' \cos q, q + q'\right),
\end{aligned}$$

or well, taking

$$\begin{aligned}
\Psi_t \colon \mathbb{R} \times \mathbb{C} &\longrightarrow \mathbb{R} \times \mathbb{C} \\
(p, z) &\longmapsto \Phi_t(p, z) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{t\mathrm{i}} \end{pmatrix} \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} p \\ \mathrm{e}^{t\mathrm{i}} z \end{pmatrix},
\end{aligned}$$

by

$$\begin{aligned}
&(p, z, q) \cdot \left(p', z', q'\right) \\
&\quad = \left((p, z) \cdot \Psi_q\left(p', z'\right), q + q'\right) \\
&\quad = \left((p, z) \cdot \left(p', \mathrm{e}^{q\mathrm{i}} z'\right), q + q'\right) \quad (\cdot = \text{operation in } H \equiv \mathbb{R} \times \mathbb{C}) \\
&\quad = \left(p + p' + \frac{1}{2} \operatorname{Im}\left(\bar{z} \mathrm{e}^{q\mathrm{i}} z'\right), z + \mathrm{e}^{q\mathrm{i}} z', q + q'\right).
\end{aligned}$$

4.6 The Adjoint Representation

Problem 4.70 Let G be the group defined by

$$G = \left\{A \in \mathrm{GL}(2, \mathbb{R}) : A^t A = r^2 I,\ r > 0,\ \det A > 0\right\}.$$

(i) Find the explicit expression of the elements of G.

(ii) Find its Lie algebra.
(iii) Calculate the *adjoint representation* of G.

Solution

(i) Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}).$$

By imposing $A^t A = r^2 I$ we obtain:

$$a_{11}^2 + a_{21}^2 = a_{21}^2 + a_{22}^2 = r^2, \qquad (\star)$$

$$a_{11}a_{12} + a_{21}a_{22} = 0. \qquad (\star\star)$$

From ($\star$) we deduce

$$a_{11} = r\cos\alpha, \qquad a_{12} = r\cos\beta, \qquad a_{21} = r\sin\alpha, \qquad a_{22} = r\sin\beta,$$

and then equation ($\star\star$) tells us that

$$0 = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta).$$

Hence $\beta = \alpha + \frac{k\pi}{2}$, $k \in \mathbb{Z}$. Accordingly,

$$A = \begin{pmatrix} r\cos\alpha & (-1)^k r\sin\alpha \\ r\sin\alpha & (-1)^{k-1} r\cos\alpha \end{pmatrix},$$

from which $\det A = (-1)^{k-1} r^2$. Hence $A \in G$ if and only if k is odd, and we can write

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a^2 + b^2 = r^2,$$

where $a = r\cos\alpha$, $b = r\sin\alpha$. The elements of G are usually called the similarities of the plane, as they are the product of a rotation by a homothety, both around the origin (see Problem 4.38), i.e.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Hence we have

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) : a_{11} - a_{22} = a_{12} + a_{21} = 0 \right\}.$$

(ii) By (i), the tangent space at the identity element e is

$$T_e G = \left\{ X \in M(2, \mathbb{R}) : Xf = 0,\ f = a_{11} - a_{22} \text{ or } f = a_{12} + a_{21} \right\}$$

$$\equiv \left\langle \frac{\partial}{\partial x_1^1}\bigg|_e + \frac{\partial}{\partial x_2^2}\bigg|_e, -\frac{\partial}{\partial x_2^1}\bigg|_e + \frac{\partial}{\partial x_1^2}\bigg|_e \right\rangle \equiv \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

hence the Lie algebra of G is $\mathfrak{g} = \left\{ \left(\begin{smallmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{smallmatrix} \right) \in M(2, \mathbb{R}) \right\}$.

(iii) For an arbitrary Lie group G with Lie algebra $\mathfrak{g}$, the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}\,\mathfrak{g}$ is given by

$$\mathrm{Ad}_s X = L_{s*} R_{s*}^{-1} X, \quad s \in G, \ X \in \mathfrak{g}.$$

For a matrix group, we have

$$\mathrm{Ad}_s X = s X s^{-1}.$$

As the group G of similarities of the plane is Abelian, the adjoint representation is trivial, i.e.

$$\mathrm{Ad}_s = \mathrm{id}_{\mathfrak{g}}, \quad s \in G.$$

Problem 4.71 The algebra $\mathbb{H}$ of quaternions is an algebra of dimension 4 over the field $\mathbb{R}$ of real numbers. $\mathbb{H}$ has a basis formed by four elements e_0, e_1, e_2, e_3 satisfying

$$e_0^2 = e_0, \qquad e_i^2 = -e_0, \qquad e_0 e_i = e_i e_0 = e_i, \qquad e_i e_j = -e_j e_i = e_k, \qquad (\star)$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. If $q = \sum_{i=0}^{3} a_i e_i \in \mathbb{H}$, the *conjugate* quaternion of q is defined by

$$\bar{q} = a_0 e_0 - (a_1 e_1 + a_2 e_2 + a_3 e_3),$$

and the real number $|q| = \sqrt{\sum_{i=0}^{3} a_i^2}$ is called the *norm* of q. Let $\mathbb{H}^*$ denote the multiplicative group of non-zero quaternions.

(i) Prove that $\mathbb{H}^*$ is a Lie group.

(ii) Consider the map ρ that defines a correspondence from each $p \in \mathbb{H}^*$ into the $\mathbb{R}$-linear automorphism of $\mathbb{H}$ defined by

$$\rho(p)\colon q \mapsto \rho(p) q = pq, \quad q \in \mathbb{H}.$$

Which is the representative matrix of $\rho(p)$ with respect to the given basis of $\mathbb{H}$? Compute its determinant.

(iii) Prove that ρ is a representation of $\mathbb{H}^*$ on $\mathbb{H} \equiv \mathbb{R}^4$.

(iv) Find the *group of inner automorphisms* $\mathrm{Int}\,\mathfrak{g}$ *of the Lie algebra* of $\mathbb{H}^*$.

Solution

(i) To prove that $\mathbb{H}^*$ is an abstract group is left to the reader. Given $q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}^*$, applying the multiplication rules $(\star)$, we obtain

$$q^{-1} = \frac{1}{|q|^2}(a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3) = \frac{\bar{q}}{|q|^2},$$

and then, for $p \in \mathbb{H}^*$, we have

$$\begin{aligned} qp^{-1} = \frac{1}{|p|^2}\big\{&(a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)e_0 \\ &+ (-a_0b_1 + a_1b_0 - a_2b_3 + a_3b_2)e_1 \\ &+ (-a_0b_2 + a_1b_3 + a_2b_0 - a_3b_1)e_2 \\ &+ (-a_0b_3 - a_1b_2 + a_2b_1 + a_3b_0)e_3\big\}. \end{aligned}$$

Thus the map $\mathbb{H}^* \times \mathbb{H}^* \to \mathbb{H}^*$, $(q, p) \mapsto qp^{-1}$, is C^∞, and hence $\mathbb{H}^*$ is a Lie group.

(ii) Let $q \in \mathbb{H}$, $p \in \mathbb{H}^*$, written as in (i). Then it is easy to obtain

$$\rho(q)p = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

so the above matrix is the matrix of $\rho(q)$ with respect to the basis $\{e_0, e_1, e_2, e_3\}$. We have $\det \rho(q) = |q|^2$.

(iii) A representation of $\mathbb{H}^*$ on $\mathbb{H} \equiv \mathbb{R}^4$ is a homomorphism from $\mathbb{H}^*$ to the group of automorphisms $\mathrm{GL}(4, \mathbb{R})$ of $\mathbb{R}^4$. Since $\det \rho(q) = |q|^2 \neq 0$, $\rho(q)$ is invertible. Thus ρ sends $\mathbb{H}^*$ to $\mathrm{GL}(4, \mathbb{R})$, and since $\rho(q^{-1})\rho(q)p = q^{-1}qp = p$, we have $\rho(q)^{-1} = \rho(q^{-1})$. Furthermore, we have

$$\rho\big(qq'\big)p = qq'p = \rho(q)\rho\big(q'\big)p,$$

that is, $\rho(qq') = \rho(q)\rho(q')$.

(iv) The group of inner automorphisms $\mathrm{Int}\,\mathfrak{g}$ of the Lie algebra of $\mathbb{H}^*$ is (see Definition on p. 153) the image of

$$\mathbb{H}^* \to \mathrm{Aut}\,\mathrm{Lie}\big(\mathbb{H}^*\big), \qquad q \mapsto \mathrm{Ad}_q,$$

where $\mathrm{Lie}(\mathbb{H}^*)$ stands for the Lie algebra of $\mathbb{H}^*$. We identify $\mathrm{Lie}(\mathbb{H}^*) \cong T_e\mathbb{H}^*$ to $\mathbb{H}$, and we consider the basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{H}$ above. Hence, the adjoint representation gives rise to a homomorphism $\mathbb{H}^* \to \mathrm{GL}(4, \mathbb{R})$, $q \mapsto \mathrm{Ad}_q$.

We claim that $\mathrm{Int}\,\mathfrak{g}$ is the subgroup $\mathrm{SO}(3)$ embedded in $\mathrm{GL}(4, \mathbb{R})$ as

$$\widetilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathrm{SO}(3).$$

Since

$$\mathrm{Ad}_q\, Y_e = R_{q^{-1}*}L_{q*}Y_e = (R_{q^{-1}} \circ L_q)_*Y_e, \quad Y \in \mathrm{Lie}\big(\mathbb{H}^*\big),$$

$\mathrm{Int}\,\mathfrak{g}$ is isomorphic to the group of matrices $(R_{q^{-1}} \circ L_q)_*$, $q \in \mathbb{H}$. Moreover, as R_q and L_q are linear maps on $\mathbb{H}$, we can identify $(R_{q^{-1}} \circ L_q)_*$ to $R_{q^{-1}} \circ L_q$, that is, $\mathrm{Ad}_q = R_{q^{-1}} \circ L_q$. With the same notations as above, we note that $L_q = \rho(q)$. Hence $\det L_q = |q|^2$. Similarly, it can be proved that $\det R_q = |q|^2$. Hence,

$$\det \mathrm{Ad}_q = (\det R_{q^{-1}})(\det L_q) = (\det R_q)^{-1} \det L_q = 1.$$

Therefore, $\mathrm{Int}\,\mathfrak{g}$ is contained in the special linear group $\mathrm{SL}(4, \mathbb{R})$. Let $\langle\,,\rangle$ denote the scalar product of vectors in $\mathbb{R}^4$. By using the formula for qp^{-1} in (i) we obtain $\langle q, p\rangle = |p|^2 \operatorname{Re}(qp^{-1})$, where $\operatorname{Re} q = \frac{1}{2}(q + \bar{q})$. Then, we have

$$\begin{aligned}
\langle \mathrm{Ad}_q p_1, \mathrm{Ad}_q p_2\rangle &= |\mathrm{Ad}_q p_2|^2 \operatorname{Re}\big(\mathrm{Ad}_q p_1 (\mathrm{Ad}_q p_2)^{-1}\big) \\
&= \big|qp_2q^{-1}\big|^2 \operatorname{Re}\big(qp_1p_2^{-1}q^{-1}\big) \\
&= |p_2|^2 \operatorname{Re}\big(p_1p_2^{-1}\big) = \langle p_1, p_2\rangle.
\end{aligned}$$

It follows that Ad_q is an isometry, and, consequently, it belongs to $\mathrm{O}(4)$. Furthermore, $\mathrm{Ad}_q e_0 = e_0$. Hence Ad_q leaves invariant the orthogonal subspace $\langle e_0\rangle^{\perp} = \langle e_1, e_2, e_3\rangle$. Accordingly, every Ad_q is a matrix of the form $\widetilde{A}$ above. Therefore, $\mathrm{Ad}_q \in \mathrm{SO}(3)$.

Moreover, the kernel of Ad is $\mathbb{R}^*$, the centre of $\mathbb{H}^*$. We have $\mathbb{H}^*/\mathbb{R}^+ \cong S^3 = \{q \in \mathbb{H} : |q| = 1\}$. Hence,

$$\mathbb{H}^*/\mathbb{R}^* \cong S^3/\{+1, -1\} = \mathbb{R}\mathrm{P}^3,$$

which is compact and connected. Accordingly, $\mathrm{Int}\,\mathfrak{g}$ is a compact, connected subgroup in $\mathrm{SO}(3)$. Hence it necessarily coincides with $\mathrm{SO}(3)$.

Problem 4.72 Prove that $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ and apply it to prove that $\mathrm{SU}(2)$ is a two-fold covering of $\mathrm{SO}(3)$.

Hint Apply that $\mathrm{Sp}(1)$ has centre $\mathbb{Z}_2 \cong \{\pm I\}$ and acts by conjugation on $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$.

One can find the relevant theory, for instance, in Ziller [15].

Solution It is easy to verify that any matrix of $\mathrm{SU}(2)$ has the form

$$M(a,b) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1.$$

Then the map $\mathrm{SU}(2) \to \mathrm{Sp}(1)$, $M(a,b) \mapsto q = a + b\mathbf{j}$, is an isomorphism.

We will now prove that one can see the adjoint representation of

$$\mathrm{Sp}(1) = \big\{q \in \mathbb{H} : |q| = 1\big\} = \big\{q \in \mathbb{H} : \bar{q} = q^{-1}\big\}$$

as the two-fold covering

$$\begin{aligned}
\varphi\colon \mathrm{Sp}(1) &\longrightarrow \mathrm{SO}(3) \\
q &\longmapsto \{v \mapsto qv\bar{q}\} \in \mathrm{SO}(\operatorname{Im}\mathbb{H}) \cong \mathrm{SO}(3).
\end{aligned}$$

In fact, since $|qv\bar{q}| = |v|$, the map $v \mapsto qv\bar{q}$ is an isometry of $\mathbb{H}$. Now, $(\varphi(q))(1) = 1$, and hence $\varphi(q)$ preserves the orthogonal complement $(\mathbb{R} \cdot 1)^{\perp} = \operatorname{Im}\mathbb{H}$. Moreover, $\varphi(q)$ lies in SO(3) as Sp(1) is connected. But if $q \in \ker\varphi$, then for the anticommuting imaginary units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of $\mathbb{H}$, we have $q\mathbf{i} = \mathbf{i}q$, $q\mathbf{j} = \mathbf{j}q$ and $q\mathbf{k} = \mathbf{k}q$, and, consequently, $q \in \mathbb{R}$, i.e. $q = \pm 1$. So that $\ker\varphi \cong \mathbb{Z}_2$ and the centre of Sp(1) is $\mathbb{Z}_2 = \{\pm 1\}$. Since both groups have dimension three, φ is a two-fold covering map. Hence $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$.

Problem 4.73 Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. If ad stands for the adjoint representation of $\mathfrak{g}$, that is, the differential of the adjoint representation $G \to \operatorname{Aut}\mathfrak{g}$, $s \mapsto \mathrm{Ad}_s$, prove:

(i)

$$(\exp \operatorname{ad} tX)(Y) = Y + t[X, Y] + \frac{t^2}{2!}\big[X, [X, Y]\big] + \cdots, \qquad X, Y \in \mathfrak{g}.$$

(ii)

$$\mathrm{Ad}_{\exp tX}(Y) = Y + t[X, Y] + \frac{t^2}{2!}\big[X, [X, Y]\big] + \cdots, \qquad X, Y \in \mathfrak{g}.$$

Solution

(i)

$$\begin{aligned}(\exp \operatorname{ad} tX)(Y) &= \left(I + \operatorname{ad} tX + \frac{1}{2!}(\operatorname{ad} tX)^2 + \cdots\right)(Y) \\ &= Y + [tX, Y] + \frac{1}{2!}\big[tX, [tX, Y]\big] + \cdots \\ &= Y + t[X, Y] + \frac{t^2}{2!}\big[X, [X, Y]\big] + \cdots .\end{aligned}$$

(ii) The expansion follows from the formula

$$\mathrm{Ad} \circ \exp = \exp \circ \operatorname{ad}$$

and (i) above.

Problem 4.74 Consider the Lie algebra $\mathfrak{g}$ with a basis $\{e_1, e_2, e_3\}$ having non-vanishing brackets

$$\begin{gathered}[e_1, e_3] = ae_1 + be_2, \qquad [e_2, e_3] = ce_1 + de_2, \\ ad - bc \neq 0,\ a^2 + d^2 + 2bc \neq 0.\end{gathered}$$

(i) Compute the ideal $[\mathfrak{g}, \mathfrak{g}]$. Is $\mathfrak{g}$ Abelian? Is $\mathfrak{g}$ solvable?

(ii) Compute ad_X for any $X = X^1 e_1 + X^2 e_2 + X^3 e_3$.
(iii) Compute $\mathrm{tr}(\mathrm{ad}_X)^2$. When is $\mathrm{tr}(\mathrm{ad}_X)^2 = 0$?

Solution

(i) $[\mathfrak{g}, \mathfrak{g}] = \langle e_1, e_2 \rangle$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, and thus $\mathfrak{g}$ is solvable but not Abelian.
(ii)

$$\mathrm{ad}_X = \begin{pmatrix} -aX^3 & -cX^3 & aX^1 + cX^2 \\ -bX^3 & -dX^3 & bX^1 + dX^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) $\mathrm{tr}(\mathrm{ad}_X)^2 = (a^2 + d^2 + 2bc)(X^3)^2$, and $\mathrm{tr}(\mathrm{ad}_X)^2 = 0$ only if $X \in [\mathfrak{g}, \mathfrak{g}]$.

Problem 4.75 Prove that the Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & -x & y \\ x & 0 & z \\ 0 & 0 & 0 \end{pmatrix},\ x, y, z \in \mathbb{R} \right\}$$

is solvable but not *completely solvable*.

Remark Completely solvable Lie algebras (or groups) over $\mathbb{R}$ are also called split solvable or real solvable Lie algebras (or groups), see Definition 4.6.

The relevant theory is developed, for instance, in Knapp [8].

Solution It is immediate that $\mathfrak{g}$ admits a basis

$$\left\{ e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

with non-zero brackets

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2.$$

As

$$\big[[e_1, e_2], [e_1, e_3]\big] = 0,$$

the Lie algebra is solvable.

Now, the eigenvalues of

$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

that is, the solutions of $\det(\lambda I - \mathrm{ad}_{e_1}) = 0$, are $\lambda = 0, \pm\mathrm{i}$. Since ad has some non-real eigenvalues, $\mathfrak{g}$ is not completely solvable.

Problem 4.76 Let $\mathfrak{g}$ be a two-dimensional complex Lie algebra with basis $\{X, Y\}$ such that $[X, Y] = Y$.

(i) Identify the *regular elements*.
(ii) Prove that $\mathbb{C}X$ is a *Cartan subalgebra* but that $\mathbb{C}Y$ is not.
(iii) Find the *weight-space decomposition* of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathbb{C}X$.

Hint (to (i)) Apply Definition 4.10.

Hint (to (ii)) Apply Proposition 4.7.

Remark Root-space decompositions of Lie algebras are usually described for either complex semi-simple Lie algebras or compact connected Lie groups. However, they also exist for the general case of complex Lie groups (see Knapp [8]). Notice that the example $\mathfrak{g}$ in this problem is not semi-simple; actually, it is solvable.

The relevant theory is developed, for instance, in Knapp [8].

Solution

(i) We have

$$\operatorname{ad}_X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{ad}_Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

so a generic element $zX + wY$, $z, w \in \mathbb{C}$, has the characteristic polynomial

$$\lambda^2 - z\lambda.$$

This has the lowest possible degree whenever $z \neq 0$. Hence, according to Definition 4.10, the regular elements are those in $\mathfrak{g} \setminus \langle Y \rangle$.

(ii) Both $\mathbb{C}X$ and $\mathbb{C}Y$ are Abelian and hence nilpotent. We are in the conditions of Proposition 4.9 below, so a given nilpotent subalgebra $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only $\mathfrak{h}$ coincides with its own normaliser,

$$N_{\mathfrak{g}}(\mathfrak{h}) = \left\{ A \in \mathfrak{g} : [A, \mathfrak{h}] \subseteq \mathfrak{h} \right\}.$$

Since $[X, Y] = Y$, the conclusion is immediate.

(iii) From (ii) one has that the decomposition is $\mathfrak{g} = \mathbb{C}X \oplus \mathbb{C}Y$ with $\mathbb{C}Y$ the (generalised) weight space for the linear functional α such that $\alpha(X) = 1$.

Problem 4.77 Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be a root-space decomposition for a complex semi-simple Lie algebra, and let Δ' be a subset of Δ that forms a *root system* in the $\mathbb{R}$-linear span of all $\alpha \in \Delta'$ (see Theorem 4.17 and Definition 4.18).

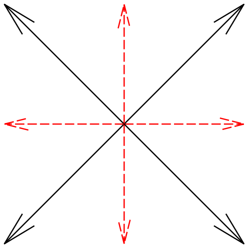


Fig. 4.3 The root system of $\mathrm{SO}(5,\mathbb{C})$. *Dashed*, the subsystem Δ'

(i) Show by example that $\mathfrak{g}' = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha$ need not be a subalgebra of $\mathfrak{g}$.

(ii) Suppose that $\Delta' \subseteq \Delta$ is a root subsystem with the following property. Whenever α and β are in Δ' and $\alpha+\beta$ is in Δ, then $\alpha+\beta$ is in Δ'. Prove that

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{h}' = \sum_{\alpha\in\Delta'} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$$

is a subalgebra of $\mathfrak{g}$ and that it is semi-simple.

Hint (to (i)) Take $\mathfrak{g} = \mathfrak{so}(5,\mathbb{C})$ and the subset $\Delta' = \{\pm\varepsilon_1, \pm\varepsilon_2\}$ of the root system of $\mathfrak{so}(5,\mathbb{C})$ (see Table on p. 562 and Fig. 4.3).

Hint (to (ii)) Apply the properties of a Cartan decomposition of a complex semi-simple Lie algebra in Theorem 4.17 and Cartan's Criterion for Semisimplicity 4.13.

The relevant theory is developed, for instance, in Helgason [7].

Solution

(i) We take, as in the hint, the given subset Δ' of the root system of $\mathfrak{so}(5,\mathbb{C})$. Let us first check that Δ' is an abstract root system (see Definition 4.18). Let V be the vector space spanned by the roots in Δ, then it is clear that Δ' also spans V.

Moreover, given the symmetry of Δ', it suffices to choose an element, say $\varepsilon_1 \in \Delta'$, and prove that the orthogonal transformation

$$s_{\varepsilon_1}(\beta) = \beta - \frac{2\langle\beta,\varepsilon_1\rangle}{|\varepsilon_1|^2}\varepsilon_1, \quad \beta \in \Delta',$$

carry Δ' to itself. Now, this transformation is the reflection of Δ' on the axis orthogonal to ε_1.

Furthermore, all the quotients $\frac{2\langle\beta,\gamma\rangle}{|\gamma|^2}$, $\beta,\gamma\in\Delta'$, equal either 0 or ± 2, anyway an integer.

So Δ' is in fact a root system.

On the other hand, in the Lie algebra $\mathfrak{g} = \mathfrak{so}(5,\mathbb{C})$ we have, for instance, that

$$[\mathfrak{g}_{\varepsilon_1}, \mathfrak{g}_{\varepsilon_2}] = \mathfrak{g}_{\varepsilon_1+\varepsilon_2}$$

because $\varepsilon_1+\varepsilon_2 \in \Delta$ (see Theorem 4.17). But $\mathfrak{g}_{\varepsilon_1+\varepsilon_2} \not\subset \mathfrak{g}'$ because $\varepsilon_1+\varepsilon_2 \notin \Delta'$, i.e. $\mathfrak{g}'$ is not a subalgebra of $\mathfrak{g}$.

(ii) That

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha$$

is a subalgebra of $\mathfrak{g}$ is immediate on account of the additional property in the statement and Proposition 4.7.

To see that $\mathfrak{g}'$ is semi-simple, if suffices to prove that the Killing form of the Lie algebra $\mathfrak{g}'$ is non-degenerate. For this, let $B_\mathfrak{g}$ and $B_{\mathfrak{g}'}$ denote the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively.

For each pair $\alpha, -\alpha \in \Delta$, select vectors $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $B_\mathfrak{g}(X_\alpha, X_{-\alpha}) = 1$ (by Proposition 4.7 and by Theorem 4.14 the form $B_\mathfrak{g}$ is non-degenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$). Consider two elements in $\mathfrak{g}'$,

$$X_i = h_i + \sum_{\alpha\in\Delta'} a^i_\alpha X_\alpha, \quad \text{where } h_i \in \mathfrak{h}' \subset \mathfrak{h},\ i = 1, 2.$$

Since by Proposition 4.7 we have $[\mathfrak{g}^\lambda, [\mathfrak{g}^\mu, \mathfrak{g}^\nu]] \subset \mathfrak{g}^{\lambda+\mu+\nu}$ for any $\lambda, \mu, \nu \in \mathfrak{h}^*$ ($\mathfrak{g}^0 = \mathfrak{h}$) and by Theorem 4.14 one has $\mathfrak{g}^\lambda = \mathfrak{g}_\lambda$, we obtain that

$$B_{\mathfrak{g}'}(X_\alpha, X_\beta) = \operatorname{tr}_{\mathfrak{g}'}(\operatorname{ad}_{X_\alpha} \circ \operatorname{ad}_{X_\beta}) = 0 \quad \text{if } \alpha, \beta \in \Delta',\ \alpha+\beta \neq 0,$$

and $B_{\mathfrak{g}'}(\mathfrak{h}', \mathfrak{g}_\alpha) = 0$ if $\alpha \neq 0$. Hence,

$$B_{\mathfrak{g}'}(X_1, X_2) = B_{\mathfrak{g}'}(h_1, h_2) + \sum_{\alpha\in\Delta'} \big(a^1_\alpha a^2_{-\alpha} + a^1_{-\alpha} a^2_\alpha\big) B_{\mathfrak{g}'}(X_\alpha, X_{-\alpha}). \qquad (\star)$$

Now, on account of Theorem 4.17, one has

$$\operatorname{ad}_{X_{-\alpha}} \operatorname{ad}_{X_\alpha}(X_\beta) = \frac{q_{\alpha,\beta}(1-p_{\alpha,\beta})}{2} \alpha(h_\alpha)(a_\beta X_\beta),$$
$$\operatorname{ad}_{X_{-\alpha}} \operatorname{ad}_{X_\alpha}(h) = \alpha(h)[X_\alpha, X_{-\alpha}] = \alpha(h) h_\alpha \quad \text{for any } h \in \mathfrak{h}.$$

Since by definition $q_{\alpha,\beta} \geqslant 0$ and $p_{\alpha,\beta} \leqslant 0$, and moreover $\alpha(h_\alpha) = B_\mathfrak{g}(h_\alpha, h_\alpha) > 0$, we obtain that

$$B_{\mathfrak{g}'}(X_{-\alpha}, X_\alpha) = \alpha(h_\alpha) + \sum_{\beta\in\Delta'} \frac{q_{\alpha,\beta}(1-p_{\alpha,\beta})}{2} \alpha(h_\alpha) > 0.$$

Since each endomorphism $\operatorname{ad}_{h_i} : \mathfrak{g}' \to \mathfrak{g}'$ is semi-simple with eigenvalues $\beta(h_i)$, $\beta \in \Delta'$ and 0, we have

$$B_{\mathfrak{g}'}(h_1, h_2) = \sum_{\beta\in\Delta'} \beta(h_1)\beta(h_2).$$

Suppose that $B_{\mathfrak{g}'}(X_1, X_2) = 0$ for all $X_2 \in \mathfrak{g}'$. Then $B_{\mathfrak{g}'}(X_1, h_2) = 0$, so

$$B_{\mathfrak{g}'}(h_1, h_2) = 0 \quad \forall h_2 \in \mathfrak{h}' = \sum_{\alpha \in \Delta'} \mathbb{R} h_\alpha.$$

Hence $\alpha(h_1) = 0$ for each $\alpha \in \Delta'$, that is, $B_{\mathfrak{g}}(h_1, h_\alpha) = 0$ for all $\alpha \in \Delta'$, thus $h_1 = 0$, and formula ($\star$) reduces to

$$0 = 0 + \sum_{\alpha \in \Delta'} \left(a_\alpha^1 a_{-\alpha}^2 + a_{-\alpha}^1 a_\alpha^2\right) B_{\mathfrak{g}'}(X_\alpha, X_{-\alpha})$$

with $B_{\mathfrak{g}'}(X_\alpha, X_{-\alpha}) > 0$, which can be further reduced taking the sum on the positive roots in Δ' to

$$0 = 2 \sum_{\alpha \in (\Delta')^+} \left(a_\alpha^1 a_{-\alpha}^2 + a_{-\alpha}^1 a_\alpha^2\right) B_{\mathfrak{g}'}(X_\alpha, X_{-\alpha}).$$

As this happens for any $a_{-\alpha}^2, a_\alpha^2$, with $\alpha \in (\Delta')^+$, it follows that $a_{-\alpha}^1 = a_\alpha^1 = 0$ for all $\alpha \in (\Delta')^+$, and we conclude that $X_1 = 0$.

Problem 4.78 With the terminology and notations in Definitions 4.28:

(i) Prove that the roots of the general linear group $\mathrm{GL}(4, \mathbb{C})$ are

$$\pm(\varepsilon_1 - \varepsilon_2), \quad \pm(\varepsilon_1 - \varepsilon_3), \quad \pm(\varepsilon_1 - \varepsilon_4), \quad \pm(\varepsilon_2 - \varepsilon_3),$$
$$\pm(\varepsilon_2 - \varepsilon_4), \quad \pm(\varepsilon_3 - \varepsilon_4),$$

each with multiplicity one.

(ii) Prove that the roots of the symplectic group $\mathrm{Sp}(\mathbb{C}^4, \Omega)$ are

$$\pm(\varepsilon_1 - \varepsilon_2), \quad \pm(\varepsilon_1 + \varepsilon_2), \quad \pm 2\varepsilon_1, \quad \pm 2\varepsilon_2,$$

each with multiplicity one.

(iii) Prove that the roots of the special orthogonal group $\mathrm{SO}(\mathbb{C}^5, B)$ are

$$\pm(\varepsilon_1 - \varepsilon_2), \quad \pm(\varepsilon_1 + \varepsilon_2), \quad \pm\varepsilon_1, \quad \pm\varepsilon_2,$$

each with multiplicity one.

(iv) Why $2\varepsilon_i$, $i = 1, 2$, are roots of $\mathrm{Sp}(\mathbb{C}^4, \Omega)$ but not of $\mathrm{SO}(\mathbb{C}^4, B)$?

Remark We follow in this problem (see below from Definitions 4.25 to Definitions 4.28) the terminology and notations of [6], which have the advantage that each of the corresponding diagonal subgroups is a maximal torus.

The relevant theory is developed in Goodman and Wallach [6].

Solution

(i) Let E^i_j be the matrix with (i, j)th entry equal to 1 and zero elsewhere. For $A = \mathrm{diag}(a_1, \ldots, a_4) \in \mathfrak{h}$, we have

$$\left[A, E^i_j\right] = \left[\mathrm{diag}(a_1, a_2, a_3, a_4), E^i_j\right] = (a_i - a_j)E^i_j = \langle \varepsilon_i - \varepsilon_j, A\rangle E^i_j.$$

Since the set $\{E^i_j\}$, $i, j \in \{1, \ldots, 4\}$, is a basis of $\mathfrak{g} = \mathfrak{gl}(4, \mathbb{R}) = M(4, \mathbb{C})$, the roots are in fact the ones in the statement, each with multiplicity 1.

(ii) Label the basis for $\mathbb{C}^4$ as $\{e_1, e_2, e_{-2}, e_{-1}\}$. Consider E^i_j for $i, j \in \{\pm 1, \pm 2\}$. Set

$$X_{\varepsilon_1-\varepsilon_2} = E^1_2 - E^{-2}_{-1}, \qquad X_{\varepsilon_2-\varepsilon_1} = E^2_1 - E^{-1}_{-2}, \qquad X_{\varepsilon_1+\varepsilon_2} = E^1_{-2} + E^2_{-1},$$

$$X_{-\varepsilon_1-\varepsilon_2} = E^{-2}_1 + E^{-1}_2, \qquad X_{2\varepsilon_1} = E^1_{-1}, \qquad X_{-2\varepsilon_1} = E^{-1}_1,$$

$$X_{2\varepsilon_2} = E^2_{-2}, \qquad X_{-2\varepsilon_2} = E^{-2}_2.$$

Then, for $A = \mathrm{diag}(a_1, a_2, a_{-2}, a_{-1}) \in \mathfrak{h}$, one has

$$[A, X_{\varepsilon_i-\varepsilon_j}] = \langle \varepsilon_i - \varepsilon_j, A\rangle X_{\varepsilon_i-\varepsilon_j}, \qquad [A, X_{\pm(\varepsilon_i+\varepsilon_j)}] = \pm\langle \varepsilon_i + \varepsilon_j, A\rangle X_{\varepsilon_i+\varepsilon_j}.$$

Hence the elements in $\mathfrak{h}^*$ in the statement are roots of $\mathfrak{sp}(\mathbb{C}^4, \Omega)$. Now,

$$\{X_{\pm(\varepsilon_1-\varepsilon_2)}, X_{\pm(\varepsilon_1+\varepsilon_2)}, X_{\pm 2\varepsilon_1}, X_{\pm 2\varepsilon_2}\}$$

is a basis for $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ mod $\mathfrak{h}$. So the given roots are all of the roots, each with multiplicity one.

(iii) We embed $\mathrm{SO}(\mathbb{C}^4, B)$ into $\mathrm{SO}(\mathbb{C}^5, B)$ by using the map (4.2) for $r = 2$. Since $H \subset \mathrm{SO}(\mathbb{C}^4, B) \subset \mathrm{SO}(\mathbb{C}^5, B)$ via this embedding, the roots $\pm\varepsilon_1 \pm \varepsilon_2$ of $\mathrm{ad}_{\mathfrak{h}}$ on $\mathfrak{so}(\mathbb{C}^4, B)$ also occur for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^5, B)$. Label the basis of $\mathbb{C}^5$ as $\{e_{-2}, e_{-1}, e_0, e_1, e_2\}$. Consider E^i_j for $i, j \in \{0, \pm 1, \pm 2\}$. Then one can prove that the root vectors from $\mathrm{SO}(\mathbb{C}^4, B)$ are

$$X_{\varepsilon_1-\varepsilon_2} = E^1_2 - E^{-2}_{-1}, \qquad X_{\varepsilon_2-\varepsilon_1} = E^2_1 - E^{-1}_{-2},$$

$$X_{\varepsilon_1+\varepsilon_2} = E^1_{-2} - E^2_{-1}, \qquad X_{-\varepsilon_1-\varepsilon_2} = E^{-2}_1 - E^{-1}_2.$$

Define

$$X_{\varepsilon_1} = E^1_0 - E^0_{-1}, \qquad X_{\varepsilon_2} = E^2_0 - E^0_{-2}, \qquad X_{-\varepsilon_1} = E^0_1 - E^{-1}_0,$$

$$X_{-\varepsilon_2} = E^0_2 - E^{-2}_0.$$

Then we have $X_{\pm\varepsilon_i} \in \mathfrak{g}$, $i = 1, 2$, and

$$[A, X_{\pm\varepsilon_i}] = \pm\langle \varepsilon_i, A\rangle X_{\varepsilon_i}, \qquad A \in \mathfrak{h}.$$

As $\{X_{\pm\varepsilon_i}\}$, $i = 1, 2$, is a basis for $\mathfrak{g}$ mod $\mathfrak{so}(\mathbb{C}^4, B)$, one concludes that the roots of $\mathfrak{so}(\mathbb{C}^5, B)$ are the ones in the statement, each with multiplicity one.

(iv) Both $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ and $\mathfrak{so}(\mathbb{C}^4, B)$ have the same subalgebra of diagonal matrices $\operatorname{diag}(r, s, -s, -r)$, which give rise in both cases to the roots $-2\varepsilon_i$, $i = 1, 2$. For instance,

$$\left[\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & -s & 0 \\ 0 & 0 & 0 & -r \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2r & 0 & 0 & 0 \end{pmatrix}.$$

However, the non-zero skew diagonal matrices

$$\begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & s & 0 \\ 0 & t & 0 & 0 \\ u & 0 & 0 & 0 \end{pmatrix}$$

exist in $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ and originate the roots $2\varepsilon_i$, $i = 1, 2$; but those matrices do not exist in $\mathfrak{so}(\mathbb{C}^4, B)$.

Problem 4.79 Show that the group of diagonal matrices

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}, \qquad \lambda_i = \pm 1, \ \lambda_1 \cdot \dots \cdot \lambda_n = 1,$$

is a *maximal Abelian subgroup* of $\mathrm{SO}(n)$ but is not a torus.

The relevant theory is developed, for instance, in Adams [1] or Bröcker and tom Dieck [3].

Solution Since for any such matrix A, one has $A^t A = 1$ and $\det A = 1$, that set of matrices is in fact a subgroup of $\mathrm{SO}(n)$. As the matrices are diagonal, the subgroup is Abelian. It is obviously maximal but not a torus, due to the definition of a torus (see Definitions 4.28).

Problem 4.80 Classify all the reduced root systems on $\mathbb{R}^2$.

Hint Apply Definition 4.18 and Proposition 4.19, and multiply each side of (iii) and (iv) in that proposition.

The relevant theory is developed, for instance, in Bröcker and tom Dieck [3].

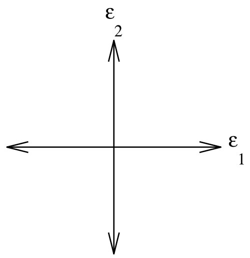


Fig. 4.4 The root system $\mathfrak{a}_1 \oplus \mathfrak{a}_1$

Solution Let $(\varepsilon_1, \varepsilon_2)$ be the usual orthonormal basis of $\mathbb{R}^2$. Since any set of roots spans $\mathbb{R}^2 = \langle \varepsilon_1, \varepsilon_2 \rangle$ and, by Proposition 4.19(i), the opposite of a root is a root, any root system should have at least a set of roots $\alpha, -\alpha, \beta, -\beta$, with α, β linearly independent.

By Proposition 4.19(iii), the possible values for the cosine of the angle between $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$ are

$$\cos\theta = \frac{|\alpha|}{|\beta|}\left\{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}\right\}. \tag{$\star$}$$

Suppose that $|\alpha| \leqslant |\beta|$. Then from Proposition 4.19(iv) we have moreover that

$$\cos\theta = \frac{|\beta|}{|\alpha|}\left\{0, \pm\frac{1}{2}\right\}. \tag{$\star\star$}$$

Multiplying each side of ($\star$) and ($\star\star$), one gets, as possible values of $\cos\theta$,

$$\cos\theta = \left\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}\right\},$$

that is, for $\theta \in [0, 2\pi]$, the possible values of θ are

$$\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}.$$

It is customary and useful, and supposes no restriction, to take $\langle \alpha, \beta \rangle \leqslant 0$. Notice that it suffices to change α by $-\alpha$ if necessary. This reduces the possibilities to

$$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}.$$

Consider first the case $\theta = \frac{\pi}{2}$. Then $\langle \alpha, \beta \rangle = 0$, and we can take

$$\Delta = \{\pm\varepsilon_1, \pm\varepsilon_2\},$$

which is (see Fig. 4.4) the root system $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ corresponding to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$.

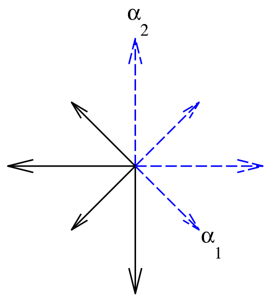


Fig. 4.5 The root system Δ of $\mathfrak{sp}(2, \mathbb{C})$. *Dashed*, the positive roots. *Labeled*, the simple roots

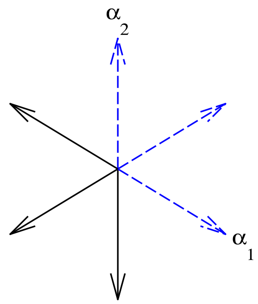


Fig. 4.6 The root system of $\mathrm{SL}(3, \mathbb{C})$. *Dashed*, the positive roots. *Labeled*, the simple roots

Suppose now that $\theta = \frac{2\pi}{3}$ and take $|\alpha| = 1$. Then from ($\star\star$) we have $|\beta| = 1$, so on account of Proposition 4.19(v), the corresponding root system is

$$\left\{\pm\varepsilon_1, \pm\left(\frac{1}{2}\varepsilon_1 \pm \frac{\sqrt{3}}{2}\varepsilon_2\right)\right\},$$

which is (see Fig. 4.6) the root system $\mathfrak{a}_2$ corresponding to $\mathrm{SL}(3, \mathbb{C})$.

Let now $|\alpha| = 1$ and $\theta = \frac{3\pi}{4}$. Then from ($\star\star$) it follows that

$$\frac{\sqrt{2}}{2} = |\beta|\frac{1}{2},$$

i.e. $|\beta| = \sqrt{2}$, so the corresponding root system is

$$\{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\},$$

which corresponds to either (see Fig. 4.3) the root system $\mathfrak{b}_2$ of the simple group $\mathrm{SO}(5, \mathbb{C})$ or (see Fig. 4.5) to its isomorphic root system $\mathfrak{c}_2$ of the simple group $\mathrm{Sp}(2, \mathbb{C})$.

Consider finally $|\alpha| = 1$ and $\theta = \frac{5\pi}{6}$; then from ($\star\star$) it follows that

$$\frac{\sqrt{3}}{2} = |\beta|\frac{1}{2},$$

that is, $|\beta| = \sqrt{3}$. Due to the fact that a root system contains, for any root, its opposite and their symmetric roots under reflections, we have that the corresponding root system is

$$\left\{\pm\varepsilon_1, \pm\left(\frac{1}{2}\varepsilon_1 \pm \frac{\sqrt{3}}{2}\varepsilon_2\right), \pm\left(\sqrt{3}\varepsilon_1 \pm \frac{\sqrt{3}}{2}\varepsilon_2\right), \pm\sqrt{3}\varepsilon_2\right\},$$

which is (see Fig. 4.7) the root system $\mathfrak{g}_2$ corresponding to the simple group $G_2^{\mathbb{C}}$.

Problem 4.81 Consider the next simple compact connected Lie groups:

1. SU(3).
2. SO(5).
3. Sp(2).
4. G_2.

Find in each case, by using the tables on p. 562:

(a) The *semisum* ρ *of positive roots*.
(b) The order $|W|$ of the *Weyl group* W for the corresponding root system and the *inner automorphisms of the Lie algebra generating* W.
(c) The *Cartan matrix*.
(d) The *Dynkin diagram*.

Hint (to 1(a)–4(a)) See Definition in 4.20.

Hint (to 1(b)–4(b)) Apply Theorem 4.24 on the properties of the Weyl group and Theorem 4.29 on the properties satisfied by the entries of the matrices of the elements of the Lie subalgebra $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ of G_2.

Hint (to 1(c,d)–4(c,d)) See Definitions 4.21.

The relevant theory is developed, for instance, in Adams [1].

Solution

1. Since SU(3) is a (compact) real form of $SL(3,\mathbb{C})$ (see p. 559), the root system Δ of SU(3) is (see p. 160) of type $\mathfrak{a}_n$ in the table in p. 562, for $n = 2$. That is, that given in Fig. 4.6, i.e.

$$\Delta = \left\{\pm(\varepsilon_i - \varepsilon_j);\ i, j = 1, 2, 3,\ i \neq j\right\}.$$

(Notice that the vertices of the corresponding hexagon lye on a plane in the three-dimensional vector space $V = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ with orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.) The subset of positive roots is

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3\},$$

and hence

$$\rho = \varepsilon_1 - \varepsilon_3.$$

On the other hand, according to the table in p. 562, the set of simple roots is given by

$$\Pi = \{\alpha_1, \alpha_2\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}.$$

To compute $|W|$, recall that the Weyl group W is generated by the reflections with respect to the hyperplanes $\mathbf{p}_1$ and $\mathbf{p}_2$ (the lines if $n = 2$) orthogonal to the vectors α_1 and α_2 in the two-dimensional space $V_2 = \langle\alpha_1, \alpha_2\rangle \subset V$. Since an orientation of V_2 is not preserved by a reflection (because any reflection has determinant -1), the Weyl group W contains a subgroup W' of index two (i.e. $|W/W'| = 2$). But each orthogonal transformation of $V_2 = \mathbb{R}^2$ preserving an orientation is a rotation. Taking into account that the angle between the planes $\mathbf{p}_1$ and $\mathbf{p}_2$ equals $\frac{\pi}{3}$ (see Problem 4.80), we obtain that the group W' is generated by the rotation of angle $\frac{2\pi}{3}$. Therefore $|W'| = 3$ and $|W| = 6$.

To describe the inner automorphisms of the Lie algebra $\mathfrak{su}(3)$ generating the Weyl group W, recall that the Lie algebra $\mathfrak{su}(3)$ consists (see the table on p. 560) of the traceless skew-Hermitian 3×3 matrices and define the inner product $\langle\, , \rangle$ on $\mathfrak{su}(3)$ by

$$\langle X, Y\rangle = \frac{1}{4\pi^2}\operatorname{tr}\bigl({}^t\bar{X}Y\bigr) = -\frac{1}{4\pi^2}\operatorname{tr} XY,$$

which is clearly SU(3)-invariant. Moreover, restricted to the Lie algebra

$$\mathfrak{t} = \left\{\begin{pmatrix} 2\pi \mathrm{i} x_1 & 0 & 0 \\ 0 & 2\pi \mathrm{i} x_2 & 0 \\ 0 & 0 & 2\pi \mathrm{i} x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}, x_1 + x_2 + x_2 = 0\right\}$$

of a maximal torus T (see, for instance, [1, p. 83], [3, p. 170]), this inner product has the form

$$x_1^2 + x_2^2 + x_3^2.$$

Since this is the usual inner product, reflection is the usual reflection.

It is easy to verify that the matrix $E_j^k \in \mathfrak{sl}(3, \mathbb{C})$, $1 \leqslant k, j \leqslant 3, k \neq j$, containing a unique non-zero element (which is equal to 1) in the kth row and the jth column is a root vector with respect the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{sl}(3, \mathbb{C})$:

$$\bigl[E_j^k, \operatorname{diag}(2\pi \mathrm{i} x_1, 2\pi \mathrm{i} x_2, 2\pi \mathrm{i} x_3)\bigr] = (2\pi \mathrm{i} x_k - 2\pi \mathrm{i} x_j)E_j^k.$$

Note also that this root vector is not an element of the compact Lie algebra $\mathfrak{su}(3) \subset \mathfrak{sl}(3, \mathbb{C})$. The corresponding root equals $\varepsilon_k - \varepsilon_j$, where by definition

$$\varepsilon_1 = \operatorname{diag}(2\pi \mathrm{i}, 0, 0), \qquad \varepsilon_2 = \operatorname{diag}(0, 2\pi \mathrm{i}, 0), \qquad \varepsilon_3 = \operatorname{diag}(0, 0, 2\pi \mathrm{i})$$

(here the restriction to $\mathfrak{t}$ of each root belonging to the dual space of the complex Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{sl}(3, \mathbb{C})$ is identified with the element of the real Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ using the inner product on $\mathfrak{t}$).

The hyperplane of $V = \{(x_1, x_2, x_3)\} = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ orthogonal to the root $\varepsilon_1 - \varepsilon_2$ is the plane $x_1 = x_2$. The reflection with respect to this plane is given by

$$x_1' = x_2, \qquad x_2' = x_1, \qquad x_3' = x_3.$$

This is induced by an inner automorphism, namely by conjugation with an element of SU(3). In fact, we can write, omitting the factors 2π,

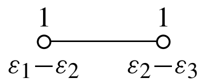
$$\begin{pmatrix} \mathrm{i}x_2 & 0 & 0 \\ 0 & \mathrm{i}x_1 & 0 \\ 0 & 0 & \mathrm{i}x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i}x_1 & 0 & 0 \\ 0 & \mathrm{i}x_2 & 0 \\ 0 & 0 & \mathrm{i}x_3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As this happens similarly for the other roots, the Weyl group contains the symmetric group $\mathfrak{S}_3$ on $\varepsilon_1, \varepsilon_2, \varepsilon_3$. But since $|W| = 6 = 3!$, W is this symmetric group.

On account of the set of simple roots Π of $\mathfrak{so}(3)$, the Cartan matrix is given by

$$C = \begin{pmatrix} \frac{2\langle \alpha_1, \alpha_1 \rangle}{|\alpha_1|^2} & \frac{2\langle \alpha_1, \alpha_2 \rangle}{|\alpha_1|^2} \\ \frac{2\langle \alpha_2, \alpha_1 \rangle}{|\alpha_2|^2} & \frac{2\langle \alpha_2, \alpha_2 \rangle}{|\alpha_2|^2} \end{pmatrix} = \begin{pmatrix} \frac{2\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}{|\varepsilon_1 - \varepsilon_2|^2} & \frac{2\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3 \rangle}{|\varepsilon_1 - \varepsilon_2|^2} \\ \frac{2\langle \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_2 \rangle}{|\varepsilon_1 - \varepsilon_3|^2} & \frac{2\langle \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_3 \rangle}{|\varepsilon_1 - \varepsilon_3|^2} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 2\cos\frac{2\pi}{3} \\ 2\cos\frac{2\pi}{3} & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the Dynkin diagram by



2. Since SO(5) is a (compact) real form of SO(5, $\mathbb{C}$) (see p. 559), the root system Δ of SO(5) is (see p. 160) of type $\mathfrak{b}_n$ in the table on p. 562, for $n = 2$. That is, that given in Fig. 4.3, i.e.

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j;\ i, j = 1, 2,\ i \neq j\} \cup \{\pm\varepsilon_i;\ i = 1, 2\},$$

and the set of positive roots is

$$\Delta^+ = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_1, \varepsilon_2\},$$

so

$$\rho = \frac{1}{2}(3\varepsilon_1 + \varepsilon_2).$$

According to the table on p. 562, the set of simple roots is given by

$$\Pi = \{\alpha_1, \alpha_2\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2\}.$$

To compute $|W|$ as above, it is sufficient to compute $|W'|$, where W' is the subgroup of index two in W generated by the rotations. Taking into account that the angle between the planes $\mathbf{p}_1$ and $\mathbf{p}_2$ (defined by α_1 and α_2 in $\langle \alpha_1, \alpha_2 \rangle$) in this case equals $\frac{\pi}{4}$ (see Problem 4.80), we obtain that the group W' is generated by the rotation of angle $\frac{\pi}{2}$. Therefore $|W'| = 4$ and $|W| = 8$.

To describe the inner automorphisms of the Lie algebra $\mathfrak{so}(5)$ generating the Weyl group W, recall that the Lie algebra $\mathfrak{so}(5)$ consists (see table on p. 560) of the skew-symmetric real 5×5 matrices and define the inner product $\langle\, , \rangle$ on $\mathfrak{so}(5)$ by

$$\langle X, Y \rangle = \frac{1}{8\pi^2} \operatorname{tr}\bigl({}^t XY\bigr) = -\frac{1}{8\pi^2} \operatorname{tr} XY,$$

which is clearly SO(5)-invariant. Moreover, restricted to the Lie algebra

$$\mathfrak{t} = \left\{ A(x_1, x_2) = \begin{pmatrix} 0 & -2\pi x_1 & 0 & 0 & 0 \\ 2\pi x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\pi x_2 & 0 \\ 0 & 0 & 2\pi x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ x_1, x_2 \in \mathbb{R} \right\}$$

of a maximal torus T (see, for instance, [1, p. 89], [3, p. 171]), it has the form

$$x_1^2 + x_2^2.$$

Since this is the usual inner product, reflection is the usual reflection.

We can find as above the root vectors of the complex Lie algebra $\mathfrak{so}(5, \mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{so}(5, \mathbb{C})$ (corresponding to the roots from Δ) to prove that $\varepsilon_1 = A(1, 0)$ and $\varepsilon_2 = A(0, 1)$. But since in this case the expressions for the root vectors are more complicated, here we only present directly the inner automorphisms of $\mathfrak{so}(5)$ that induce the reflections corresponding to the simple roots α_1 and α_2 (see Theorem 4.24).

Put $\varepsilon_1' = A(1, 0)$ and $\varepsilon_2' = A(0, 1)$. In fact, we can write, omitting the factors 2π,

$$\begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that this inner automorphism of $\mathfrak{so}(5)$, namely the conjugation by the given element of the Lie group SO(5), acts in the space $V = \{(x_1, x_2)\} = \langle \varepsilon_1', \varepsilon_2' \rangle$

as the map given by

$$x'_1 = x_2, \qquad x'_2 = x_1,$$

i.e. as the reflection with respect to the line (hyperplane) $x_1 = x_2$ orthogonal in V to the vector $\varepsilon'_1 - \varepsilon'_2 \in V$.

Next we can write, again omitting the factors 2π,

$$\begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is an inner automorphism of $\mathfrak{so}(5)$ that induces in V a reflection given by

$$x'_1 = x_1, \qquad x'_2 = -x_2,$$

i.e. the reflection with respect to the hyperplane of V orthogonal to the vector ε'_2.

As this happens similarly for the other analogous roots in each case, the Weyl group consists of the transformations

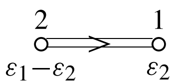
$$\varepsilon_1 \mapsto \pm\varepsilon_1, \qquad \varepsilon_2 \mapsto \pm\varepsilon_2 \quad \text{and} \quad \varepsilon_1 \mapsto \pm\varepsilon_2, \qquad \varepsilon_2 \mapsto \pm\varepsilon_1.$$

Note also that by Theorem 4.24 the simple roots α_1, α_2 are proportional to the vectors $\varepsilon'_1 - \varepsilon'_2$ and ε'_2 because $\angle(\varepsilon'_1 - \varepsilon'_2, \varepsilon'_2) = \frac{3\pi}{4}$ (see Problem 4.80). Taking into account that the bases $\varepsilon'_1, \varepsilon'_2$ and $\varepsilon_1, \varepsilon_2$ are orthonormal and $|\alpha_1| = \sqrt{2}|\alpha_2|$, $\angle(\alpha_1, \alpha_2) = \frac{3\pi}{4}$, we conclude that $\varepsilon_1 = \pm\varepsilon'_1$ and $\varepsilon_2 = \pm\varepsilon'_2$.

It is easy to verify that the Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and the Dynkin diagram by



3. Since $\mathrm{Sp}(2)$ is a (compact) real form of $\mathrm{Sp}(2, \mathbb{C})$ (see p. 559), the root system Δ of $\mathrm{Sp}(2)$ is (see p. 160) of type $\mathfrak{c}_n$ in the table on p. 562, for $n = 2$. That is, that given in Fig. 4.5, i.e.

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \; ; \; i, j = 1, 2, i \neq j\} \cup \{\pm 2\varepsilon_i \; ; \; i = 1, 2\},$$

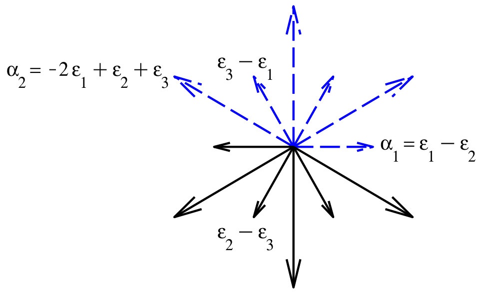


Fig. 4.7 The root system of G_2. *Dashed*, the positive roots. *Labeled* as α_1, α_2, the simple roots

and the set of positive roots is

$$\Delta^+ = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\};$$

hence,

$$\rho = 2\varepsilon_1 + \varepsilon_2.$$

According to the table on p. 560, the set of simple roots is

$$\Pi = \{\alpha_1, \alpha_2\} = \{\varepsilon_1 - \varepsilon_2, 2\varepsilon_2\}.$$

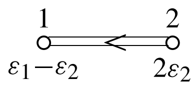
As for the order of the Weyl group and its action on the Cartan subalgebra, note that the root systems of Sp(2) and SO(5) are isomorphic (compare Figs. 4.5 and 4.3) and note also that actually, the Lie group Sp(2) is isomorphic to the universal covering group Spin(5) of SO(5) (see, for instance, Adams [1, Proposition 5.1] and Postnikov [11, Lect. 13]). Thus, from (ii) above we have that

$$|W| = 8.$$

It is easy to verify that the Cartan matrix is given by

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

and the Dynkin diagram by



4. Since G_2 is a (compact) real form of $G_2^{\mathbb{C}}$ (see p. 559), the root system Δ of G_2 is (see p. 160) of type $\mathfrak{g}$ in the table on p. 562. That is, that given in Fig. 4.7, i.e.

$$\Delta = \big\{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\big\}$$
$$\cup \big\{\pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\big\}.$$

(Notice that the corresponding twelve points lye on a plane in the three-dimensional vector space $V = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ generated by the orthonormal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$.) The set of positive roots is

$$\begin{aligned}\Delta^+ = \{&\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\} \\ &\cup \{2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\},\end{aligned}$$

and so

$$\rho = \frac{1}{2}(\varepsilon_1 - \varepsilon_3).$$

According to the table in p. 560, the set of simple roots is given by

$$\Pi = \{\alpha_1, \alpha_2\} = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}.$$

To compute $|W|$, it is sufficient to compute $|W'|$, where W' is the subgroup of index two in W generated by the rotations. Taking into account that the angle between the planes $\mathbf{p}_1$ and $\mathbf{p}_2$ (defined by α_1 and α_2 in $\langle \alpha_1, \alpha_2 \rangle$) in this case equals $\frac{\pi}{6}$ (see Problem 4.80), we obtain that the group W' is generated by the rotation of angle $\frac{\pi}{3}$. Therefore $|W'| = 6$ and $|W| = 12$.

To describe the inner automorphisms of the Lie algebra $\mathfrak{g}_2$ generating the Weyl group W, recall that the Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ consists of the skew-symmetric real 7×7 matrices (a_{ij}) satisfying the conditions recalled in the hint above.

The Lie algebra $\mathfrak{t}'$ of a maximal torus T' of $\mathrm{SO}(7)$ is usually written as

$$\mathfrak{t}' = \left\{ \begin{pmatrix} 0 & -2\pi x_1 & 0 & 0 & 0 & 0 & 0 \\ 2\pi x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\pi x_2 & 0 & 0 & 0 \\ 0 & 0 & 2\pi x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\pi x_3 & 0 \\ 0 & 0 & 0 & 0 & 2\pi x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Now, we can make a change of coordinates in such a way that $\mathfrak{t}'$ is expressed by

$$\mathfrak{t}' = \left\{ \begin{pmatrix} 0 & -2\pi x_1 & 0 & 0 & 0 & 0 & 0 \\ 2\pi x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\pi x_2 \\ 0 & 0 & 0 & 0 & 0 & -2\pi x_3 & 0 \\ 0 & 0 & 0 & 0 & 2\pi x_3 & 0 & 0 \\ 0 & 0 & 0 & -2\pi x_2 & 0 & 0 & 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Then, according to the equations in the hint above, the elements in $\mathfrak{t}' \cap \mathfrak{g}_2$ are those with matrix as the previous one but with the additional condition $x_1 + x_2 + x_3 = 0$, and they constitute the Lie algebra $\mathfrak{t}$ of a maximal torus T of G_2.

Define the inner product $\langle\,,\rangle$ on $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ by

$$\langle X, Y\rangle = \frac{1}{8\pi^2}\operatorname{tr}\bigl({}^tXY\bigr) = -\frac{1}{8\pi^2}\operatorname{tr} XY.$$

This is clearly G_2-invariant (as it is SO(7)-invariant) and, restricted to the Lie algebra

$$\mathfrak{t} = \bigl\{x_1\varepsilon_1' + x_2\varepsilon_2' + x_3\varepsilon_3' : x_1, x_2, x_3 \in \mathbb{R},\ x_1 + x_2 + x_3 = 0\bigr\}$$

$$= \left\{\begin{pmatrix} 0 & -2\pi x_1 & 0 & 0 & 0 & 0 & 0 \\ 2\pi x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\pi x_2 \\ 0 & 0 & 0 & 0 & 0 & -2\pi x_3 & 0 \\ 0 & 0 & 0 & 0 & 2\pi x_3 & 0 & 0 \\ 0 & 0 & 0 & -2\pi x_2 & 0 & 0 & 0 \end{pmatrix} : \right.$$

$$\left. x_1, x_2, x_3 \in \mathbb{R},\ x_1 + x_2 + x_3 = 0 \right\}$$

of a maximal torus T, it is the restriction of

$$x_1^2 + x_2^2 + x_3^2$$

to $x_1 + x_2 + x_3 = 0$. Since this is the usual inner product with the standard orthonormal base $\varepsilon_1', \varepsilon_2', \varepsilon_3'$, reflection is the usual reflection.

The hyperplane in $V = \{(x_1, x_2, x_3)\} = \langle\varepsilon_1', \varepsilon_2', \varepsilon_3'\rangle$ orthogonal to the vector $\varepsilon_1' - \varepsilon_2'$ is the plane $x_1 = x_2$, and the reflection with respect to this plane is given by

$$x_1' = x_2, \qquad x_2' = x_1, \qquad x_3' = x_3.$$

We will show that this reflection is induced by some inner automorphism of G_2. In fact, we can write, omitting the factors 2π,

$$\begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that the first matrix M on the right-hand side belongs to $\mathrm{SO}(7)$. (Notice that other possibilities for a matrix in $\mathrm{SO}(7)$ to satisfy the previous equation are possible, for instance, the matrix M' given as M but with $M'_{33} = M'_{66} = 1$, but one can check that $M' \notin \mathrm{G}_2$.)

To see that M actually belongs to the Lie subgroup $\mathrm{G}_2 \subset \mathrm{SO}(7)$, it suffices to see that it is an element of the group $\mathrm{Aut}\,\mathbb{O}$ of automorphisms of the octonions.

To this end, recall that each element of the algebra $\mathbb{O}$ admits a unique expression as $q_1 + q_2\mathbf{e}$ with $q_1, q_2 \in \mathbb{H}$, where $\mathbb{H}$ is the quaternion algebra. Then the multiplication in $\mathbb{O}$ is defined by the standard multiplication relations in $\mathbb{H}$ and by the relations

$$q_1(q_2\mathbf{e}) = (q_2q_1)\mathbf{e}, \qquad (q_1\mathbf{e})q_2 = (q_1\bar{q}_2)\mathbf{e}, \qquad (q_1\mathbf{e})(q_2\mathbf{e}) = -\bar{q}_2q_1. \qquad (\star)$$

On the other hand, each element of the quaternion algebra $\mathbb{H}$ admits a unique expression as $z_1 + z_2\mathbf{j}$ with $z_1, z_2 \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers. Then the multiplication and conjugation in $\mathbb{H}$ are defined uniquely by the relations

$$z\mathbf{j} = \mathbf{j}\bar{z}, \qquad \mathbf{j}^2 = -1, \qquad \overline{z_1 + z_2\mathbf{j}} = \bar{z}_1 - z_2\mathbf{j}, \quad z, z_1, z_2 \in \mathbb{C}. \qquad (\star\star)$$

In particular, denoting by $\mathbf{i}$ the imaginary unit of $\mathbb{C}$, we have

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \qquad \mathbf{ij} = -\mathbf{ji}, \qquad \bar{\mathbf{i}} = -\mathbf{i}, \qquad \bar{\mathbf{j}} = -\mathbf{j}.$$

The algebra $\mathbb{O}$ can be also considered as the real space $\mathbb{R}^8$ with the following basis:

$$e_0 = 1, \qquad e_1 = \mathbf{i}, \qquad e_2 = \mathbf{j}, \qquad e_3 = \mathbf{ij}, \qquad e_4 = \mathbf{e}, \qquad e_5 = \mathbf{ie},$$
$$e_6 = \mathbf{je}, \qquad e_7 = (\mathbf{ij})\mathbf{e}.$$

Moreover, the algebra $\mathbb{O}$ is a normed algebra with respect to the inner product in $\mathbb{R}^8$ with orthonormal basis $\{e_p\}$, $p=0,\ldots,7$, i.e. $|uv|=|u||v|$ for any $u,v\in\mathbb{O}$. Remark also that

$$e_p^2=-1,\quad p=1,\ldots,7,\quad \text{and}\quad e_pe_l=-e_le_p,\quad p,l=1,\ldots,7,\ p\neq l,$$

i.e. $e_1,\ldots,e_7$ are (anticommuting) imaginary units of $\mathbb{O}$.

Now the matrix M can be viewed as the matrix $\operatorname{diag}(1,M)\in SO(8)$ acting, under this identification $\mathbb{O}\equiv\mathbb{R}^8$, by

$$r_0e_0+\sum_{p=1}^{7}r_ie_i\longmapsto r_0e_0+r_7e_1+r_4e_2-r_3e_3+r_2e_4+r_5e_5-r_6e_6+r_1e_7,$$
$$r_i\in\mathbb{R},$$

i.e.

$$\begin{array}{llll} e_0\mapsto e_0, & e_1\mapsto e_7, & e_2\mapsto e_4, & e_3\mapsto -e_3,\\ e_4\mapsto e_2, & e_5\mapsto e_5, & e_6\mapsto -e_6, & e_7\mapsto e_1. \end{array}$$

To prove that $M\in\operatorname{Aut}\mathbb{O}$, we find a new canonical basis $\{e'_p\}$, $p=0,\ldots,7$, in $\mathbb{O}$ constructing some additional automorphism of $\mathbb{O}$. Recall only that for any elements $\xi,\eta,\zeta\in\mathbb{O}$ such that

$$|\xi|=|\eta|=|\zeta|=1,\quad \eta\perp\xi,\ \zeta\perp\xi,\ \zeta\perp\eta,\ \zeta\perp\xi\eta,$$

there exists a unique automorphism $\Phi\colon\mathbb{O}\to\mathbb{O}$ such that (cf. [11, Lect. 15, Lemma 1])

$$\Phi(\mathbf{i})=\xi,\qquad \Phi(\mathbf{j})=\eta,\qquad \Phi(\mathbf{e})=\zeta.$$

It is evident that the spaces

$$\langle e_0,e_3,e_5,e_6\rangle=\langle 1,\mathbf{ij},\mathbf{ie},\mathbf{je}\rangle,\qquad \langle e_1,e_2,e_4,e_7\rangle=\big\langle\mathbf{i},\mathbf{j},\mathbf{e},(\mathbf{ij})\mathbf{e}\big\rangle$$

are invariant with respect of the mapping M. So to construct our automorphism, we choose ξ,η in the first space and ζ in the second. Indeed, putting $\xi=\mathbf{je}=e_6$, $\eta=\mathbf{ie}=e_5$ and $\zeta=-\mathbf{e}=-e_4$ and taking into account that $e_6e_5=(\mathbf{je})(\mathbf{ie})=$

$\mathbf{ij} = e_3$, we obtain the automorphism Φ of $\mathbb{O}$ for which $\Phi(e_0) = e_0$ and

$$\begin{aligned}
\Phi(e_1) &= \Phi(\mathbf{i}) = \mathbf{je} = e_6, \\
\Phi(e_2) &= \Phi(\mathbf{j}) = \mathbf{ie} = e_5, \\
\Phi(e_3) &= \Phi(\mathbf{ij}) = (\mathbf{je})(\mathbf{ie}) = \mathbf{ij} = e_3, \\
\Phi(e_4) &= \Phi(\mathbf{e}) = -\mathbf{e} = -e_4, \\
\Phi(e_5) &= \Phi(\mathbf{ie}) = -(\mathbf{je})\mathbf{e} = \mathbf{j} = e_2, \\
\Phi(e_6) &= \Phi(\mathbf{je}) = -(\mathbf{ie})\mathbf{e} = \mathbf{i} = e_1, \\
\Phi(e_7) &= \Phi\big((\mathbf{ij})\mathbf{e}\big) = -\big((\mathbf{je})(\mathbf{ie})\big)\mathbf{e} = -(\mathbf{ij})\mathbf{e} = -e_7.
\end{aligned}$$

Now, in the new canonical basis $e'_p = \Phi(e_p)$, $p = 0, \dots, 7$, we obtain that the map M is defined by the following relations:

$$\begin{array}{llll}
e'_0 \mapsto e'_0, & e'_1 \mapsto -e'_1, & e'_2 \mapsto e'_2, & e'_3 \mapsto -e'_3, \\
e'_4 \mapsto -e'_5, & e'_5 \mapsto -e'_4, & e'_6 \mapsto -e'_7, & e'_7 \mapsto -e'_6.
\end{array}$$

Identifying the space $\mathbb{O}$ with $\mathbb{C}^4$ by putting $(z_1, z_2, z_3, z_4) = (z_1 + z_2\mathbf{j}) + (z_3 + z_4\mathbf{j})\mathbf{e}$, we obtain the map

$$(z_1, z_2, z_3, z_4) \mapsto (\bar{z}_1, \bar{z}_2, a\bar{z}_3, a\bar{z}_4) \quad \text{with } a = -\mathbf{i}. \tag{$\star\star\star$}$$

Using relations ($\star$) and ($\star\star$), we obtain that the product in $\mathbb{C}^4 \equiv \mathbb{O}$ is defined by the relation

$$\begin{aligned}
z \cdot u &= (z_1, z_2, z_3, z_4)(u_1, u_2, u_3, u_4) \\
&= (z_1u_1 - z_2\bar{u}_2 - z_3\bar{u}_3 - \bar{z}_4u_4, z_1u_2 + z_2\bar{u}_1 + \bar{z}_3u_4 - z_4\bar{u}_3, \\
&\quad z_1u_3 - \bar{z}_2u_4 + z_3\bar{u}_1 + z_4\bar{u}_2, \bar{z}_1u_4 + z_2u_3 - z_3u_2 + z_4u_1).
\end{aligned}$$

Using this relation, we can find the four coordinates of the product $M(z)M(u)$ (on the left) and of the element $M(zu)$ (on the right):

$$\begin{array}{ll}
\bar{z}_1\bar{u}_1 - \bar{z}_2u_2 - (a\bar{z}_3)(-au_3) - (-az_4)(a\bar{u}_4), & \bar{z}_1\bar{u}_1 - \bar{z}_2u_2 - \bar{z}_3u_3 - z_4\bar{u}_4, \\
\bar{z}_1\bar{u}_2 + \bar{z}_2u_1 + (-az_3)(a\bar{u}_4) - (a\bar{z}_4)(-au_3), & \bar{z}_1\bar{u}_2 + \bar{z}_2u_1 + z_3\bar{u}_4 - \bar{z}_4u_3, \\
\bar{z}_1(a\bar{u}_3) - z_2(a\bar{u}_4) + (a\bar{z}_3)u_1 + (a\bar{z}_4)u_2, & a(\bar{z}_1\bar{u}_3 - z_2\bar{u}_4 + \bar{z}_3u_1 + \bar{z}_4u_2), \\
z_1(a\bar{u}_4) + \bar{z}_2(a\bar{u}_3) - (a\bar{z}_3)\bar{u}_2 + (a\bar{z}_4)\bar{u}_1, & a(z_1\bar{u}_4 + \bar{z}_2\bar{u}_3 - \bar{z}_3\bar{u}_2 + \bar{z}_4\bar{u}_1).
\end{array} \tag{$\dagger$}$$

Now it is evident that $M(z)M(u) = M(zu)$ for $z, u \in \mathbb{O}$, i.e. M belongs in fact to G_2.

In turn, the hyperplane orthogonal in V to the vector $-2\varepsilon'_1 + \varepsilon'_2 + \varepsilon'_3$ is the plane $\mathbf{p}$ of the equation

$$-2x_1 + x_2 + x_3 = 0.$$

Given a point $P = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, its symmetric point with respect to $\mathbf{p}$ is given by $P' = 2Q - P$, where Q stands for the foot of the perpendicular traced to $\mathbf{p}$ from P, whose equations are $(\tilde{x}_1 - 2t, \tilde{x}_2 + t, \tilde{x}_3 + t)$, t being a real number parametrising the points of the straight line, since the normal vector to $\mathbf{p}$ is $(-2, 1, 1)$. Imposing now that the point $(\tilde{x}_1 - 2t, \tilde{x}_2 + t, \tilde{x}_3 + t)$ satisfies the equation of $\mathbf{p}$, we obtain $t = \frac{1}{6}(2\tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3)$, from which we get $Q = \frac{1}{3}(2(\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3), 2\tilde{x}_1 + 5\tilde{x}_2 - \tilde{x}_3, 2\tilde{x}_1 - \tilde{x}_2 + 5\tilde{x}_3)$ and finally

$$P' = \left(\frac{-\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3}{3}, \frac{2\tilde{x}_1 + 2\tilde{x}_2 - \tilde{x}_3}{3}, \frac{2\tilde{x}_1 - \tilde{x}_2 + 2\tilde{x}_3}{3} \right).$$

Since the reflection with respect to the plane $\mathbf{p}$ preserves the subspace $\{x_1 + x_2 + x_3 = 0\} \subset V$, we obtain (dropping the tilde $\tilde{}$) that for each point $P = (x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = 0$, its image is the point $P' = (-x_1, -x_3, -x_2)$. In other words, the reflection in the space $\{x_1 + x_2 + x_3 = 0\} \subset V$ is given by

$$x_1' = -x_1, \qquad x_2' = -x_3, \qquad x_3' = -x_2.$$

This is induced by an inner automorphism of G_2. In fact, we can write, omitting the factors 2π,

$$\begin{aligned}
&\begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

It is obvious that the first matrix M on the right-hand side belongs to $\mathrm{SO}(7)$. (Notice that, as before, other possibilities for a matrix in $\mathrm{SO}(7)$ to satisfy the previous equation are possible, for instance, the matrix M' given as M but with $M'_{11} = M'_{33} = 1$, but one can check that $M' \notin \mathrm{G}_2$.)

The matrix M can be viewed as the matrix $\operatorname{diag}(1, M) \in \mathrm{SO}(8)$ acting, under the identification $\mathbb{O} \equiv \mathbb{R}^8$, by

$$e_0 + \sum_{i=1}^{7} r_i e_i \longmapsto r_0 e_0 - r_1 e_1 + r_2 e_2 - r_3 e_3 + r_5 e_4 + r_4 e_5 + r_7 e_6 + r_6 e_7, \quad r_i \in \mathbb{R}.$$

Hence the map M is defined by the following relations:

$$\begin{aligned} &e_0 \mapsto e_0, \quad &&e_1 \mapsto -e_1, \quad &&e_2 \mapsto e_2, \quad &&e_3 \mapsto -e_3, \\ &e_4 \mapsto e_5, \quad &&e_5 \mapsto e_4, \quad &&e_6 \mapsto e_7, \quad &&e_7 \mapsto e_6. \end{aligned}$$

Identifying the space $\mathbb{O}$ with $\mathbb{C}^4$ by putting $(z_1, z_2, z_3, z_4) = (z_1 + z_2\mathbf{j}) + (z_3 + z_4\mathbf{j})\mathbf{e}$, we obtain that

$$(z_1, z_2, z_3, z_4) \mapsto (\bar{z}_1, \bar{z}_2, \mathbf{i}\bar{z}_3, \mathbf{i}\bar{z}_4),$$

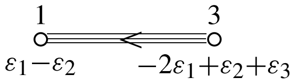
i.e. this map coincides with the map ($\star\star\star$) with $a = \mathbf{i}$. Now from relations ($\dagger$) it follows that $M(uv) = M(u)M(v)$, so actually $M \in \mathrm{G}_2$.

Now by Theorem 4.24 the simple roots α_1, α_2 are proportional to the vectors $\alpha'_1 = \varepsilon'_1 - \varepsilon'_2$ and $\alpha'_2 = -2\varepsilon'_1 + \varepsilon'_2 + \varepsilon'_3$ because $\angle(\alpha'_1, \alpha'_2) = \frac{5\pi}{6}$ (see Problem 4.80). Taking into account that the bases $\varepsilon'_1, \varepsilon'_2$ and $\varepsilon_1, \varepsilon_2$ are orthonormal and $|\alpha_1| = \frac{1}{\sqrt{3}}|\alpha_2|$, $\angle(\alpha_1, \alpha_2) = \frac{5\pi}{6}$, we conclude that $\varepsilon_1 = \pm\varepsilon'_1$ and $\varepsilon_2 = \pm\varepsilon'_2$.

The Cartan matrix is given by

$$\begin{aligned} C &= \begin{pmatrix} \frac{2\langle\alpha_1,\alpha_1\rangle}{|\alpha_1|^2} & \frac{2\langle\alpha_1,\alpha_2\rangle}{|\alpha_1|^2} \\ \frac{2\langle\alpha_2,\alpha_1\rangle}{|\alpha_2|^2} & \frac{2\langle\alpha_2,\alpha_2\rangle}{|\alpha_2|^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\langle\varepsilon_1-\varepsilon_2,\varepsilon_1-\varepsilon_2\rangle}{|\varepsilon_1-\varepsilon_2|^2} & \frac{2\langle\varepsilon_1-\varepsilon_2,-2\varepsilon_1+\varepsilon_2+\varepsilon_3\rangle}{|\varepsilon_1-\varepsilon_2|^2} \\ \frac{2\langle-2\varepsilon_1+\varepsilon_2+\varepsilon_3,\varepsilon_1-\varepsilon_2\rangle}{|-2\varepsilon_1+\varepsilon_2+\varepsilon_3|^2} & \frac{2\langle-2\varepsilon_1+\varepsilon_2+\varepsilon_3,-2\varepsilon_1+\varepsilon_2+\varepsilon_3\rangle}{|-2\varepsilon_1+\varepsilon_2+\varepsilon_3|^2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2\sqrt{3}\cos\frac{5\pi}{6} \\ 2\frac{1}{\sqrt{3}}\cos\frac{5\pi}{6} & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \end{aligned}$$

and the Dynkin diagram by



4.7 Lie Groups of Transformations

Problem 4.82 Consider:

(a) $M = (0, 4\pi) \subset \mathbb{R}$, with the differentiable structure induced by the usual one on $\mathbb{R}$.
(b) S^1 with the usual differentiable structure as a closed submanifold of $\mathbb{R}^2$.
(c) The map $f: M \to S^1$, $s \mapsto f(s) = (\cos s, \sin s)$.

Prove:

(i) The equivalence relation $\sim$ in M given by $s \sim t$ if and only if $f(s) = f(t)$ induces on the set $M/\!\sim$ a structure of quotient manifold diffeomorphic to S^1.
(ii) The manifold $M/\!\sim$ cannot be obtained by the action of a group of transformations acting on M.

Solution

(i) The differentiable map $\tilde{f}: M \to \mathbb{R}^2$ given by $\tilde{f}(s) = (\cos s, \sin s)$ is differentiable and defines the map $f: M \to S^1$. Since S^1 is an embedded submanifold of $\mathbb{R}^2$, f is differentiable.

Furthermore f is a submersion as the rank of f at any s is equal to the rank of the matrix $(-\sin s, \cos s)$, which is equal to 1. Moreover, the associated quotient manifold is diffeomorphic to S^1. In fact, as the equivalence relation is defined by $s \sim t$ if and only if $f(s) = f(t)$, we have to prove that on $M/\!\sim$ there is a differentiable structure such that the map $\pi: M \to M/\!\sim$ is a submersion. In fact, denote by $[s]$ the equivalence class of s under $\sim$. Then the map

$$h: M/\!\sim\, \to S^1, \qquad [s] \mapsto (\cos s, \sin s),$$

is clearly bijective, and thus $M/\!\sim$ admits only one differentiable structure with which $h: M/\!\sim\, \to S^1$ is a diffeomorphism. The following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & S^1 \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle h} & \\ & M/\!\sim & \end{array}$$

is obviously commutative, and since f is a submersion and h is a diffeomorphism, we deduce that π is a submersion. Consequently $M/\!\sim$ is a quotient manifold of M.

(ii) Let us suppose that there exists a group of transformations G acting on M by

$$\theta: G \times M \to M, \qquad (g, s) \mapsto \theta(g, s) = gs,$$

such that from this action we would have the previous quotient manifold. Then, given $g \in G$, as $gs \sim s$, it would be

$$\begin{cases} gs = s \text{ or } s + 2\pi, & s \in (0, 2\pi), \\ g(2\pi) = 2\pi, \\ gs = s \text{ or } s - 2\pi, & s \in (2\pi, 4\pi). \end{cases} \tag{$\star$}$$

Consider the continuous map $h\colon M \to \mathbb{R}$, $s \mapsto h(s) = gs - s$. By ($\star$) above, $h(M) \subset \{-2\pi, 0, 2\pi\}$. Moreover, we know that $0 \in h(M)$ because $h(2\pi) = 0$. But since M is connected and h continuous, $h(M)$ is connected. We conclude that $h(M) = 0$, that is, $gs = s$ for all $s \in M$. As this holds for every $g \in G$, the associated quotient manifold would be M, which cannot be homeomorphic to $M/\!\sim$, because $M/\!\sim$ is compact and M is not.

Problem 4.83 Given $\mathbb{R}^2$ with its usual differentiable structure, show:

1. The additive group $\mathbb{Z}$ of the integers acts on $\mathbb{R}^2$ as a transformation group by the action

$$\begin{aligned} \theta\colon \qquad & \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2 \\ & \big(n, (x, y)\big) \mapsto \theta\big(n, (x, y)\big) = (x + n, y). \end{aligned}$$

2. The quotient space $\mathbb{R}^2/\mathbb{Z}$ of $\mathbb{R}^2$ by that action admits a structure of quotient manifold.
3. $S^1 \times \mathbb{R}$ admits a structure of quotient manifold of $\mathbb{R}^2$, diffeomorphic to $\mathbb{R}^2/\mathbb{Z}$ as above.

Solution

1. $\mathbb{Z}$ acts on $\mathbb{R}^2$ as a transformation group by the given action. In fact, for each n, the map

$$\theta_n : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto \theta(x, y) = (x + n, y),$$

is C^∞. Moreover,

$$\theta\big(n_1, \theta\big(n_2, (x, y)\big)\big) = \theta\big(n_1, (x + n_2, y)\big) = (x + n_1 + n_2, y) = \theta\big(n_1 + n_2, (x, y)\big).$$

2. $\mathbb{Z}$ acts freely on $\mathbb{R}^2$, because if $\theta(n, (x, y)) = (x, y)$, i.e. $(x + n, y) = (x, y)$, we have $n = 0$, which is the identity element of $\mathbb{Z}$.

   Furthermore, the action of $\mathbb{Z}$ is properly discontinuous. In fact, we have to verify the two conditions in Definition 4.31:

   (i) Given $(x_0, y_0) \in \mathbb{R}^2$, let us consider $U = (x_0 - \varepsilon, x_0 + \varepsilon) \times \mathbb{R}$ with $0 < \varepsilon < \frac{1}{2}$. Then, if $(x_1, y_1) \in U \cap \theta_n(U)$, we have

$$x_0 - \varepsilon < x_1 < x_0 + \varepsilon, \qquad x_0 + n - \varepsilon < x_1 < x_0 + n + \varepsilon,$$

   from which $|n| < 2\varepsilon$, that is, $n = 0$.

(ii) Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$, such that $(x_0, y_0) \not\sim (x_1, y_1)$, that is, such that:
(a) $y_0 \neq y_1$, or (b) $y_0 = y_1$, $x_1 \neq x_0 + n$, for all $n \in \mathbb{Z}$.

In the case (a), we have two different cases: $x_0 = x_1$ and $x_0 \neq x_1$, but the solution is the same; we only have to consider

$$U = \mathbb{R} \times (y_0 - \varepsilon, y_0 + \varepsilon), \qquad V = \mathbb{R} \times (y_1 - \varepsilon, y_1 + \varepsilon), \quad 0 < \varepsilon < |y_1 - y_0|/2.$$

Since $\theta_n(V) = V$, we have $U \cap \theta_n(V) = U \cap V = \emptyset$.

In the case (b), we have $\alpha = |x_1 - x_0| \notin \mathbb{Z}$. We can suppose that $x_1 > x_0$. Thus $\alpha = x_1 - x_0$. Let $m \in \mathbb{Z}$ be such that $m < \alpha < m + 1$, and consider the value

$$0 < \varepsilon < \min\big\{(\alpha - m)/2, (m + 1 - \alpha)/2\big\}.$$

So, it suffices to consider

$$U = (x_0 - \varepsilon, x_0 + \varepsilon) \times \mathbb{R}, \qquad V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \mathbb{R}.$$

Let us see that $U \cap \theta_n(V) = \emptyset$ for all $n \in \mathbb{Z}$. It is clear that the only values of n that could give a non-empty intersection are $n = -m$ and $n = -(m + 1)$.

If $n = -m$, then if

$$(x_2, y_2) \in U \cap \theta_{-m}(V),$$

we have that

$$x_0 - \varepsilon < x_2 < x_0 + \varepsilon, \qquad x_1 - m - \varepsilon < x_2 < x_1 - m + \varepsilon,$$

so that $x_1 - m - \varepsilon < x_0 + \varepsilon$, hence $x_1 - x_0 - m < 2\varepsilon$, thus $\alpha - m < 2\varepsilon$, thus getting a contradiction.

If $n = -(m + 1)$ and $(x_2, y_2) \in U \cap \theta_{-(m+1)}(V)$, we have that

$$x_0 - \varepsilon < x_2 < x_0 + \varepsilon, \qquad x_1 - (m + 1) - \varepsilon < x_2 < x_1 - (m + 1) + \varepsilon.$$

Thus $x_0 - \varepsilon < x_1 - (m + 1) + \varepsilon$, so $x_0 - x_1 + (m + 1) < 2\varepsilon$, hence $(m + 1) - \alpha < 2\varepsilon$, thus getting a contradiction.

We conclude that $\mathbb{R}^2/\mathbb{Z}$ admits a structure of quotient manifold of dimension 2.

3. We shall denote by $[(x, y)]$ the class of (x, y) under the previous action. It is immediate that the map

$$\begin{aligned} f \colon \mathbb{R} &\to S^1 \\ s &\mapsto f(s) = (\sin 2\pi s, \cos 2\pi s) \end{aligned}$$

is a local diffeomorphism and thus is a submersion. Since the product of submersions is a submersion, it follows that $f \times \mathrm{id}_{\mathbb{R}} \colon \mathbb{R}^2 \to S^1 \times \mathbb{R}$ is a submersion. Consider the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr, mapsto, "f \times \mathrm{id}_{\mathbb{R}}"] \arrow[dr, "\pi"'] & & S^1 \times \mathbb{R} \\ & \mathbb{R}^2/\mathbb{Z} \arrow[ur, "h"'] & \end{tikzcd}$$

$$\begin{tikzcd}
(x,y) \arrow[rr, mapsto] \arrow[dr] & & (f(x),y) \\
& \pi(x,y)=[(x,y)] \arrow[ur, "h"'] &
\end{tikzcd}$$

where h is defined by $h([(x,y)]) = (f(x),y)$. Note that the definition makes sense as if $(x_0,y_0) \sim (x_1,y_1)$, we have $y_0 = y_1$, $x_1 = x_0 + n$, and thus $f(x_0,y_0) = f(x_1,y_1)$, consequently h does not depend on the representative of a given equivalence class. Furthermore h is one-to-one. In fact:

(a) h is injective, because if $(f(x_0),y_0) = (f(x_1),y_1)$ then $\sin 2\pi x_0 = \sin 2\pi x_1$, $\cos 2\pi x_0 = \cos 2\pi x_1$, and $y_0 = y_1$, hence $x_0 = x_1 + n$, $y_0 = y_1$, so $[(x_0,y_0)] = [(x_1,y_1)]$.

(b) h is surjective since $h \circ \pi$ is.

Problem 4.84 Consider $M = \mathbb{R}^2$ with its usual differentiable structure and let $\mathbb{Z}$ be the additive group of integer numbers. Prove:

1. $\mathbb{Z}$ acts on $\mathbb{R}^2$ as a transformation group by the C^∞ action

$$\theta\colon \mathbb{Z}\times\mathbb{R}^2 \to \mathbb{R}^2, \qquad \big(n,(x,y)\big) \mapsto \big(x+n,(-1)^n y\big).$$

2. $\mathbb{R}^2/\mathbb{Z}$ is a quotient manifold.

Remark $\mathbb{R}^2/\mathbb{Z}$ is diffeomorphic to the infinite Möbius strip (see Problem 1.31).

Solution

1. θ is an action of $\mathbb{Z}$ on $\mathbb{R}^2$, because $\theta(0,(x,y)) = (x,y)$ and

$$\theta\big(n_1,\theta\big(n_2,(x,y)\big)\big) = \big(x+n_1+n_2,(-1)^{n_1+n_2}y\big) = \theta\big(n_1+n_2,(x,y)\big).$$

Furthermore, the action is C^∞. In fact, since $\mathbb{Z}$ is a discrete group, we only have to prove that for each $n\in\mathbb{Z}$, the action $\theta_n : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y)\mapsto (x+n,(-1)^n y)$, is a diffeomorphism, but this is clear.

2. Since $\mathbb{Z}$ is discrete, we only have to prove that the action θ is free and properly discontinuous.

(i) The action θ is free, because if $\theta(n,(x,y)) = (x,y)$, then $n = 0$.

(ii) The action of $\mathbb{Z}$ is properly discontinuous. In fact:

(a) Given $(x_0,y_0)\in\mathbb{R}^2$, let $U = (x_0-\varepsilon, x_0+\varepsilon)\times\mathbb{R}$ with $0<\varepsilon<\frac{1}{2}$. Then, given $(x_1,y_1)\in U\cap\theta_n(U)$, one has that

$$x_0-\varepsilon < x_1 < x_0+\varepsilon, \qquad x_0+n-\varepsilon < x_1 < x_0+n+\varepsilon,$$

from which $n = 0$.

(b) Now, let $(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2$ such that $(x_0,y_0)\nsim(x_1,y_1)$, where $\sim$ denotes the equivalence relation given by the present action.

For the sake of simplicity, we can assume that (x_0,y_0) and (x_1,y_1) are in the same quadrant of $\mathbb{R}^2$. We have two possibilities:

(α) $y_1 \neq y_0$.
(β) $y_1 = y_0$, and $x_1 \neq x_0 + 2n$ for all $n \in \mathbb{Z}$.

For (α), it suffices to consider $U = \mathbb{R} \times (y_0 - \varepsilon, y_0 + \varepsilon)$ and $V = \mathbb{R} \times (y_1 - \varepsilon, y_1 + \varepsilon)$ with $0 < \varepsilon < |y_1 - y_0|/2$. Let

$$V^* = \bigl\{(x_0, y_0) \in \mathbb{R}^2 : (x_0, -y_0) \in V\bigr\}.$$

Then $U \cap \theta_n(V) \subset U \cap (V \cup V^*) = \emptyset$.

In case (β), we can assume that $x_1 > x_0$, and we have two possibilities:

(β_1) $x_1 - x_0 \neq n$, for all n.
(β_2) $x_1 - x_0 = n_0 =$ an odd integer.

Case (β_1) admits a solution similar to that given for (b) in Problem 4.83 for the case $S^1 \times \mathbb{R}$.

In case (β_2), it suffices to consider the open balls $U = B((x_0, y_0), \varepsilon)$ and

$$V = B\bigl((x_1, y_0), \varepsilon\bigr), \quad 0 < \varepsilon < \min\bigl(1/2, (x_1 - x_0)/2, y_0/2\bigr).$$

In fact, it is easily checked that if either $n = -n_0$ or $n \neq -n_0$, the wanted intersection is empty.

Consequently $\mathbb{R}^2/\mathbb{Z}$ is a quotient manifold.

Problem 4.85 Find the one-parameter subgroups of $\mathrm{GL}(2, \mathbb{R})$ corresponding to

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute the corresponding actions on $\mathbb{R}^2$ and their infinitesimal generators from the natural action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$.

Solution The one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$ corresponding to the element $X \in \mathfrak{gl}(n, \mathbb{R}) = M(n, \mathbb{R})$ is $\mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$, $t \mapsto \mathrm{e}^{tX}$. Thus,

$$\mathrm{e}^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad \mathrm{e}^{tB} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The group $\{\mathrm{e}^{tA}\}$ acts on $\mathbb{R}^2$, and the orbit of the point (x_0, y_0) is the circle with centre (0,0) and radius $r = \sqrt{x_0^2 + y_0^2}$.

The group $\{\mathrm{e}^{tB}\}$ acts on $\mathbb{R}^2$ giving as orbit of each point (x_0, y_0) the straight line $(x_0 + ty_0, y_0)$, which reduces to $(x_0, 0)$ if $y_0 = 0$.

The infinitesimal generator of $(x, y) \mapsto (x \cos t + y \sin t, -x \sin t + y \cos t)$ is the vector field

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (x \cos t + y \sin t) \frac{\partial}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (-x \sin t + y \cos t) \frac{\partial}{\partial y} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},$$

and the infinitesimal generator of $(x, y) \mapsto (x + ty, y)$ is the vector field

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (x+ty)\frac{\partial}{\partial x} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 y\frac{\partial}{\partial y} = y\frac{\partial}{\partial x}.$$

Problem 4.86 Find, in terms of the vector b, the matrix A and its eigenvalues when the Euclidean motion

$$f : x \to Ax + b, \quad A \in \mathrm{O}(3),\ b = \left(b^1, b^2, b^3\right),$$

of $\mathbb{R}^3$ has a fixed point.

Solution The equation $f(x) = x$ for some $x \in \mathbb{R}^3$ is the same as $b = (I - A)(x)$, where I stands for the identity. Thus f has a fixed point if and only if $b \in \mathrm{im}(I - A)$. Then:

1. If $+1$ is not an eigenvalue of A, then $\ker(I - A) = \{0\}$, and so $I - A$ is an automorphism of $\mathbb{R}^3$. In this case, $b \in \mathrm{im}(I - A)$.
2. Suppose that $Au = u$ for some non-zero $u \in \mathbb{R}^3$. We can assume that u is a unit vector. Then $\mathbb{R}^3 = \langle u\rangle \oplus \langle u\rangle^\perp$, where $\langle u\rangle = \{\lambda u : \lambda \in \mathbb{R}\}$; and A acts on the plane $\langle u\rangle^\perp$ as an isometry (in fact, from $Au = u$ it follows that $g(u, v) = g(u, Av)$, where g stands for the Euclidean metric of $\mathbb{R}^3$; thus, as $g(Au, Av) = g(u, v)$, $g(u, v) = 0$ implies $g(u, Av) = 0$). We have $b = \lambda u + b'$, $b' \in \langle u\rangle^\perp$, and $x = \alpha u + x'$ for all $x \in \mathbb{R}^3$. Thus $b = (I - A)(x)$ if and only if $\lambda = 0$ and $b' = (I - A)(x')$. Denote by A' the restriction of A to $\langle u\rangle^\perp$. If $+1$ is not an eigenvalue of A', we are done. In the other case, making an orthonormal change of basis, we will have $A' = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $A' = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. That is, if $+1$ is an eigenvalue of A, then we have:
   (a) If $A = I$, then f has no fixed points except for $b = 0$.
   (b) If A is a mirror symmetry, f has no fixed points except when b is orthogonal to the plane of symmetry.
   (c) If A is neither the identity nor a mirror symmetry, f has no fixed points except when $g(b, u) = 0$, that is, when b is orthogonal to the rotation axis of A (in this case).

   Note that the multiplicity of the eigenvalue $+1$ is 3, 2 or 1, in the cases (a), (b) and (c), respectively.

Problem 4.87 Let $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half-plane and consider $(x, y) \in H^2$ as $z = x + \mathrm{i}y \in \mathbb{C}$ under the identification $\mathbb{R}^2 \cong \mathbb{C}$. Prove that the group of fractional linear transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z},\ ad - bc = 1,$$

does not act freely on H^2.

Hint Compute, for instance, the isotropy group of i.

Solution The isotropy group is given by the condition $\frac{az+b}{cz+d} = z$, that is, $az + b = cz^2 + \mathrm{d}z$. For example, for $z = \mathrm{i}$, one has $a\mathrm{i} + b = d\mathrm{i} - c$, so we have $a = d$, $b = -c$, hence the isotropy group of i is the group of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with a, b integers such that $a^2 + b^2 = 1$. Hence, the solutions are $(a, b) = (1, 0)$, $(-1, 0)$, $(0, 1)$ or $(0, -1)$, and the subgroup is not the identity.

Problem 4.88

(i) Prove that the map

$$\theta \colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \qquad (a, x) \mapsto ax,$$

is a C^∞ action of $\mathbb{R}^+$ on $\mathbb{R}$. Is it free?

(ii) The action θ induces the equivalence relation $\sim$ in $\mathbb{R}$ defined by $x \sim y$ if there exists $a \in \mathbb{R}^+$ such that $\theta(a, x) = y$ or, equivalently, if there exists $a \in \mathbb{R}^+$ such that $ax = y$. Prove that $\mathbb{R}/\mathbb{R}^+$ is not a quotient manifold of $\mathbb{R}$.

Solution

(i) We have

$$\theta(1, x) = x, \qquad \theta\big(a, \theta\big(a', x\big)\big) = aa'x = \theta\big(aa', x\big).$$

Moreover, θ is C^∞, as

$$\big(\mathrm{id}_{\mathbb{R}} \circ \theta \circ (\mathrm{id}_{\mathbb{R}^+} \times \mathrm{id}_{\mathbb{R}})^{-1}\big)(a, t) = at$$

is C^∞.

The action θ is not free: For $x = 0$ and any $a \in \mathbb{R}^+$, we have $ax = 0$.

(ii) If $\mathbb{R}/\mathbb{R}^+$ were a quotient manifold of $\mathbb{R}$, then the natural map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{R}^+$, $x \mapsto [x]$ would be a submersion. But $\mathbb{R}/\mathbb{R}^+$ has only three points: $[-1]$, $[0]$ and $[1]$. If it were a manifold, it would be discrete, so disconnected. Thus π cannot be even continuous.

Problem 4.89 Show that

$$(x, y) \mapsto \theta_t(x, y) = \big(x\mathrm{e}^{2t}, y\mathrm{e}^{-3t}\big)$$

defines a C^∞ action of $\mathbb{R}$ on $\mathbb{R}^2$ and find its infinitesimal generator.

Solution We have $\theta_0(x, y) = (x, y)$ and

$$\theta_{t'}\theta_t(x, y) = \big(x\mathrm{e}^{2(t+t')}, y\mathrm{e}^{-3(t+t')}\big) = \theta_{t+t'}(x, y),$$

and hence θ is a C^∞ action of $\mathbb{R}$ on $\mathbb{R}^2$. The infinitesimal generator X is

$$X = \left.\frac{\mathrm{d}(x\mathrm{e}^{2t})}{\mathrm{d}t}\right|_{t=0} \frac{\partial}{\partial x} + \left.\frac{\mathrm{d}(y\mathrm{e}^{-3t})}{\mathrm{d}t}\right|_{t=0} \frac{\partial}{\partial y} = 2x\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y}.$$

Problem 4.90 Let

$$S^3 = \{q = x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k} \in \mathbb{H} : |q| = 1\}$$

act on itself by right translations.

Prove that the fundamental vector fields $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$ associated to the elements $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ are, respectively,

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}, \qquad Y = -z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t},$$

$$Z = -t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}.$$

Solution Identify the vector space of purely imaginary quaternions to the tangent space $T_1 S^3$. The flow generated by $\mathbf{i}^*$ is $R_{\exp(t\mathbf{i})}(q)$, $q \in S^3$. Hence,

$$\mathbf{i}^*_q(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x \circ R_{\exp(t\mathbf{i})})(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x\big(q \exp(t\mathbf{i})\big) = x(q\mathbf{i})$$

$$= x\big\{\big(x(q) + y(q)\mathbf{i} + z(q)\mathbf{j} + t(q)\mathbf{k}\big)\mathbf{i}\big\} = -y(q).$$

Similarly we obtain

$$\mathbf{i}^*_q(y) = x(q), \qquad \mathbf{i}^*_q(z) = t(q), \qquad \mathbf{i}^*_q(t) = -z(q),$$

so

$$\mathbf{i}^* = X.$$

The other cases are obtained analogously.

Remark The vector fields given in Problem 1.94 are $^*\mathbf{i}$, $^*\mathbf{j}$, $^*\mathbf{k}$, which are the fundamental vector fields with respect to left translations of S^3 on itself, instead of the right action above.

Problem 4.91 Let $G \times M \to M$, $(g, p) \mapsto g \cdot p$, be a differentiable action of a Lie group G on a differentiable manifold M. Let $\sim$ be the equivalence relation induced by this action, i.e.

$$p \sim q \quad \Leftrightarrow \quad \exists g \in G \quad \text{such that} \quad q = g \cdot p.$$

Let $N = \{(p, q) \in M \times M : p \sim q\}$. Assume that N is a closed embedded submanifold of $M \times M$. Prove that the map $\pi : N \to M$, $\pi(p, q) = p$, is a submersion.

Remark According to the Theorem of the Closed Graph 1.16, this problem proves that the quotient manifold $M/G = M/{\sim}$ of a group action exists if and only if the graph of $\sim$ is a closed embedded submanifold of $M \times M$.

Solution Let $(p_0, q_0) \in N$ be an arbitrary point. Hence there exists $g \in G$ such that $q_0 = g \cdot p_0$. Let $s\colon M \to M \times M$ be the differentiable map $\sigma(p) = (p, g \cdot p)$. This map takes values in N, and hence it induces, by virtue of the assumption, a differentiable map $\sigma\colon M \to N$, which is a section of π, i.e. $\pi \circ \sigma = \mathrm{id}_N$. As $\sigma(p_0) = (p_0, q_0)$, we conclude that π is a submersion at (p_0, q_0).

Problem 4.92 Let $\Psi : G \times M \to M$, $(g, p) \mapsto gp$, be a (left) action of a Lie group G on a manifold M. For an arbitrary element X of the Lie algebra $\mathfrak{g}$ of the Lie group G, denote by $\hat{X}$ the vector field on M generated by the one-parameter subgroup $\exp tX \subset G$, i.e.

$$\hat{X}_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)p.$$

Prove:

(i) The map $\mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto \hat{X}$, is linear, and for each vector $X \in \mathfrak{g}$, the vector field $\hat{X}$ is a smooth vector field with associated one-parameter group of diffeomorphisms

$$\varphi_t = \exp tX : M \to M.$$

(ii) For any $g \in G$ and $X \in \mathfrak{g}$, we have

$$g \cdot \hat{X} = \widehat{\mathrm{Ad}_g X}.$$

(iii) For arbitrary $X, Y \in \mathfrak{g}$, we have

$$\widehat{[X, Y]} = -[\hat{X}, \hat{Y}]$$

(the map $X \mapsto \hat{X}$ is an anti-homomorphism).

(iv) The linear subspace $\{X \in \mathfrak{g} : \hat{X}_p = 0\} \subset \mathfrak{g}$ is the Lie algebra $\mathfrak{g}_p$ of the isotropy group $G_p = \{g \in G : gp = p\}$. Moreover, $G_{gp} = gG_p g^{-1}$ and $\mathfrak{g}_{gp} = \mathrm{Ad}_g(\mathfrak{g}_p)$.

Hint (to (iii)) Apply the geometric interpretation of the Lie bracket of two vector fields in Proposition 1.18.

Solution

(i) Consider the tangent map $\Psi_* : TG \times TM \to TM$ and its restriction to $T_eG \times TM \to TM$, where e is the identity element of the Lie group G. In particular, for arbitrary $X \in \mathfrak{g} = T_eG$, we obtain that $\Psi_{*(e,p)}(X, 0) = \hat{X}_p$ because X is the tangent vector to the curve $\exp tX \subset G$ at the identity e (at $t = 0$). The map $X \mapsto \hat{X}$ is linear as the maps $\Psi_{*(e,p)}$ are linear for all $p \in M$. The vector field $\hat{X}$ is smooth as so is the map Ψ (it is easy to prove this fact using local coordinates on G and M). The group $\exp tX$ is the one-parameter group associated with $\hat{X}$ because

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\exp(t_0 + t)X\big)p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)\big((\exp t_0 X)p\big).$$

(ii) By definition, $(g \cdot \hat{X})_p = g_*(\hat{X}_{g^{-1}p})$, where $g_*\colon T_{g^{-1}p}M \to T_pM$. In other words,

$$(g \cdot \hat{X})_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g\big((\exp tX)\big(g^{-1}p\big)\big).$$

Taking into account that

$$g(\exp tX)g^{-1} = \exp t(\mathrm{Ad}_g X)$$

and that $g_1(g_2p) = (g_1g_2)p$, we obtain the following equation:

$$(g \cdot \hat{X})_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp t(\mathrm{Ad}_g X)p,$$

i.e.

$$g \cdot \hat{X} = \widehat{\mathrm{Ad}_g X}.$$

(iii) It suffices to prove that $\widehat{[X,Y]}_p = -[\hat{X},\hat{Y}]_p$ for any point $p \in M$. As we proved above, the map $\psi\colon \mathfrak{g} \to T_pM$, $X \mapsto \hat{X}_p$, is linear (for any fixed point p), and, consequently, for its tangent map at any point $Y \in \mathfrak{g}$, we have $\psi_{Y*} = \psi$. Here we have used the natural identification of the tangent space $T_0\mathfrak{g}$ with $\mathfrak{g}$. Therefore (see Definitions 4.5),

$$\begin{aligned}
\psi\big([X,Y]\big) = \psi_{Y*}\big([X,Y]\big) = \psi_{Y*}(\mathrm{ad}_X Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi(\mathrm{Ad}_{\exp tX} Y) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\widehat{\mathrm{Ad}_{\exp tX} Y})_p \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big((\exp tX)\cdot\hat{Y}\big)_p \quad \text{(by (ii))} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big((\exp tX)_*\hat{Y}\big)_p \\
&= -[\hat{X},\hat{Y}]_p \quad \text{(by Proposition 1.18).}
\end{aligned}$$

(iv) It is clear that if $X \in \mathfrak{g}_p$, then $\exp(tX)p = p$, and, consequently, $\hat{X}_p = 0$. Suppose now that $\hat{X}_p = 0$ for some $X \in \mathfrak{g}$. Since

$$\exp(t+t_0)X = \exp tX \exp t_0X = \exp t_0X \exp tX,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} (\exp tX)p = (\exp t_0X)_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)p\right) = (\exp t_0X)_*\hat{X}_p = 0$$

for all $t_0 \in \mathbb{R}$, that is, $(\exp tX)p = p$. Thus $\exp tX \subset G_p$ and, consequently, $X \in \mathfrak{g}_p$.

It is clear that $G_{gp} \subset gG_pg^{-1}$ and $g^{-1}G_{gp}g \subset G_p$. Therefore $G_{gp} = gG_pg^{-1}$ and, consequently,

$$\mathfrak{g}_{gp} = \mathrm{Ad}_g(\mathfrak{g}_p)$$

(an infinitesimal version of the first identity).

Problem 4.93 Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on a symplectic manifold (M, Ω). Suppose that the form Ω is exact, i.e. $\Omega = \mathrm{d}\theta$, where θ is a one-form on M. Suppose also that each diffeomorphism $g \in G$ preserves the form θ, i.e. $g^*\theta = \theta$. For each $X \in \mathfrak{g}$, put

$$f_X = \theta(\hat{X}),$$

where $\hat{X}$ is the vector field on M associated with the one-parameter group $\exp tX$. Prove that this action of G on M is Hamiltonian with *momentum map*

$$\mu(p)(X) = \theta(\hat{X}_p),$$

i.e. for all $X \in \mathfrak{g}$ and $g \in G$:

(i) $\hat{X}$ is the Hamiltonian vector field of the function f_X.
(ii) $(g^{-1})^* f_X = f_{\mathrm{Ad}_g X}$.

Solution

(i) Using the well-known identity $L_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X$ and the G-invariance of the form θ ($L_{\hat{X}}\theta = 0$), we obtain that

$$i_{\hat{X}}\Omega = i_{\hat{X}}\,\mathrm{d}\theta = L_{\hat{X}}\theta - \mathrm{d}i_{\hat{X}}\theta = -\mathrm{d}i_{\hat{X}}\theta = -\mathrm{d}f_X.$$

(ii) Taking into account that $g \cdot \hat{X} = \widehat{\mathrm{Ad}_g X}$ (see Problem 4.92), we obtain that

$$\bigl(g^{-1}\bigr)^* f_X = \bigl(g^{-1}\bigr)^*(i_{\hat{X}}\theta) = i_{g\cdot\hat{X}}\bigl(\bigl(g^{-1}\bigr)^*\theta\bigr) = i_{g\cdot\hat{X}}\theta = i_{\widehat{\mathrm{Ad}_g X}}\theta = f_{\mathrm{Ad}_g X}.$$

Problem 4.94 Let Ω be a skew-symmetric non-degenerate bilinear form on a real linear space V of dimension $2n$. Consider the Lie group

$$\widetilde{G} = \bigl\{g \in \mathrm{Aut}(V) : \Omega(gv, gw) = \Omega(v, w),\, v, w \in V\bigr\}$$

with Lie algebra

$$\tilde{\mathfrak{g}} = \bigl\{X \in \mathrm{End}(V) : \Omega(Xv, w) = -\Omega(v, Xw),\, v, w \in V\bigr\}$$

(isomorphic to $\mathrm{Sp}(n, \mathbb{R})$ and $\mathfrak{sp}(n, \mathbb{R})$, respectively). Let G be some closed Lie subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. Consider Ω as a symplectic differential form on the manifold V, independent of $v \in V$.

Prove that the natural linear action of the Lie group G on V is Hamiltonian with momentum map

$$\mu(v)(X) = \frac{1}{2}\Omega(v, Xv), \quad v \in V, \ X \in \mathfrak{g}.$$

Solution Since the action of G on V is linear, we obtain that

$$\hat{X}_v = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)v = Xv.$$

Here we identify naturally the tangent space $T_v V$ with the linear space V. For any $X \in \mathfrak{g}$, put

$$f_X(v) = \frac{1}{2}\Omega(v, Xv).$$

Such a map $X \mapsto f_X$ is linear. Since each operator $X \in \mathfrak{g}$ is skew-symmetric with respect to the form Ω, we have

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f_X(v+tw) &= -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{1}{2}\Omega\big(v+tw, X(v+tw)\big) \\ &= -\frac{1}{2}\big(\Omega(w, Xv) + \Omega(v, Xw)\big) = \Omega(Xv, w) = (i_{\hat{X}}\Omega)(w) \end{aligned}$$

for all $w \in T_v V = V$, i.e. $\hat{X}$ is the Hamiltonian vector field of f_X. Similarly, taking into account that $\mathrm{Ad}_g X = gXg^{-1}$, we obtain the G-equivariance of the map $X \mapsto f_X$:

$$\big((g^{-1})^* f_X - f_{\mathrm{Ad}_g X}\big)(v) = \frac{1}{2}\Omega\big(g^{-1}v, Xg^{-1}v\big) - \frac{1}{2}\Omega\big(v, gXg^{-1}v\big) = 0.$$

Problem 4.95 Let $\varphi\colon M \to M$ be a diffeomorphism of a manifold M. Denote by $\pi : T^*M \to M$ the natural projection of the cotangent bundle T^*M onto M. The map φ on M induces the diffeomorphism of T^*M onto T^*M as follows: A point ω_x of T^*M is determined by the point $x = \pi(\omega_x) \in M$ and is a linear function on the tangent space T_xM to M at the point x. The tangent map φ_{*x} of φ at x maps T_xM into $T_{\varphi(x)}M$. Its dual, φ_x^*, maps a linear function on $T_{\varphi(x)}M$ into a linear function on T_xM. Thus φ_x^{*-1} maps T_x^*M into $T_{\varphi(x)}^*M$, and the induced diffeomorphism $\tilde{\varphi}$ on T^*M is given by

$$\tilde{\varphi}(\omega_x) = \varphi_x^{*-1}\omega_x,$$

where $\pi(\varphi_x^{*-1}\omega_x) = \varphi(x)$.

Prove:

(i) The diffeomorphism $\tilde{\varphi}$ preserves the canonical one-form ϑ on T^*M, i.e.

$$\tilde{\varphi}^*\vartheta = \vartheta.$$

(ii) The diffeomorphism $\tilde{\varphi}$ preserves the canonical symplectic form $\Omega = \mathrm{d}\vartheta$ on T^*M, i.e.

$$\tilde{\varphi}^*\Omega = \Omega.$$

Solution

(i) By definition of the canonical form ϑ, for any point $\omega_x \in T_x^*M$, $x \in M$, and any tangent vector $X \in T_{\omega_x}T^*M$, we have $\vartheta_{\omega_x}(X) = \omega_x(\pi_{*\omega_x}(X))$ and recall that $\pi_{*\omega_x}(X) \in T_xM$ because $\pi(\omega_x) = x$. Note that by the definition of the map $\tilde{\varphi}$,

$$\pi \circ \tilde{\varphi} = \varphi \circ \pi \quad \text{and, consequently,} \quad \pi_{*\tilde{\varphi}(\omega_x)} \circ \tilde{\varphi}_{*\omega_x} = \varphi_{*x} \circ \pi_{*\omega_x}.$$

Since, by definition, $(\varphi_x^{*-1}\omega_x)(\varphi_{*x}X) = \omega_x(X)$, where $X \in T_xM$, we obtain that

$$\begin{aligned}
\big(\tilde{\varphi}^*\vartheta\big)_{\omega_x}(X) &= \vartheta_{\tilde{\varphi}(\omega_x)}(\tilde{\varphi}_{*\omega_x}X) && \text{(by the definition of } \tilde{\varphi}^*) \\
&= \tilde{\varphi}(\omega_x)\big(\pi_{*\tilde{\varphi}(\omega_x)}(\tilde{\varphi}_{*\omega_x}X)\big) && \text{(by the definition of } \vartheta) \\
&= \tilde{\varphi}(\omega_x)\big(\varphi_{*x}(\pi_{*\omega_x}X)\big) \\
&= \big(\varphi_x^{*-1}\omega_x\big)\big(\varphi_{*x}(\pi_{*\omega_x}X)\big) && \text{(by the definition of } \tilde{\varphi}) \\
&= \omega_x(\pi_{*\omega_x}X) = \vartheta_{\omega_x}(X).
\end{aligned}$$

(ii) It is sufficient to note that $\mathrm{d} \circ \tilde{\varphi}^* = \tilde{\varphi}^* \circ \mathrm{d}$.

Problem 4.96 Let $f \colon M \to \mathbb{R}$ be a smooth function on a manifold M. Prove that the map

$$\tilde{f} \colon \omega_x \mapsto \omega_x + \mathrm{d}f(x), \quad \omega_x \in T_x^*M,\ x \in M,$$

defines a diffeomorphism of the cotangent bundle T^*M preserving the canonical symplectic form $\Omega = \mathrm{d}\vartheta$ on T^*M.

Solution Given local coordinates $q = (q^1, \dots, q^n)$ on M, they induce local coordinates $(q, p) = (q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*M putting

$$\omega_x = \sum_i p_i(\omega_x)\,\mathrm{d}q^i\big|_x, \quad \omega_x \in T^*M,\ x \in M.$$

But since $\vartheta = \sum_i p_i \mathrm{d}q^i$ and

$$\tilde{f}(q, p) = \left(q^1, \dots, q^n, p_1 + \frac{\partial f(q)}{\partial q^1}, \dots, p_n + \frac{\partial f(q)}{\partial q^n}\right)$$

are the local expressions of ϑ and $\tilde{f}$ in these coordinates, respectively, we have that

$$\tilde{f}^*\vartheta = \sum_{i=1}^n \left(p_i + \frac{\partial f(q)}{\partial q^i}\right)\mathrm{d}q^i = \vartheta + \mathrm{d}\big(\pi^* f\big),$$

where $\pi : T^*M \to M$ is the canonical projection. It is evident that the inverse map $\tilde{f}^{-1}$ is the map $\tilde{g}$, where $g = -f$. The map $\tilde{f}$ is symplectic because

$$\tilde{f}^*\Omega = \tilde{f}^*(\mathrm{d}\vartheta) = \mathrm{d}(\tilde{f}^*\vartheta) = \mathrm{d}(\vartheta + \mathrm{d}(\pi^* f)) = \mathrm{d}\vartheta = \Omega.$$

Problem 4.97 Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g} = T_e G$, denote by X^l (resp. X^r) the left (resp. right) G-invariant vector field on G such that $X^l_e = X$ (resp. $X^r_e = X$).

Prove:

(i) Each left (resp. right) G-invariant vector field on the Lie group G is associated with the natural right (resp. left) action of G on G. For arbitrary vectors $X, Y \in \mathfrak{g}$, we have

$$[X^l, Y^l] = [X, Y]^l, \qquad [X^r, Y^r] = -[X, Y]^r.$$

(ii) Any left (resp. right) G-invariant q-form ω on the Lie group G is smooth.

Solution

(i) Put

$$l_g(g') = gg', \qquad r_g(g') = g'g, \quad g, g' \in G.$$

Then $X^l_g = l_{g*}X$ and $X^r_g = r_{g*}X$ for arbitrary $g \in G$. The vector field X^l is the vector field on G associated with the one-parameter group $r_{\exp tX}$ of diffeomorphisms of G (see Problem 4.92(i)):

$$X^l_g = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g \exp tX = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} r_{\exp tX}(g).$$

We have

$$[X^l, Y^l] = [X, Y]^l$$

by the definition of the bracket operation in $\mathfrak{g}$.

Similarly, the vector field X^r is associated with the one-parameter group $l_{\exp tX}$:

$$X^r_g = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp tX)g = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} l_{\exp tX}(g).$$

Then

$$[X^r, Y^r] = -[X, Y]^r,$$

as the vector fields X^r, Y^r and $[X, Y]^r$ coincide with the vector fields $\hat{X}$, $\hat{Y}$ and $\widehat{[X, Y]}$ associated with the natural left action of G on itself, respectively (see Problem 4.92(iii)).

(ii) Let ω be a left (resp. right) G-invariant form on G, i.e. $l_g^*\omega=\omega$ (resp. $r_g^*\omega=\omega$) for each $g\in G$. An arbitrary smooth vector field X on G has the form $\sum f_i X_i^l$ (resp. $\sum f_i X_i^r$), where $X_1,\dots,X_n$ is a basis of the Lie algebra $\mathfrak{g}$, and each f_i is a smooth real-valued function on G. The form ω is smooth because by definition

$$\omega\bigl(X_{i_1}^l,\dots,X_{i_q}^l\bigr)=\text{const.}$$

(resp. $\omega(X_{i_1}^r,\dots,X_{i_q}^r)=\text{const.}$).

Problem 4.98 Let G be a Lie group acting transitively on a manifold M. Let G_p be the isotropy group of some point $p\in M$. The *isotropy representation*

$$G_p\to \operatorname{Aut}(T_pM),\qquad g\mapsto g_{*p},$$

induces the action of G_p on the space $\Lambda^q T_p^*M$.

Prove:

(i) Any G-invariant q-form α on the manifold M is a smooth differential form.
(ii) Any G_p-invariant q-covector $w\in\Lambda^q T_p^*M$ (on the tangent space T_pM) determines a unique smooth G-invariant differential form β on M such that $\beta_p=w$.

Solution

(i) Since the Lie group G acts transitively on M, this manifold is G-equivariantly diffeomorphic to the homogeneous space G/H, where $H=G_p$. Here we consider the natural left action of G on the quotient space G/H:

$$G\times G/H\to G/H,\qquad (g_1,gH)\mapsto g_1gH.$$

This G-equivariant diffeomorphism is defined by the map $gH\mapsto gp$ (see Theorem 4.34). Therefore, to prove (i), it is sufficient to consider the case where $M=G/H$ and α is a (left) G-invariant form on G/H, i.e. $g^*\alpha=\alpha$ for all $g\in G$.

Let $\pi: G\to G/H$ be the canonical projection. This map is G-equivariant with respect to the natural left actions of G on G and G/H. Therefore the form $\pi^*\alpha$ is a left G-invariant form on G:

$$l_g^*\bigl(\pi^*\alpha\bigr)=(\pi\circ l_g)^*\alpha=(g\circ\pi)^*\alpha=\pi^*\bigl(g^*\alpha\bigr)=\pi^*\alpha,$$

where $l_g\colon G\to G$, $g'\mapsto gg'$. The form $\pi^*\alpha$ as a left G-invariant form on G is smooth (see Problem 4.97(ii)). Since the projection π is a submersion, the form α is also smooth. Indeed, by the Theorem of the Rank 1.11, for any point $g\in G$, there exist a neighbourhood U of g, coordinates $x^1,\dots,x^n$ on U, and coordinates $x^1,\dots,x^m$ ($m\leqslant n$) on the open subset $\pi(U)\subset G/H$ such that for the restriction $\pi|U$, we have, in these coordinates, $\pi(x^1,x^2,\dots,x^n)=$

$(x^1, x^2, \dots, x^m)$. Therefore the restrictions $\alpha|\pi(U)$ and $\pi^*\alpha|U$ (forms) are described by the same expression

$$\sum_{1\le i_1<\cdots<i_q\le m} a_{i_1\dots i_q}\big(x^1,\dots,x^m\big)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q}$$

and, consequently, are smooth simultaneously.

(ii) Put

$$\beta_{gp}(g_*X_1,\dots,g_*X_q) = w(X_1,\dots,X_q)$$

for arbitrary tangent vectors $X_1,\dots,X_q \in T_pM$ and $g \in G$. Such a q-form β on $M = G\cdot p$ is well defined because w is G_p-invariant, i.e.

$$w(X_1,\dots,X_q) = w(g_{*p}X_1,\dots,g_{*p}X_q)$$

for all $g \in G_p$. By definition this form β on M is G-invariant and, consequently, smooth (by (i)).

Problem 4.99 Let $\mathfrak{g}$ be a Lie algebra with connected Lie group G. Let

$$\mathrm{Ad}^\sharp\colon G \to \mathrm{Aut}\big(\mathfrak{g}^*\big), \qquad g \mapsto \mathrm{Ad}^\sharp_g,$$

where

$$\big(\mathrm{Ad}^\sharp_g\,\alpha\big)(X) = \alpha(\mathrm{Ad}_{g^{-1}}X), \quad \alpha\in\mathfrak{g}^*,\ X\in\mathfrak{g},$$

is the *coadjoint representation* of G in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

Consider on $\mathfrak{g}^*$ the *coadjoint action of* G putting

$$g\cdot\alpha = \mathrm{Ad}^\sharp_g\,\alpha.$$

Let

$$\mathscr{O}_\alpha = \big\{\mathrm{Ad}^\sharp_g\,\alpha : g\in G\big\}$$

be the *coadjoint orbit* of a covector $\alpha\in\mathfrak{g}^*$, and let ϕ be the corresponding projection $G\to\mathscr{O}_\alpha$, $\phi(g) = \mathrm{Ad}^\sharp_g\,\alpha$. Let Ω be a 2-form on $\mathscr{O}_\alpha$ such that

$$\Omega(\hat X_\beta, \hat Y_\beta) = -\beta\big([X,Y]\big) \quad \text{at each point } \beta\in\mathscr{O}_\alpha.$$

Here $\hat X$, $\hat Y$ denote the vector fields on $\mathscr{O}_\alpha$ corresponding to the one-parameter groups

$$\mathrm{Ad}^\sharp_{\exp tX}, \qquad \mathrm{Ad}^\sharp_{\exp tY}, \quad X,Y\in\mathfrak{g}.$$

The values of these vector fields at each point $\beta\in\mathscr{O}_\alpha$ span the tangent space $T_\beta\mathscr{O}_\alpha$.

Prove:

(i) The Lie algebra $\mathfrak{g}_\beta$ of the isotropy group

$$G_\beta = \left\{ g \in G : \operatorname{Ad}_g^\sharp \beta = \beta \right\}$$

of an element $\beta \in \mathscr{O}_\alpha$ is the annihilator of the tangent space $T_\beta \mathscr{O}_\alpha \subset \mathfrak{g}^*$ in $\mathfrak{g}$, and

$$\mathfrak{g}_\beta = \left\{ X \in \mathfrak{g} : \beta\big([X, Y]\big) = 0, \forall Y \in \mathfrak{g} \right\}.$$

(ii) The 2-form Ω is well defined and non-degenerate.
(iii) The 2-form Ω is G-invariant and, consequently, smooth.
(iv) The 2-form Ω is closed.
(v) The form Ω defines a symplectic structure on the orbit $\mathscr{O}_\alpha$ (the *Lie–Poisson symplectic structure*), and $\phi^* \Omega = \mathrm{d}\alpha^l$, where α^l is the unique left G-invariant one-form on G such that $\alpha_e^l = \alpha$.

Solution

(i) Taking into account that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \operatorname{Ad}_{\exp tX} Y = \operatorname{ad}_X Y = [X, Y],$$

and identifying naturally the tangent space $T_\beta \mathfrak{g}^*$ with $\mathfrak{g}^*$ and the tangent space $T_\beta \mathscr{O}_\alpha$ with some subspace of $\mathfrak{g}^*$, we obtain that $\hat{X}_\beta = \operatorname{ad}_X^\sharp \beta$, where by definition

$$\operatorname{ad}_X^\sharp \beta(Y) = \beta(-\operatorname{ad}_X Y) = \beta\big([-X, Y]\big), \quad Y \in \mathfrak{g}.$$

Therefore,

$$T_\beta \mathscr{O}_\alpha = \{\hat{Y}_\beta, Y \in \mathfrak{g}\} = \left\{ \operatorname{ad}_Y^\sharp \beta, Y \in \mathfrak{g} \right\}, \quad \beta \in \mathscr{O}_\alpha,$$

and

$$\begin{aligned} \mathfrak{g}_\beta &= \{X \in \mathfrak{g} : \hat{X}_\beta = 0\} = \left\{ X \in \mathfrak{g} : \operatorname{ad}_X^\sharp \beta = 0 \right\} \\ &= \left\{ X \in \mathfrak{g} : \beta\big([X, Y]\big) = 0 \ \forall Y \in \mathfrak{g} \right\} = \left\{ X \in \mathfrak{g} : \operatorname{ad}_Y^\sharp \beta(X) = 0 \ \forall Y \in \mathfrak{g} \right\} \end{aligned}$$

(see Problem 4.92(iv)).

(ii) Similarly, if $\hat{X}_\beta = \hat{X}'_\beta$ for $X, X' \in \mathfrak{g}$, then $X' - X \in \mathfrak{g}_\beta$, and

$$\begin{aligned} \Omega(\hat{X}_\beta, \hat{Y}_\beta) &= -\beta\big([X, Y]\big) = -\beta\big([X, Y]\big) - \beta\big(\big[X' - X, Y\big]\big) = -\beta\big(\big[X', Y\big]\big) \\ &= \Omega\big(\hat{X}'_\beta, \hat{Y}_\beta\big). \end{aligned}$$

Thus the form Ω is well defined. This form is non-degenerate because $\Omega(\hat{X}_\beta, \hat{Y}_\beta) = 0$ for all $Y \in \mathfrak{g}$ if and only if $\beta([X, Y]) = 0 \ \forall Y \in \mathfrak{g}$, i.e. $\hat{X}_\beta = 0$ ($X \in \mathfrak{g}_\beta$).

(iii) The form Ω is G-invariant because $g_{*\beta}\hat{X}_\beta = \widehat{\mathrm{Ad}_g X}_{g\beta}$ (see Problem 4.92(ii)):

$$\begin{aligned}(g^*\Omega)_\beta(\hat{X}_\beta, \hat{Y}_\beta) &= \Omega_{g\beta}(g_{*\beta}\hat{X}_\beta, g_{*\beta}\hat{Y}_\beta) = \Omega_{g\beta}\big((\widehat{\mathrm{Ad}_g X})_{g\beta}, (\widehat{\mathrm{Ad}_g Y})_{g\beta}\big)\\ &= (-g\beta)\big([\mathrm{Ad}_g X, \mathrm{Ad}_g Y]\big) = \big(-\mathrm{Ad}_g^\sharp \beta\big)\big(\mathrm{Ad}_g\big([X,Y]\big)\big)\\ &= -\beta\big([X,Y]\big) = \Omega_\beta(\hat{X}_\beta, \hat{Y}_\beta).\end{aligned}$$

Thus, by Problem 4.98(i), the form Ω is smooth.

(iv) We shall give two proofs. By the definition of $\mathrm{d}\Omega$ (see formula (7.2)), for arbitrary vectors $X, Y, Z \in \mathfrak{g}$ we have

$$\begin{aligned}\mathrm{d}\Omega(\hat{X}, \hat{Y}, \hat{Z}) &= \hat{X}\big(\Omega(\hat{Y}, \hat{Z})\big) - \hat{Y}\big(\Omega(\hat{X}, \hat{Z})\big) + \hat{Z}\big(\Omega(\hat{X}, \hat{Y})\big)\\ &\quad - \Omega\big([\hat{X}, \hat{Y}], \hat{Z}\big) + \Omega\big([\hat{X}, \hat{Z}], \hat{Y}\big) - \Omega\big([\hat{Y}, \hat{Z}], \hat{X}\big)\\ &= -\hat{X}\big(\beta\big([Y,Z]\big)\big) + \hat{Y}\big(\beta\big([X,Z]\big)\big) - \hat{Z}\big(\beta\big([X,Y]\big)\big)\\ &\quad \text{(by the definition of } \Omega)\\ &\quad + \Omega\big(\widehat{[X,Y]}, \hat{Z}\big) - \Omega\big(\widehat{[X,Z]}, \hat{Y}\big) + \Omega\big(\widehat{[Y,Z]}, \hat{X}\big)\\ &\quad \text{(by Problem 4.92(iii))}\\ &= -\mathrm{ad}_X^\sharp \beta\big([Y,Z]\big) + \mathrm{ad}_Y^\sharp \beta\big([X,Z]\big) - \mathrm{ad}_Z^\sharp \beta\big([X,Y]\big)\\ &\quad - \beta\big(\big[[X,Y],Z\big]\big) + \beta\big(\big[[X,Z],Y\big]\big) - \beta\big(\big[[Y,Z],X\big]\big)\\ &\quad \text{(by the definition of } \Omega)\\ &= -2\beta\big(\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big]\big)\\ &= 0 \quad \text{(by the Jacobi identity)}.\end{aligned}$$

We give the second proof of (iv) considering the orbit $\mathscr{O}_\alpha$ as the homogeneous manifold G/G_α with the natural left G-action on it. Denote by Φ the corresponding diffeomorphism

$$G/G_\alpha \to \mathscr{O}_\alpha, \qquad gG_\alpha \mapsto \mathrm{Ad}_g^\sharp \alpha.$$

Let π be the canonical projection of G onto G/G_α. It is clear that for any vector $X \in \mathfrak{g}$,

$$(\Phi \circ \pi)_{*e}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\Phi\circ\pi)(\exp tX) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}^\sharp_{\exp tX}\alpha = \hat{X}_\alpha,$$

and, consequently, for $X, Y \in \mathfrak{g} = T_eG$, we have

$$\begin{aligned}\big((\Phi\circ\pi)^*\Omega\big)_e(X,Y) &= \Omega\big((\Phi\circ\pi)_{*e}X, (\Phi\circ\pi)_{*e}Y\big) = \Omega(\hat{X}_\alpha, \hat{Y}_\alpha)\\ &= -\alpha\big([X,Y]\big).\end{aligned}$$

There exists a unique left G-invariant one-form α^l on G such that $\alpha^l_e = \alpha$ (see Problem 4.98(ii)). Since Ω is a G-invariant form on the orbit $\mathscr{O}_\alpha$ and the maps π and Φ are G-equivariant, the form $(\Phi \circ \pi)^*\Omega$ is a left G-invariant form on G. But by definition $\Phi \circ \pi = \phi$. Therefore for arbitrary left-invariant vector fields X^l and Y^l on the Lie group G with $X^l_e = X$ and $Y^l_e = Y$, we have

$$\begin{aligned}(\phi^*\Omega)(X^l, Y^l) &= (\phi^*\Omega)_e(X, Y) = -\alpha([X, Y]) = -\alpha^l([X, Y]^l) \\ &= -\alpha^l([X^l, Y^l]).\end{aligned}$$

Now taking into account that by the definition of $\mathrm{d}\alpha^l$ (see formula (7.1)),

$$\mathrm{d}\alpha^l(X^l, Y^l) = X^l(\alpha^l(Y^l)) - Y^l(\alpha^l(X^l)) - \alpha^l([X^l, Y^l]) = -\alpha^l([X^l, Y^l]),$$

we obtain that $\phi^*\Omega = \mathrm{d}\alpha^l$. Hence the form $\phi^*\Omega$ is exact and, in particular, closed. Since $0 = \mathrm{d}(\phi^*\Omega) = \phi^*\mathrm{d}\Omega$ and the map $\phi = \Phi \circ \pi$ is a submersion of G onto $\mathscr{O}_\alpha$, the form Ω on $\mathscr{O}_\alpha$ is also closed.

(v) The closed non-degenerate two-form Ω defines a symplectic structure on the orbit $\mathscr{O}_\alpha$ (the Lie–Poisson symplectic structure).

Problem 4.100 Let

$$\Psi : G \times M \to M, \qquad (g, p) \mapsto gp,$$

be a (left) action of a Lie group G on a manifold M. Denote by $\mathfrak{g}$ the Lie algebra of G. For an arbitrary point $p \in M$, denote by $\widetilde{M}$ its G-orbit $Gp = \Psi(G, p)$.

Prove:

(i) The subset $\widetilde{M} \subset M$ is a submanifold of M.

(ii) The restriction map $\widetilde{\Psi} = \Psi|(G \times \widetilde{M})$ defines a smooth action of G on $\widetilde{M}$. For any $X, Y \in \mathfrak{g}$, the vector fields $\hat{X}_\Psi$ and $\hat{Y}_\Psi$ on M are tangent to the orbit $\widetilde{M}$ at each point of $\widetilde{M} \subset M$, and

$$\hat{X}_\Psi|\widetilde{M} = \hat{X}_{\widetilde{\Psi}}, \qquad [\hat{X}_\Psi, \hat{Y}_\Psi]|\widetilde{M} = [\hat{X}_{\widetilde{\Psi}}, \hat{Y}_{\widetilde{\Psi}}].$$

Here $\hat{X}_\Psi$ (resp. $\hat{X}_{\widetilde{\Psi}}$) denotes the vector field on M (resp. on $\widetilde{M}$) associated with the one-parameter subgroup $\exp tX \subset G$ and the action Ψ on M (resp. the action $\widetilde{\Psi}$ on $\widetilde{M}$).

Solution

(i) Since $\widetilde{M} \cong G/G_p$, where G_p is the isotropy group (which is closed) of a point p, the set $\widetilde{M}$ naturally has a manifold structure. To prove that $\widetilde{M}$ is a submanifold of M, it is sufficient to show that the natural one-to-one map $\phi\colon G/G_p \to M$, $gG_p \mapsto gp$, is an immersion. Indeed,

$$\operatorname{rank} \phi_{*o} = \dim G - \dim G_p = \dim \widetilde{M},$$

where $o = \pi(e)$, and $\pi : G \to G/G_p$ is the natural projection, because for $X \in \mathfrak{g}$, $\hat{X}(p) = 0$ if and only if X is an element of the Lie algebra $\mathfrak{g}_p$ of the Lie group G_p (see Problem 4.92(iv)).

The rank of the map ϕ is independent of the choice of a point because by definition $\phi(\pi(gg')) = g\phi(\pi(g'))$ for all $g, g' \in G$ and, consequently, $\phi_{*\pi(g)} = g_{*p} \circ \phi_{*o}$, where $g_{*p} : T_pM \to T_{gp}M$ is a non-degenerate linear map.

(ii) The restriction map $\widetilde{\Psi} = \Psi|(G \times \widetilde{M})$ is smooth because any one-to-one immersion is locally an embedding (see the solution of Problem 2.40). Since the subset $\widetilde{M} \subset M$ is G-invariant, $\widetilde{\Psi}$ defines a (left) action on $\widetilde{M}$. Now taking into account Problem 2.40, we conclude.

4.8 Homogeneous Spaces

Problem 4.101 Prove that $\mathrm{O}(n+1)/\mathrm{O}(n)$ and $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ are homogeneous spaces and that the sphere S^n is diffeomorphic to each of them.

Solution By means of the map

$$\mathrm{O}(n) \to \mathrm{O}(n+1), \qquad A \mapsto \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right),$$

$\mathrm{O}(n)$ is a closed Lie subgroup of $\mathrm{O}(n+1)$, so that the quotient space $\mathrm{O}(n+1)/\mathrm{O}(n)$, with the usual C^∞ structure, is a homogeneous space.

We will prove:

(i) There exists a C^∞ action of $\mathrm{O}(n+1)$ (resp. $\mathrm{SO}(n+1)$) on S^n.
(ii) This action is transitive.
(iii) The isotropy group H_p is isomorphic to $\mathrm{O}(n)$ (resp. $\mathrm{SO}(n)$) for some $p \in S^n$ (see Fig. 4.8 for the case $\mathrm{SO}(2)$).

Now, we have:

(i) The action $\mathrm{GL}(n+1,\mathbb{R}) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(A, v) \mapsto Av$, is C^∞, and its restriction $\mathrm{O}(n+1) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is also C^∞. As the action of the orthogonal group preserves the length of vectors, the restriction $\mathrm{O}(n+1) \times S^n \to \mathbb{R}^{n+1}$ takes values in S^n and is C^∞.
(ii) Given any pair $p, q \in S^n$, there exists $A \in \mathrm{O}(n+1)$ with $q = Ap$. For, let $\{e_i\}$, $\{\bar{e}_i\}$ be orthonormal bases with respect to the Euclidean metric of $\mathbb{R}^{n+1}$ satisfying $e_1 = p$, $\bar{e}_1 = q$. Then one takes as A the matrix of the change of basis, so that, in fact, $A \in \mathrm{O}(n+1)$.
(iii) We choose, for the sake of simplicity, $p = (1, 0, \dots, 0)$. By definition,

$$H_p = \{A \in \mathrm{O}(n+1) : Ap = p\};$$

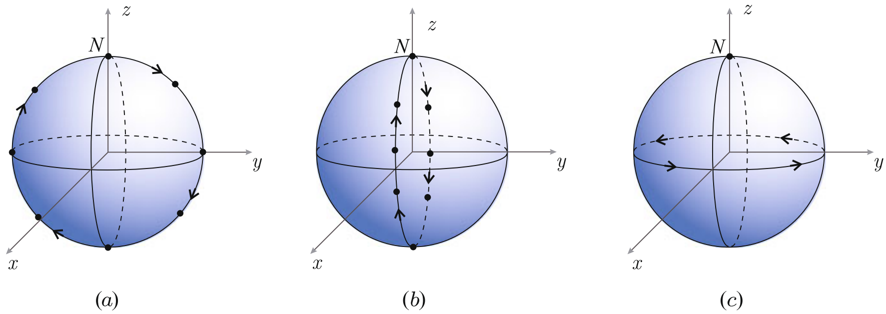


Fig. 4.8 The sphere S^2 viewed as the homogeneous space SO(3)/SO(2). The north pole rotates under rotations around either the x- or the y-axis but not under rotations around the z-axis

thus, if $A = (a_{ij})$, we have $a_{11} = 1$, $a_{i1} = 0$, $i = 2, \dots, n$. Moreover, as $A \in H_p \subset \mathrm{O}(n+1)$, we have ${}^tAA = I$, hence $p = {}^tAAp = {}^tAp$, so that $a_{1i} = 0$, $i = 2, \dots, n$. Thus,

$$H_p = \left\{ \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array} \right) \in \mathrm{O}(n+1) \right\},$$

but ${}^tAA = I$, so ${}^tBB = I$, i.e. $B \in \mathrm{O}(n)$. Thus $H_p \cong \mathrm{O}(n)$.

Hence, one has a diffeomorphism $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$. One also has $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$, because the above arguments are valid taking orthonormal bases $\{e_i\}$ and $\{\bar{e}_i\}$ with the same orientation, satisfying $e_1 = p$, $\bar{e}_1 = q$, which is always possible.

Problem 4.102 Prove that $\mathrm{U}(n)/\mathrm{U}(n-1)$ and $\mathrm{SU}(n)/\mathrm{SU}(n-1)$ are homogeneous spaces and that the sphere S^{2n-1} is diffeomorphic to each of them.

Solution By means of the map

$$\mathrm{U}(n-1) \to \mathrm{U}(n), \qquad A \mapsto \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array} \right),$$

$\mathrm{U}(n-1)$ is a closed subgroup of $\mathrm{U}(n)$, and thus the quotient space $\mathrm{U}(n)/\mathrm{U}(n-1)$, with the usual C^∞ structure, is a homogeneous space.

Consider S^{2n-1} as the unit sphere of $\mathbb{C}^n$ with the usual Hermitian product $\langle\,,\rangle$, that is, $\langle \sum_i \lambda^i e_i, \sum_j \mu^j e_j \rangle = \sum_i \lambda^i \bar{\mu}^i$, so

$$S^{2n-1} = \left\{ \left(z^1, \dots, z^n\right) \in \mathbb{C}^n : \sum_i z^i \bar{z}^i = 1 \right\}.$$

The isometry group of the metric $\langle\,,\,\rangle$ is $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) : {}^t\bar{A}A = I\}$. Hence, similarly to Problem 4.101 we have:

(i) The map $\mathrm{U}(n) \times S^{2n-1} \to S^{2n-1}$, being the restriction of the C^∞ map $\mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^n \to \mathbb{C}^n$, is differentiable.
(ii) The action of $\mathrm{U}(n)$ on S^{2n-1} is transitive.
(iii) The isotropy subgroup of $p = (1, 0, \dots, 0) \in S^{2n-1}$ is isomorphic to $\mathrm{U}(n-1)$.
Hence one has a diffeomorphism $S^{2n-1} \cong \mathrm{U}(n)/\mathrm{U}(n-1)$, and similarly to Problem 4.101, one proves that $S^{2n-1} \cong \mathrm{SU}(n)/\mathrm{SU}(n-1)$.

Problem 4.103 Prove that S^1 and S^3 are Lie groups by two different methods: First, from Problem 4.102. Then, by using the fact that S^1 and S^3 can be respectively identified to the unit complex numbers and to the unit quaternions.

Solution From the diffeomorphisms

$$S^{2n-1} \cong \mathrm{U}(n)/\mathrm{U}(n-1) \cong \mathrm{SU}(n)/\mathrm{SU}(n-1)$$

in Problem 4.102, for $n = 1$, one has $S^1 \cong \mathrm{U}(1)$, and for $n = 2$, we have

$$S^3 \cong \mathrm{U}(2)/\mathrm{U}(1) = \mathrm{SU}(2).$$

So S^1 and S^3 are Lie groups.

That $S^1 \cong \mathrm{U}(1)$ was already seen in Problem 4.48. As for S^3, we have

$$S^3 = \left\{ (x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1 \right\} \equiv \left\{ q \in \mathbb{H} : |q| = 1 \right\}.$$

Now, given $q, q' \in \mathbb{H}$, one can check that $|qq'| = |q||q'|$; hence if $q, q' \in S^3$ as above, then $qq' \in S^3$. Moreover, from the rules of multiplication in $\mathbb{H}$ (Problem 4.71(i)) we conclude that S^3 is a Lie group. One can also obtain this applying Cartan's Criterion on Closed Subgroups of a Lie group to $S^3 \subset \mathbb{H}^*$.

Problem 4.104 Let $V_k(\mathbb{R}^n)$ denote the set of k-frames $(e_1, \dots, e_k)$ in $\mathbb{R}^n$ which are orthonormal with respect to the Euclidean metric g of $\mathbb{R}^n$. Prove:

(i) $V_k(\mathbb{R}^n)$ is a closed embedded C^∞ submanifold of $\mathbb{R}^{nk}$ (called the *Stiefel manifold* of orthonormal k-frames in $\mathbb{R}^n$).
(ii) $\mathrm{O}(n)/\mathrm{O}(n-k)$ is a homogeneous space diffeomorphic to $V_k(\mathbb{R}^n)$.

Solution

(i) Let us denote by x_j^i, $i = 1, \dots, n$, $j = 1, \dots, k$, the coordinate functions on $\mathbb{R}^{nk}$, that is, $x_j^i(e_1, \dots, e_k)$ is the ith component of e_j in the standard basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$. The equations defining $V_k(\mathbb{R}^n)$ are

$$f_{ij} = \sum_{h=1}^{n} x_i^h x_j^h - \delta_{ij} = 0, \quad 1 \leqslant i \leqslant j \leqslant k.$$

We shall now prove that the differentials of the functions f_{ij} are linearly independent, so concluding. For this, we first consider that the action

$$\mathrm{O}(n) \times V_k\big(\mathbb{R}^n\big) \to V_k\big(\mathbb{R}^n\big), \qquad \big(A, (e_1, \dots, e_k)\big) \mapsto (Ae_1, \dots, Ae_k), \qquad (\star)$$

is transitive, since given two g-orthonormal k-bases of $\mathbb{R}^n$, they can be completed to two orthonormal bases of $\mathbb{R}^n$, and there is always a matrix $A \in \mathrm{O}(n)$ that defines a correspondence between them.

Moreover, since $(f_{ij} + \delta_{ij})(e_1, \dots, e_k)$ is nothing but the scalar product of e_i and e_j, we clearly have

$$(f_{ij} + \delta_{ij})\big(A \cdot (e_1, \dots, e_k)\big) = (f_{ij} + \delta_{ij})(e_1, \dots, e_k)$$

for all $A \in \mathrm{O}(n)$, $(e_1, \dots, e_k) \in V_k(\mathbb{R}^n)$. Thus, it suffices to see that the differentials of the functions f_{ij} are linearly independent at a point $(e_1, \dots, e_k) \in V_k(\mathbb{R}^n)$. Take the point represented by the $n \times k$ matrix whose first n rows are the identity matrix I_k and the other $n-k$ rows are zero, that is, $x_i^h(e_1, \dots, e_k) = \delta_{hi}$. Then, it is immediate that

$$(\mathrm{d}f_{ij})_{(e_1,\dots,e_k)} = \big(\mathrm{d}x_j^i + \mathrm{d}x_i^j\big)_{(e_1,\dots,e_k)}.$$

As $i \leqslant j$, we are done.

(ii) By means of the map

$$\mathrm{O}(n-k) \to \mathrm{O}(n), \qquad A \mapsto \begin{pmatrix} I_k & 0 \\ 0 & A \end{pmatrix},$$

$\mathrm{O}(n-k)$ is a closed Lie subgroup of $\mathrm{O}(n)$, hence the quotient space $\mathrm{O}(n)/\mathrm{O}(n-k)$, with the usual C^∞ structure, is a homogeneous space.

We have

$$V_k\big(\mathbb{R}^n\big) = \big\{(e_1, \dots, e_k) \in \big(\mathbb{R}^n\big)^k : g(e_i, e_j) = \delta_{ij}\big\} \subset \big(S^{n-1}\big)^k.$$

In particular, $V_1(\mathbb{R}^n) = S^{n-1} \cong \mathrm{O}(n)/\mathrm{O}(n-1)$, as we proved in Problem 4.101. The action $(\star)$ is obviously differentiable. We have seen that it is also transitive. To determine the isotropy group of a point we choose, for the sake of simplicity, the point $p = (e_1, \dots, e_k)$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the ith place. Then, since $H_p = \{A \in \mathrm{O}(n) : Ap = p\}$, a calculation similar to that in Problem 4.105 shows that

$$H_p = \begin{pmatrix} I_k & 0 \\ 0 & A \end{pmatrix}, \qquad A \in \mathrm{O}(n-k).$$

Consequently H_p is isomorphic to $\mathrm{O}(n-k)$, and $V_k(\mathbb{R}^n)$ is diffeomorphic to $\mathrm{O}(n)/\mathrm{O}(n-k)$.

Problem 4.105 Prove that $\mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$ is a homogeneous space diffeomorphic to the C^∞ manifold $G_k(\mathbb{R}^n)$ of k-planes through the origin of $\mathbb{R}^n$, called the *Grassmann manifold* of k-planes in $\mathbb{R}^n$. Analyse the particular case $G_1(\mathbb{R}^n)$.

Solution By means of the map

$$\mathrm{O}(k) \times \mathrm{O}(n-k) \to \mathrm{O}(n), \qquad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$\mathrm{O}(k) \times \mathrm{O}(n-k)$ is a closed Lie subgroup of $\mathrm{O}(n)$, and thus the quotient space $\mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$, with the usual C^∞ structure, is a homogeneous space.

The map

$$V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \qquad p = \{e_1, \dots, e_k\} \mapsto \langle e_1, \dots, e_k \rangle,$$

which defines a correspondence between each k-basis of $\mathbb{R}^n$ and the k-plane it spans, is surjective, since given a k-plane, we always can choose a g-orthonormal k-basis, g being the Euclidean metric of $\mathbb{R}^n$. The map

$$\mathrm{O}(n) \times G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \qquad \big(A, \langle e_1, \dots, e_k \rangle\big) \mapsto \langle Ae_1, \dots, Ae_k \rangle,$$

is C^∞. The action is transitive, as given two k-planes in $\mathbb{R}^n$ and a g-orthonormal k-basis of each of them, we can complete both bases to g-orthonormal bases of $\mathbb{R}^n$; but there is always an element $A \in \mathrm{O}(n)$ that transforms the one into the other, and thus it transforms the k-plane generated by the initial k-basis in the k-plane generated by the other k-basis.

In order to determine the isotropy group of a point, we choose $p = \langle e_1, \dots, e_k \rangle$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the ith place. It is easy to see that the elements of $\mathrm{O}(n)$ leaving p invariant are those of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A \in \mathrm{O}(k),\ B \in \mathrm{O}(n-k).$$

Hence $H_p \cong \mathrm{O}(k) \times \mathrm{O}(n-k)$, and thus

$$G_k(\mathbb{R}^n) \cong \mathrm{O}(n)\big/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big).$$

For $k = 1$, we have 1-planes, that is, straight lines through the origin of $\mathbb{R}^n$, and $G_1(\mathbb{R}^n)$ is then the real projective space $\mathbb{R}\mathrm{P}^{n-1}$. We thus have

$$\begin{aligned}\mathbb{R}\mathrm{P}^{n-1} &\cong G_1\bigl(\mathbb{R}^n\bigr) \cong \mathrm{O}(n)\big/\bigl(\mathrm{O}(1)\times \mathrm{O}(n-1)\bigr) \cong \mathrm{O}(n)\big/\bigl(\mathbb{Z}_2 \times \mathrm{O}(n-1)\bigr)\\ &\cong \mathrm{SO}(n)/\mathrm{O}(n-1),\end{aligned}$$

where the last equivalence follows from an argument as in Problem 4.101. Hence, real projective spaces are homogeneous spaces.

Problem 4.106 Show that $\mathrm{GL}(n,\mathbb{R})$ acts transitively on $\mathbb{R}\mathrm{P}^{n-1}$ and determine the isotropy group of $[e_1]$, $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

Solution If the points $p, q \in \mathbb{R}\mathrm{P}^{n-1}$ are given by $p = [\lambda]$, $q = [\mu]$, where $\lambda, \mu \in \mathbb{R}^n$ are two non-zero vectors, then there exists $A \in \mathrm{GL}(n,\mathbb{R})$ such that $A\lambda = \mu$, as λ (resp. μ) can be completed to a basis $v_1 = \{\lambda, v_2, \dots, v_n\}$ (resp. $v_1' = \{\mu, v_2', \dots, v_n'\}$) of $\mathbb{R}^n$ and A is the isomorphism $Av_i = v_i'$, $i = 1, \dots, n$. The isotropy group of $[e_1]$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ of elements B such that $B(\lambda_1, 0, \dots, 0) = (\mu_1, 0, \dots, 0)$, that is,

$$H = \left\{ B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{2n} & \cdots & b_{nn} \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$

So $\mathbb{R}\mathrm{P}^{n-1} = \mathrm{GL}(n,\mathbb{R})/H$. (Note that $\dim \mathbb{R}\mathrm{P}^{n-1} = \dim \mathrm{GL}(n,\mathbb{R}) - \dim H = n-1$, as expected.)

Problem 4.107 The *punctured Euclidean space* $\mathbb{R}^n \setminus \{0\}$ is homogeneous since $\mathrm{GL}(n,\mathbb{R})$ acts transitively on it.

(i) Determine the isotropy group H of $(1, 0, \dots, 0) \in \mathbb{R}^n \setminus \{0\}$.
(ii) Is the homogeneous space $\mathrm{GL}(n,\mathbb{R})/H$ reductive?

The relevant theory is developed, for instance, in Poor [10, Chap. 6].

Solution

(i)

$$H = \left\{ \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix} : v \in \mathbb{R}^{n-1},\, B \in \mathrm{GL}(n-1,\mathbb{R}) \right\}.$$

(ii) No, as we shall see giving two proofs.

1*st proof* The Lie algebra $\mathfrak{h}$ of H is, as it is easily checked by using the exponential map,

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} : v \in \mathbb{R}^{n-1},\, A \in \mathfrak{gl}(n-1,\mathbb{R}) \right\}.$$

Suppose that $\mathfrak{gl}(n,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$. Since $\dim \mathfrak{h} = (n-1)n$, one has $\dim \mathfrak{m} = n$. Let $E^i_j \in \mathfrak{gl}(n,\mathbb{R})$ be the matrix $(E^i_j)^h_k = \delta_{hi}\delta_{kj}$, so that $\{E^i_j\}^n_{i,j=1}$ is a basis of $\mathfrak{gl}(n,\mathbb{R})$.

First suppose that $n = 2$. Then the matrix E^1_1 can be written as

$$E^1_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} + \begin{pmatrix} 1 & -v \\ 0 & -a \end{pmatrix}, \quad a, v \in \mathbb{R},$$

with

$$E^{1\mathfrak{h}}_1 = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} \in \mathfrak{h}, \qquad E^{1\mathfrak{m}}_1 = \begin{pmatrix} 1 & -v \\ 0 & -a \end{pmatrix} \in \mathfrak{m}.$$

By virtue of the hypothesis, we have

$$\left[E^{1\mathfrak{h}}_1, E^{1\mathfrak{m}}_1\right] = \begin{pmatrix} 0 & -v \\ 0 & 0 \end{pmatrix} \in \mathfrak{m},$$

but this matrix also belongs to $\mathfrak{h}$, and hence $v = 0$. Moreover, E^1_2 belongs to $\mathfrak{h}$. Consequently,

$$\left[E^1_2, E^{1\mathfrak{m}}_1\right] = \begin{pmatrix} 0 & -(a+1) \\ 0 & 0 \end{pmatrix} \in \mathfrak{m},$$

and since this bracket also belongs to $\mathfrak{h}$, it follows that $a = -1$. Summarising, one has $E^{1\mathfrak{m}}_1 = I_2 \in \mathfrak{m}$. On the other hand, one has the decomposition

$$E^2_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -v \\ 1 & -a \end{pmatrix}, \quad a, v \in \mathbb{R},$$

with

$$E^{2\mathfrak{h}}_1 = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} \in \mathfrak{h}, \qquad E^{2\mathfrak{m}}_1 = \begin{pmatrix} 0 & -v \\ 1 & -a \end{pmatrix} \in \mathfrak{m}.$$

Again from $E^1_2 \in \mathfrak{h}$ we deduce that

$$\left[E^1_2, E^{2\mathfrak{m}}_1\right] = \begin{pmatrix} 1 & -a \\ 0 & -1 \end{pmatrix} \in \mathfrak{m},$$

and since I_2 and $[E^1_2, E^{2\mathfrak{m}}_1]$ are linearly independent, one concludes that

$$\mathfrak{m} = \left\langle I_2, \left[E^1_2, E^{2\mathfrak{m}}_1\right]\right\rangle,$$

which is impossible as in this case the matrix $E^{2\mathfrak{m}}_1$ could not belong to $\mathfrak{m}$, since its (2, 1)th entry is not null.

For $n \geqslant 3$, we have

$$E^2_1 = \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} + \begin{pmatrix} 0 & -v \\ u & -A \end{pmatrix},$$

$$A \in \mathfrak{gl}(n-1,\mathbb{R}),\ v \in \mathbb{R}^{n-1},\ {}^t u = (1,0,\dots,0) \in \mathbb{R}^{n-1},$$

with

$$E_1^{2\mathfrak{h}} = \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} \in \mathfrak{h}, \qquad E_1^{2\mathfrak{m}} = \begin{pmatrix} 0 & -v \\ u & -A \end{pmatrix} \in \mathfrak{m}.$$

As

$$E_3^1 = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \quad w = (0,1,\dots,0) \in \mathbb{R}^{n-1},$$

belongs to $\mathfrak{h}$, one has that

$$\left[E_3^1, E_1^{2\mathfrak{m}}\right] = \begin{pmatrix} 0 & -wA \\ 0 & -(u^i w^j) \end{pmatrix}$$

belongs to $\mathfrak{m}$, and since it also belongs to $\mathfrak{h}$, it is the null matrix. We are thus arrived at a contradiction, for the square matrix $(u^i w^j)$ of order $n-1$ never vanishes. □

2nd proof Another proof, this time unified for $n \geqslant 2$, and which uses representation theory, is the following. First, we identify $\mathfrak{gl}(n-1,\mathbb{R})$ with the subalgebra

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(n-1,\mathbb{R}) \right\}.$$

Then $\mathfrak{gl}(n,\mathbb{R})$ decomposes as a $\mathfrak{gl}(n-1,\mathbb{R})$-module into

$$\begin{aligned} \mathfrak{gl}(n,\mathbb{R}) = {} & \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\} \oplus \mathfrak{gl}(n-1,\mathbb{R}) \\ & \oplus \mathbb{R}E_1^1 \oplus \left\{ \begin{pmatrix} 0 & 0 \\ {}^t v & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\}, \end{aligned}$$

which is a sum of four non-isomorphic irreducible $\mathfrak{gl}(n-1,\mathbb{R})$-modules.

Every $\mathfrak{h}$-submodule of $\mathfrak{gl}(n,\mathbb{R})$ is in particular a $\mathfrak{gl}(n-1,\mathbb{R})$-module and hence a direct sum of some of the four $\mathfrak{gl}(n-1,\mathbb{R})$-submodules above. Thus, the only possibility for $\mathfrak{m}$ is

$$\mathfrak{m} = \mathbb{R}E_1^1 \oplus \left\{ \begin{pmatrix} 0 & 0 \\ {}^t v & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\},$$

but $[\mathfrak{h},\mathfrak{m}] \not\subseteq \mathfrak{m}$, from which we conclude that the space is not reductive. □

Problem 4.108 The *complex projective space* $\mathbb{C}\mathrm{P}^n$, which is the set of complex lines through the origin in the complex $(n+1)$-space $\mathbb{C}^{n+1}$, is diffeomorphic to the homogeneous space $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$.

(i) Does $\mathrm{SU}(n+1)$ act effectively on $\mathbb{C}\mathrm{P}^n$?
(ii) Write $\mathbb{C}\mathrm{P}^n$ as a homogeneous space G/H such that G acts effectively on $\mathbb{C}\mathrm{P}^n$.

Remark We recall that the centre $\mathbb{Z}_{n+1}$ of $\mathrm{SU}(n+1)$ consists of the diagonal matrices $\mathrm{diag}(\lambda, \dots, \lambda)$, λ being an $(n+1)$th root of 1.

Solution

(i) The answer is no, since the isotropy group $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ contains the centre $\mathbb{Z}_{n+1}$ of $\mathrm{SU}(n+1)$.

(ii) Let us compute the subgroup N. A matrix

$$s = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}, \qquad A \in \mathrm{U}(n),\ \lambda = \frac{1}{\det A},$$

belongs to N if and only if $g^{-1}sg \in \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ for all $g \in \mathrm{SU}(n+1)$. Let $\{v_1, \dots, v_{n+1}\}$ be the standard basis of $\mathbb{C}^{n+1}$, and let $g \in \mathrm{SU}(n+1)$ be the matrix given by

$$\begin{aligned} g(v_r) &= (\cos\alpha)v_r + (\sin\alpha)v_{n+1}, \\ g(v_{n+1}) &= -(\sin\alpha)v_r + (\cos\alpha)v_{n+1}, \\ g(v_i) &= v_i, \quad 1 \leqslant i \leqslant n,\ i \neq r, \end{aligned}$$

where $1 \leqslant r \leqslant n$ is a fixed index, and $\alpha \in \mathbb{R}$. Then, we must have $(g^{-1}sg)(v_{n+1}) = \mu v_{n+1}$ for some $\mu \in \mathbb{C}^*$, as $g^{-1}sg \in \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, or equivalently, $s(g(v_{n+1})) = \mu g(v_{n+1})$, and by expanding we have:

$$\begin{aligned} s\big(-(\sin\alpha)v_r + (\cos\alpha)v_{n+1}\big) &= -(\sin\alpha)A(v_r) + (\cos\alpha)\lambda v_{n+1} \\ &= \mu\big(-(\sin\alpha)v_r + (\cos\alpha)v_{n+1}\big). \end{aligned}$$

Hence $\lambda = \mu$ and $Av_r = \lambda v_r$ for all $r = 1, \dots, n$. Therefore, $A = \lambda I_n$, and since $1 = \lambda \det A = \lambda^{n+1}$, we conclude that $N = \mathbb{Z}_{n+1}$, which is the centre of $\mathrm{SU}(n+1)$. Accordingly, we can write

$$\mathbb{C}\mathrm{P}^n \cong \big(\mathrm{SU}(n+1)/\mathbb{Z}_{n+1}\big) \big/ \big(\mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big)/\mathbb{Z}_{n+1}\big).$$

The group $G = \mathrm{SU}(n+1)/\mathbb{Z}_{n+1}$ acts effectively on $\mathbb{C}\mathrm{P}^n$.

Problem 4.109 Consider the Lie group $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$ of automorphisms of the octonion algebra $\mathbb{O}$.

Prove:

(i) The Lie group G_2 is connected and simply connected.
(ii) The sphere S^6 is isomorphic to the homogeneous space $\mathrm{G}_2/\mathrm{SU}(3)$.
(iii) The real Stiefel manifold $V_2(\mathbb{R}^7)$ is isomorphic to the homogeneous space $\mathrm{G}_2/\mathrm{SU}(2)$.

Hint (to (i)–(iii)) Use the fact that for any elements $\xi, \eta, \zeta \in \mathbb{O}$ such that

$$|\xi| = |\eta| = |\zeta| = 1, \quad \eta \perp \xi,\ \zeta \perp \xi,\ \zeta \perp \eta,\ \zeta \perp \xi\eta,$$

there exists a unique automorphism $\Phi : \mathbb{O} \to \mathbb{O}$ such that

$$\Phi(\mathbf{i}) = \xi, \qquad \Phi(\mathbf{j}) = \eta, \qquad \Phi(\mathbf{e}) = \zeta$$

(see [11, Lect. 15, Lemma 1]), where $\mathbf{i}, \mathbf{j}, \mathbf{e}$ are the standard generators of the octonion algebra $\mathbb{O}$, and $|\cdot|$ denotes the norm in $\mathbb{O}$.

Hint (to (i)) To prove the simple connectedness of G_2, use the following well-known statement on the topology of homogeneous spaces (cf. [11, Lect. 12, Proposition 8]): "For a connected Lie group G and a closed connected subgroup H, the following holds: if the quotient space G/H is simply connected, then the fundamental group $\pi_1(G)$ of G is isomorphic to a quotient group of the fundamental group $\pi_1(H)$."

The relevant theory is developed, for instance, in Postnikov [11].

Solution

(i) The octonion algebra $\mathbb{O}$ can be considered as the real space $\mathbb{R}^8$ with the following basis:

$$\begin{aligned} &e_0 = 1, \quad e_1 = \mathbf{i}, \quad e_2 = \mathbf{j}, \quad e_3 = \mathbf{ij}, \quad e_4 = \mathbf{e}, \quad e_5 = \mathbf{ie}, \\ &e_6 = \mathbf{je}, \quad e_7 = (\mathbf{ij})\mathbf{e}. \end{aligned} \tag{$\star$}$$

Recall also that

$$e_p^2 = -1, \quad p = 1, \dots, 7 \quad \text{and} \quad e_p e_l = -e_l e_p, \quad p, l = 1, \dots, 7, \ p \neq l,$$

i.e. $e_1, \dots, e_7$ are (anticommuting) imaginary units of $\mathbb{O}$. This basis $\{e_0, \dots, e_7\}$ is orthonormal with respect to the scalar product $\langle u, v\rangle = (u\overline{v} + v\overline{u})/2$, where $\bar{\cdot}$ denotes conjugation in $\mathbb{O}$ (see Problem 4.81), that is, $\overline{e}_0 = e_0$ and $\overline{e}_p = -e_p$ for $p = 1, \dots, 7$. Any element of G_2 leaves invariant this scalar product, so that we may write $\mathrm{G}_2 \subset \mathrm{O}(8)$. But the unit e_0 is fixed under G_2. Therefore G_2 leaves invariant the subspace orthogonal to e_0, $V_7 = \langle e_1, \dots, e_7\rangle$, of purely imaginary octonions, and so $\mathrm{G}_2 \subset \mathrm{O}(7)$.

By the first hint above the group G_2 acts transitively on the sphere $S^6 \subset V_7$, i.e.

$$S^6 \cong \mathrm{G}_2/K, \quad \text{where } K = \{\Phi \in \mathrm{G}_2 : \Phi e_1 = e_1\}.$$

For any automorphism $\Phi \in K$, the element $\eta = \Phi e_2$ is orthogonal to e_1, and, consequently, it is an element of some sphere S^5. Moreover, by the first hint above,

$$S^5 \cong K/L, \quad \text{where } L = \{\Phi \in \mathrm{G}_2 : \Phi e_1 = e_1, \Phi e_2 = e_2\} \subset K.$$

But for any automorphism $\Phi \in L$, the element $\zeta = \Phi e_4$ is orthogonal to the elements e_1, e_2 and $e_3 = e_1 e_2$ (since they are preserved by the automorphism

Φ), that is, ζ belongs to some sphere $S^3 \subset \langle e_4, e_5, e_6, e_7\rangle$. Moreover, by the first hint above the map $\Phi \mapsto \Phi e_4$ is a diffeomorphism (by the uniqueness of Φ) of the Lie group L onto $S^3: \ L \cong S^3$.

In particular, since the Lie group L is connected and the sphere S^5, where K acts transitively, is connected, the Lie group K is connected (any connected component of K acts transitively on S^5). Similarly, considering the transitive action of G_2 on S^6 with connected isotropy group K, we obtain the connectedness of G_2.

On the other hand, according to the result recalled in the second hint, the group K is simply connected because the sphere $S^5 \cong K/L$ and the group $L \cong S^3$ are simply connected. Similarly the group G_2 is simply connected because the sphere $S^6 \cong G_2/K$ and the group K are simply connected.

(ii) Let us prove that $K \cong SU(3)$. Indeed, since the algebra $\mathbb{O}$ is alternative, that is, $u(uv) = (uu)v$ for any $u, v \in \mathbb{O}$ (cf. [11, Lect. 15]), the operator

$$I : \mathbb{O} \to \mathbb{O}, \qquad v \mapsto \mathbf{i}v,$$

defines a complex structure on the space $\mathbb{O}$. The six-dimensional subspace $V_6 = \langle e_2, \dots, e_7\rangle$ is invariant under I and can be regarded as a complex vector space with basis e_2, e_4, e_6 (see ($\star$)). The scalar product $\langle \cdot, \cdot\rangle$ in $\mathbb{O}$ satisfies $\langle Iu, Iv\rangle = \langle u, v\rangle$ because by the central Moufang identity in $\mathbb{O}$,

$$w(uv)w = (wu)(vw), \quad u, v, w \in \mathbb{O},$$

(cf. [11, Lect. 15, Lemma 2]), we have

$$(\mathbf{i}u)\overline{(\mathbf{i}v)} + (\mathbf{i}v)\overline{(\mathbf{i}u)} = (\mathbf{i}u)(-\overline{v}\mathbf{i}) + (\mathbf{i}v)(-\overline{u}\mathbf{i}) = -\mathbf{i}(u\overline{v} + v\overline{u})\mathbf{i} = u\overline{v} + v\overline{u}$$

as $u\overline{v} + v\overline{u} \in \mathbb{R}$ and $\mathbf{i}^2 = -1$. It is easy to verify that the form

$$\gamma(u, v) = \langle u, v\rangle + \langle Iu, v\rangle \mathbf{i}$$

on V_6 defines a positive definite Hermitian form and that the basis e_2, e_4, e_6 is orthonormal with respect to γ. Obviously, any automorphism $\Phi \in K$ maps V_6 onto itself and preserves the form γ (because $\Phi e_1 = e_1$, $e_1 = \mathbf{i}$). Thus the restriction map $\Phi \mapsto \Phi|_{V_6}$ is an embedding of K in SU(3), so that we have

$$K \subset U(3) \subset SO(6) \subset SO(7).$$

By a dimension argument, K is a codimension one subgroup of U(3). Let us prove that $K \cong SU(3)$. Since K is a connected and simply connected Lie group, it is sufficient to prove that the Lie algebra $\mathfrak{k}$ of K is the Lie algebra $\mathfrak{su}(3) \subset \mathfrak{u}(3)$. Indeed,

$$\mathfrak{u}(3) = \mathfrak{z} \oplus \mathfrak{su}(3),$$

where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{u}(3)$ and $\mathfrak{su}(3)$ is the maximal semisimple ideal. Consider the projection (homomorphism) π of $\mathfrak{u}(3)$ onto $\mathfrak{su}(3)$ along the ideal $\mathfrak{z}$. Let $\mathfrak{k}'$ be the image of $\mathfrak{k} \subset \mathfrak{u}(3)$ under this projection.

Suppose that $\mathfrak{k}' \neq \mathfrak{su}(3)$. Then $\mathfrak{k}'$ is a codimension one subalgebra of $\mathfrak{su}(3)$. The restriction to $\mathfrak{k}'$ of the adjoint representation of $\mathfrak{su}(3)$ is completely reducible as $\mathfrak{k}'$ is a compact Lie algebra. Thus $\mathfrak{su}(3) = \mathfrak{z}' \oplus \mathfrak{k}'$, where $\mathfrak{z}'$ is a one-dimensional $\mathfrak{k}'$-module. If the Lie algebra $\mathfrak{k}'$ were semi-simple, then this module would be trivial, i.e. $[\mathfrak{z}', \mathfrak{k}'] = 0$, and, consequently, $\mathfrak{z}'$ would be an ideal of $\mathfrak{su}(3)$ because $[\mathfrak{z}', \mathfrak{su}(3)] = 0$. This lead us to a contradiction, as $\mathfrak{su}(3)$ is a simple Lie algebra. If the compact Lie algebra $\mathfrak{k}'$ were not semi-simple, then it would be $\mathfrak{k}' = \mathfrak{c} \oplus \mathfrak{s}$, where $\mathfrak{c}$ is its non-trivial centre, and $\mathfrak{s}$ is its maximal semi-simple ideal. Therefore $\operatorname{rank}\mathfrak{s} \leqslant 1$ because the rank of the Lie algebra $\mathfrak{su}(3)$ equals 2 and $\dim\mathfrak{c} \geqslant 1$. But the unique compact semi-simple Lie algebra of rank 1 is isomorphic to $\mathfrak{su}(2)$. This leads us to a contradiction again as

$$\dim\mathfrak{su}(2) + 1 < \dim\mathfrak{su}(3) - 1\big(= \dim\mathfrak{k}'\big).$$

Thus $\mathfrak{k}' = \mathfrak{su}(3)$, and the homomorphism $\pi|_{\mathfrak{k}} \colon \mathfrak{k} \to \mathfrak{su}(3)$ is an isomorphism. Since the Lie algebra $\mathfrak{k}$ is semi-simple, any of its one-dimensional modules is trivial, that is,

$$\mathfrak{k} = \mathfrak{k}' = \mathfrak{su}(3).$$

(iii) We have proved that $K = \mathrm{SU}(3)$. In other words, $K = \{\Phi \in \mathrm{G}_2 : \Phi e_1 = e_1\}$ and also

$$K = \big\{g \in \operatorname{End}(V_7) : ge_1 = e_1,\ g(V_6) = V_6,\ gI = Ig \text{ on } V_6 \subset V_7,\ \det\nolimits_{\mathbb{C}} g = 1\big\}.$$

Now, it is evident that the subgroup $L = \{\Phi \in \mathrm{G}_2 : \Phi e_1 = e_1, \Phi e_2 = e_2\}$ of K is defined by the relations

$$\begin{aligned} L = \big\{g \in \operatorname{End}(V_7) &: ge_1 = e_1,\ g(V_6) = V_6,\ gI = Ig \text{ on } V_6 \subset V_7, \\ &\det\nolimits_{\mathbb{C}} g = 1,\ ge_2 = e_2\big\}, \end{aligned}$$

i.e. it is isomorphic to the Lie group $\mathrm{SU}(2)$.

The Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$ acts naturally on the real Stiefel manifold $V_2(\mathbb{R}^7)$. By the first hint above this action is transitive. Moreover, by definition the stabiliser of the pair $(e_1, e_2) \in V_2(\mathbb{R}^7)$ in G_2 is the Lie group $L \cong \mathrm{SU}(2)$. Thus,

$$V_2\big(\mathbb{R}^7\big) \cong \mathrm{G}_2/\mathrm{SU}(2).$$

Problem 4.110 Consider the octonion algebra $\mathbb{O}$ with its standard scalar product $\langle u, v\rangle = (u\overline{v} + v\overline{u})/2$. Let $\mathrm{SO}(8)$ be the Lie group of all operators on $\mathbb{O}$ preserving this scalar product with Lie algebra

$$\mathfrak{so}(8) = \big\{A \colon \mathbb{O} \to \mathbb{O},\ A \text{ linear} : \langle Au, v\rangle = -\langle u, Av\rangle\big\}.$$

Let $\mathfrak{so}(7)$ be its Lie subalgebra consisting of the maps $A \in \mathfrak{so}(8)$ satisfying $Ae_0 = 0$, where e_0 is the unit of $\mathbb{O}$. It is known that $\mathfrak{so}(8)$ is the unique simple real Lie algebra with an outer automorphism of order 3. In fact, if $\mathrm{Aut}(\mathfrak{so}(8))$ (resp. $\mathrm{Int}(\mathfrak{so}(8))$) denotes the group of all (resp. inner) automorphisms of $\mathfrak{so}(8)$, then the *group of outer automorphisms*,

$$\mathrm{Out}\big(\mathfrak{so}(8)\big) = \mathrm{Aut}\big(\mathfrak{so}(8)\big)/\,\mathrm{Int}\big(\mathfrak{so}(8)\big) \cong \mathfrak{S}_3,$$

the symmetric group on three letters. Let $\lambda, \kappa \in \mathrm{Aut}(\mathfrak{so}(8))$ be such that their images in $\mathrm{Aut}(\mathfrak{so}(8))/\,\mathrm{Int}(\mathfrak{so}(8))$ generate this group and satisfy the relations

$$\lambda^3 = 1, \qquad \kappa^3 = 1, \qquad \kappa\lambda\kappa = \lambda^2.$$

We may choose λ and κ as follows (see Freudenthal [5]). Define κ by

$$\kappa(A)u = \overline{(A\overline{u})}, \quad A \in \mathfrak{so}(8),\ u \in \mathbb{O}.$$

Then λ is uniquely defined by the *principle of triality* which states that for all $A \in \mathfrak{so}(8)$ and $u, v \in \mathbb{O}$, we have

$$\big(\lambda(A)u\big)v + u\big(\lambda^2(A)v\big) = \kappa(A)(uv). \tag{$\star$}$$

Consider the Lie subalgebra

$$\mathfrak{s} = \lambda\big(\mathfrak{so}(7)\big) \subset \mathfrak{so}(8).$$

There exists a unique connected (not necessary closed) subgroup S of $\mathrm{SO}(8)$ with Lie algebra $\mathfrak{s}$.

Prove:

(i) The orbit $Se_0 \subset \mathbb{O}$ is isomorphic to the sphere S^7, and the isotropy group S_{e_0} of the unit $e_0 \in \mathbb{O}$ is the Lie group $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$.

(ii) The connected Lie group S is simply connected, so that $S \cong \mathrm{Spin}(7)$,

$$S^7 \cong \mathrm{Spin}(7)/\mathrm{G}_2,$$

and the restriction $\lambda|_{\mathfrak{so}(7)} : \mathfrak{so}(7) \to \mathfrak{so}(8)$ is the eight-dimensional *basic spinor representation of* $\mathfrak{so}(7)$.

Hint (to (i)) Using the definition of the automorphism κ, show that $\kappa(\mathfrak{so}(7)) = \mathfrak{so}(7)$ and, using the principle of triality, show that for $A \in \mathfrak{so}(7)$ with $\lambda(A) \in \mathfrak{so}(7)$, we have $\lambda(A) = \lambda^2(A) = \kappa(A)$.

Hint (to (i)) To prove the connectedness of S_{e_0}, use the following well-known statement of the topology of homogeneous spaces (cf. Postnikov [11, Lect. 12, Proposition 6]): "Let G be a connected Lie group with a closed subgroup H, and let H^0 be the identity component of the Lie group H. Then the natural map $G/H^0 \to G/H$, $gH^0 \mapsto gH$, is a covering map."

Hint (to (ii)) To prove the simple connectedness of S, see the second hint in Problem 4.109.

One can find the relevant theory developed, for instance, in Freudenthal [5] and Postnikov [11].

Solution

(i) Since S is a Lie subgroup of SO(8), the orbit Se_0 is a subset of the sphere $S^7 = \{u \in \mathbb{O} : \langle u, u\rangle = 1\}$ containing the unit $e_0 \in \mathbb{O}$. Let us prove that the Lie algebra

$$\mathfrak{s}_{e_0} = \left\{\lambda(A),\ A \in \mathfrak{so}(7) : \lambda(A)e_0 = 0\right\} \subset \mathfrak{so}(7) \subset \mathfrak{so}(8)$$

of the isotropy group S_{e_0} is a Lie subalgebra of the Lie algebra

$$\mathfrak{g}_2 = \left\{B \in \mathfrak{so}(8) : (Bu)v + u(Bv) = B(uv),\ u, v \in \mathbb{O}\right\} \subset \mathfrak{so}(7)$$

of derivations of the octonion algebra. Indeed, since $\overline{e}_0 = e_0$, by the definition of κ we have that $\kappa(A)e_0 = 0$ for any $A \in \mathfrak{so}(7)$, i.e. $\kappa(\mathfrak{so}(7)) = \mathfrak{so}(7)$. Putting $u = v = e_0$ in relation ($\star$), we obtain that

$$\text{for} \quad A \in \mathfrak{so}(7): \quad \lambda(A)e_0 = 0 \quad \text{if and only if} \quad \lambda^2(A)e_0 = 0.$$

Now putting $u = e_0$ in ($\star$), we obtain that

$$\text{if} \quad \lambda(A)e_0 = 0,\ A \in \mathfrak{so}(8), \quad \text{then} \quad \lambda^2(A)v = \kappa(A)v, \quad \text{where } v \in \mathbb{O}.$$

Similarly, putting $v = e_0$ in ($\star$), we obtain that

$$\text{if} \quad \lambda^2(A)e_0 = 0,\ A \in \mathfrak{so}(8), \quad \text{then} \quad \lambda(A)u = \kappa(A)u, \quad \text{where } u \in \mathbb{O}.$$

In particular, if $A \in \mathfrak{so}(7)$ and $\lambda(A) \in \mathfrak{so}(7)$, then $\lambda^2(A) \in \mathfrak{so}(7)$ and

$$\lambda(A)u = \lambda^2(A)u = \kappa(A)u \quad \text{for all } u \in \mathbb{O},$$

and consequently, by ($\star$),

$$\big(\lambda(A)u\big)v + u\big(\lambda(A)v\big) = \lambda(A)(uv), \quad u, v \in \mathbb{O},$$

so that $\mathfrak{s}_{e_0} \subset \mathfrak{g}_2$. But the orbit Se_0 is a submanifold of the sphere S^7, in particular $(\dim \mathfrak{so}(7) - \dim \mathfrak{s}_{e_0}) \leqslant 7$. Then $\dim \mathfrak{s}_{e_0} \geqslant (21 - 7 = 14)$. Taking into account that $\dim \mathfrak{g}_2 = 14$ and $\mathfrak{s}_{e_0} \subset \mathfrak{g}_2$, we obtain that $\mathfrak{s}_{e_0} = \mathfrak{g}_2$ and the orbit Se_0 is an open connected subset of S^7. Showing now that this orbit is a closed subset of S^7, we obtain that $Se_0 = S^7$. To this end, it is sufficient to show that S is a closed (and, consequently, compact) subgroup of SO(8).

The normaliser of the Lie subalgebra $\mathfrak{so}(7)$ in $\mathfrak{so}(8)$ coincides with this algebra $\mathfrak{so}(7)$ because as it is easy to check the algebra $\mathfrak{so}(8)$ as an $\mathfrak{so}(7)$-module is a direct sum of its subalgebra $\mathfrak{so}(7)$ and a simple seven-dimensional module U_7 (the standard irreducible representation of $\mathfrak{so}(7)$ of lowest dimension), i.e. $\mathfrak{so}(8) = \mathfrak{so}(7) \oplus U_7$, where $[\mathfrak{so}(7), U_7] \subset U_7$. Since λ is an automorphism of

$\mathfrak{so}(8)$, the Lie algebra $\mathfrak{s} = \lambda(\mathfrak{so}(7))$ also coincides with its normaliser $\hat{\mathfrak{s}}$ in $\mathfrak{so}(8)$. Then the closed Lie subgroup $\hat{S} = \{h \in \mathrm{SO}(8) : \mathrm{Ad}_h(\mathfrak{s}) = \mathfrak{s}\}$ is a Lie subgroup of SO(8) with Lie algebra $\hat{\mathfrak{s}} = \mathfrak{s}$. Thus, S being a connected component of $\hat{S}$ (containing the unit element), it is also a closed subgroup of SO(8).

Let us show that $S_{e_0} = \mathrm{G}_2$. Let $S_{e_0}^0$ be the identity component of S_{e_0}. By the second hint above the natural map $S/S_{e_0}^0 \to S/S_{e_0}$, $hS_{e_0}^0 \mapsto hS_{e_0}$, is a covering map. But since the sphere $S^7 = S/S_{e_0}$ is simply connected, we obtain that the group S_{e_0} is connected. But the unique connected Lie subgroup of SO(8) with Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}(7) \subset \mathfrak{so}(8)$ is the Lie group $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$. Thus $S_{e_0} = \mathrm{G}_2$.

(ii) Let us prove that $S \cong \mathrm{Spin}(7)$. According to the result recalled in the second hint of Problem 4.109, the group S is simply connected because the sphere $S^7 \cong S/S_{e_0}$ and the group $S_{e_0} = \mathrm{G}_2$ are simply connected. Since the Lie algebra of S is isomorphic to $\mathfrak{so}(7)$, we obtain that $S \cong \mathrm{Spin}(7)$ and the corresponding representation

$$\mathfrak{so}(7) \to \mathfrak{so}(8), \qquad A \to \lambda(A),$$

is the basic spinor representation of $\mathfrak{so}(7)$.

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Chapter 5
Fibre Bundles

Abstract This chapter deals with problems on fibre bundles, principal bundles, connections on bundles, characteristic classes and almost complex manifolds, after giving a survey of definitions and results on such topics. Examples of the topics presented are: Fundamental vector fields on principal bundles and the complex and quaternionic Hopf bundles; principal U(1)-bundles and connections on them; Chern classes of the complex Hopf bundle and the tautological line bundle over the complex projective space $\mathbb{C}\mathrm{P}^1$. These topics, which might be deemed 'mysterious' by newcomers, are rendered clearer with the help of the explicit computations of some characteristic forms, classes or numbers, carried out in this chapter. The Section on characteristic classes includes two new problems on Godbillon–Vey class in the present edition. After that, linear connections, including a few calculations with indices, are shown. Furthermore, some holonomy groups and geodesics are studied, and then almost complex and complex manifolds, along with some examples of Nijenhuis tensor. The last section is devoted to almost symplectic manifolds, Hamilton's equations, and the relation with principal U(1)-bundles.

> The preceding definitions of tensor and tensor field are essentially equivalent to the classical definitions. The novelty of our treatment lies in the assignment of a topology to the set of tensors (of a prescribed type) at the various points of X. This is done in such a way as to form a bundle space under the natural projection into X. In most applications, Y is a linear space and G is a linear group; hence B is a differentiable manifold. The advantage of our approach is that a tensor field becomes a function in the ordinary sense. Its continuity and differentiability need not be given special definitions.
>
> NORMAN STEENROD, *The Topology of Fibre Bundles*, Princeton Mathematical Series, no. 14, P.U.P., 1999. (With kind permission from Princeton University Press.)
>
> In recent years the works of Stiefel, Whitney, Pontrjagin, Steenrod, Feldbau, Ehresmann, etc. have added considerably to our knowledge of the topology of manifolds with a differentiable structure, by introducing the notion of so-called fibre bundles. The topological invariants thus introduced on a manifold, called the characteristic cohomology classes, are to a certain extent

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_5,

susceptible of characterization, at least in the case of Riemannian manifolds, by means of the local geometry. Of those characterizations the generalized Gauss–Bonnet formula of Allendoerfer–Weil is probably the most notable example. (...) In the works quoted above, special emphasis has been laid on the sphere bundles, because they are the fibre bundles which arise from manifolds with a differentiable structure. Of equal importance are the manifolds with a complex analytic structure which play an important rôle in the theory of analytic functions of several complex variables and in algebraic geometry. The present paper will be devoted to a study of the fibre bundles of the complex tangent vectors of complex manifolds and their characteristic classes in the sense of Pontrjagin. It will be shown that there are certain basic classes from which all the other characteristic classes can be obtained by operations of the cohomology ring. These basic classes are then identified with the classes obtained by generalizing Stiefel–Whitney's classes to complex vectors. In the sense of de Rham the cohomology classes can be expressed by exact exterior differential forms which are everywhere regular on the (real) manifold.

SHIING-SHEN CHERN, "*Characteristic classes of Hermitian manifolds*," Ann. of Math. *47* (1946), no. 1, p. 85. (With kind permission from the Annals of Mathematics.)

5.1 Some Definitions and Theorems on Fibre Bundles

Definitions 5.1 (See Poor [7] and Problem 2.17 Above) Let

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

be a fibre bundle. The *vertical bundle on E* is the real vector bundle with total space

$$\mathscr{V}E := \pi_*^{-1}(0) = \bigcup_{\xi \in E} \ker(\pi_*|_\xi : E_\xi \to T_{\pi(\xi)}M) \subset TE.$$

Proposition 5.2 *If $E \xrightarrow{\pi} M$ is a vector bundle, then the vertical vector bundle $\mathscr{V}E$ is isomorphic to the pull-back vector bundle π^*E over E, that is, $\mathscr{V}E$ is isomorphic to E along π.*

Denote the isomorphism in Proposition 5.2 by $\mathscr{I}$, so that

$$\begin{aligned} \mathscr{I} :\ \pi^*E &\longrightarrow \mathscr{V}E \\ (\zeta, \xi) &\longmapsto \mathscr{I}_\zeta \xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\zeta + t\xi). \end{aligned}$$

Denote the vector bundle isomorphism from $\mathscr{V}E$ to E along π by pr_2,

$$\begin{aligned} \mathrm{pr}_2 \colon \mathscr{V}E &\longrightarrow E \\ \mathscr{I}_\zeta \xi &\longmapsto \xi. \end{aligned}$$

A *connection on a vector bundle* $\pi \colon E \to M$ is a vector sub-bundle $\mathscr{H}$ of $TE \to E$ such that:

(a) The sub-bundle $\mathscr{H}$ is complementary to the vertical bundle $\mathscr{V}E \to E$, that is,

$$TE = \mathscr{H} \oplus \mathscr{V}E.$$

(b) The sub-bundle $\mathscr{H}$ is homogeneous, i.e.

$$h_{\lambda *}\mathscr{H}_\xi = \mathscr{H}_{\lambda\xi}, \quad \lambda \in \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \ \xi \in E,$$

where $h_\lambda \xi = \lambda\xi$.

A connection on the tangent bundle $TM \to M$ is usually referred to as a *connection* on M.

Let $\mathscr{H}$ be a connection on a vector bundle E over M. Given $X \in TE = \mathscr{H} \oplus \mathscr{V}E$, it decomposes accordingly as $X = X^h + X^v$. The *connection map* κ of $\mathscr{H}$ is defined by

$$\begin{aligned} \kappa \colon TE &\longrightarrow E \\ X &\longmapsto \kappa(X) = \mathrm{pr}_2\big(X^v\big) = \mathrm{pr}_2\big(X - X^h\big). \end{aligned} \tag{5.1}$$

Definitions 5.3 A C^∞ *principal fibre bundle* (or simply a *principal bundle*) is a quadruple (P, π, M, G) where P, M are differentiable manifolds, G is a Lie group and π is a surjective submersion from P to M such that:

(i) G acts differentiably and freely on the right on P,

$$P \times G \to P.$$

For $g \in G$, one also writes $R_g \colon P \to P$ for the map $R_g u = ug$.

(ii) M is the quotient space of P by equivalence under G, so that for $p \in M$, G acts simply transitively on $\pi^{-1}(p)$.

(iii) P is locally trivial, that is, for any $p \in M$, there is an open neighbourhood U of p and a C^∞ map $\Phi_U \colon \pi^{-1}(U) \to G$ such that Φ_U commutes with R_g for every $g \in G$ and the map $\pi^{-1}(U) \to U \times G$ given by $u \mapsto (\pi(u), \Phi_U(u))$ is a diffeomorphism.

P is called the *bundle space* or the *total space*, π the *projection map*, M the *base space*, and G the *structure group*. For $p \in M$, $\pi^{-1}(p)$ is called the *fibre* over p. Each fibre is diffeomorphic to G via the map $j_u \colon G \to \pi^{-1}(\pi(u)) \subset P$, defined by $j_u(g) = R_g u$.

Let G be a Lie group acting on a differentiable manifold M on the right. Each element $A \in \mathfrak{g}$ induces a vector field $A^* \in \mathfrak{X}(M)$, corresponding to the action of the 1-parameter group $a_t = \exp tA$ on M. A^* is called the *fundamental vector field corresponding to* A.

Given a differentiable n-manifold M, a *linear frame* z at a point p is an ordered basis $(X_1, \dots, X_n)$ of the tangent space T_pM. The set FM of all linear frames at all points of M is a principal bundle called the *bundle of linear frames over* M, with projection map π sending each ordered basis of T_pM to the point p, and with group $\mathrm{GL}(n, \mathbb{R})$ acting on FM on the right.

There exists a natural $\mathbb{R}^n$-valued differential 1-form θ on FM called the *canonical form on the bundle of linear frames*, defined by

$$\theta(X) = z^{-1}(\pi_* X), \quad z \in \pi^{-1}(p),\ p \in M,\ X \in T_z(FM),$$

where the linear frame z is viewed as an isomorphism $z\colon \mathbb{R}^n \to T_pM$.

A *G-structure* on a differentiable n-manifold M is a principal sub-bundle of the bundle of linear frames FM whose structure group is a Lie subgroup $G \subseteq \mathrm{GL}(n, \mathbb{R})$.

Definition 5.4 Let (P, π, M, G) be a principal bundle, and let F be a manifold on which G acts on the left. The *fibre bundle associated to* (P, π, M, G) *with fibre* F is defined as follows. Let us consider the right action of G on the product $P \times F$ defined by $(u, f)g = (ug, g^{-1}f)$, where $p \in P$, $f \in F$, $g \in G$. The quotient space $E = (P \times F)/G$ under equivalence by G, is the bundle space of the associated fibre bundle.

The structure is as follows: The projection map $\pi_E\colon E \to M$ is defined by $\pi_E((u, f)G) = \pi(u)$. If $p \in M$, take a neighbourhood U of p as in Definitions 5.3 (3), with $\Phi_U\colon \pi^{-1}(U) \to G$. Then we have $\Psi_U\colon \pi_E^{-1}(U) \to F$ given by $\Psi_U((u, f)G) = \Phi_U(u)f$, so that $\pi_E^{-1}(U)$ is diffeomorphic to the product $U \times F$.

Definitions 5.5 Let (P, π, M, G) be a principal bundle. Denote by V_u the subspace of T_uP of vectors tangent to the fibre through $u \in P$. A *connection Γ in P* is an assignment of a subspace H_u of T_uP to each $u \in P$ such that:

(i) $T_uP = V_u \oplus H_u$;
(ii) $H_{ug} = R_{*g}H_u$,

$u \in P$, $g \in G$. The subspaces V_u and H_u are respectively called the *vertical* and the *horizontal* subspace of T_uP. We denote by v and h, respectively, the projections of T_uP onto V_u and H_u.

The connection Γ defines a differential 1-form ω on P, called the *connection form of* Γ, which takes values in the Lie algebra $\mathfrak{g}$ of G and satisfies, denoting by A^* the fundamental vector field on the total space P, corresponding to $A \in \mathfrak{g}$,

(i) $\omega(A^*) = A$, $A \in \mathfrak{g}$;
(ii) $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$, $g \in G$.

Given a $\mathfrak{g}$-valued 1-form on P satisfying the two conditions above, there is a unique connection Γ in P whose connection form is ω.

The *horizontal lift* of $X \in \mathfrak{X}(M)$ is the unique vector field $X^h \in \mathfrak{X}(P)$ which is horizontal and projects onto X, that is, $\pi_* X_u^h = X_{\pi(u)}$, for all $u \in P$.

Let ρ be a representation of G on a finite-dimensional real vector space V. Let α be a V-valued r-form on P such that $R_g^*\alpha = \rho(g^{-1})\alpha$, $g \in G$. The form $D\alpha$ defined by

$$(D\alpha)(X_1, \dots, X_{r+1}) = \big((\mathrm{d}\alpha) \circ h\big)(X_1, \dots, X_{r+1}) = \mathrm{d}\alpha(hX_1, \dots, hX_{r+1}),$$

for $X_1, \dots, X_{r+1} \in T_u P$, is called the *exterior covariant derivative of* α and D is called the *exterior covariant differentiation.*

The *curvature form* of the connection form ω is defined by $\Omega = D\omega$.

Definition 5.6 A connection in the fibre bundle of linear frames FM over the manifold M is called a *linear connection on M.*

Definition 5.7 A differentiable manifold M is called *parallelisable* if there exists a linear connection ∇ of M for which parallel transport is locally independent of curves. Such a ∇ is called a *flat connection.*

Definition 5.8 Let M be a differentiable manifold of dimension n, ∇ a linear connection on M, and $p \in M$. Let exp denote the restriction of the exponential map to T_pM. Under the identification $T_0(T_pM) \equiv T_pM$ one has $\exp_{*0} = \mathrm{id}\,|_{T_pM}$. Hence, according to the Inverse Map Theorem (see Theorem 1.12), there exists a star-shaped open neighbourhood V of 0 in T_pM and an open neighbourhood U of p in M such that $\exp(V) = U$ and $\exp\colon V \to U$ is a diffeomorphism.

A fixed basis $(u_1, \dots, u_n)$ of T_pM defines an isomorphism

$$u\colon \mathbb{R}^n \to T_pM, \qquad u(\zeta) = \sum_i u_i \zeta^i, \quad \zeta = \big(\zeta^1, \dots, \zeta^n\big) \in \mathbb{R}^n.$$

Let

$$\varphi\colon U \to A = u^{-1}(V), \qquad \varphi = u^{-1} \circ \exp^{-1}, \qquad x^i = r^i \circ \varphi,$$

where r^i stands for the ith coordinate function on $\mathbb{R}^n$. With the previous notations, (U, φ) is a chart of M, called a *normal coordinate system at p* (with respect to the connection ∇), and the functions $x^i : U \to \mathbb{R}$ are the coordinate functions of φ.

Definition 5.9 An *almost complex structure* on a differentiable manifold M is a differentiable map $J : TM \to TM$, such that:

(i) J maps linearly T_pM into T_pM for all $p \in M$;
(ii) $J^2 = -I$ on each T_pM, where I stands for the identity map.

Definitions 5.10 A *complex manifold M* is defined similarly to a differentiable manifold, but taking homeomorphisms from open subsets of M to $\mathbb{C}^n$ instead of $\mathbb{R}^n$, and the changes of charts $\varphi_\alpha \circ \varphi_\beta^{-1}$ being holomorphic functions on $\mathbb{C}^n$. The number n is called the *complex dimension* of M and one writes $\dim_{\mathbb{C}} M = n$. A maximal set of charts is now called a *complex structure*. A complex manifold is a differentiable manifold, as it follows from the identification $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ obtained taking $z^k = x^k + \mathrm{i}y^k$, for $x^k, y^k \in \mathbb{R}$.

A complex manifold admits an almost complex structure J, taking the linear map J_p at any $p \in M$ defined by

$$J_p\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = \left.\frac{\partial}{\partial y^k}\right|_p, \qquad J_p\left(\left.\frac{\partial}{\partial y^k}\right|_p\right) = -\left.\frac{\partial}{\partial x^k}\right|_p,$$

where $z^k = x^k + \mathrm{i}y^k$ are the coordinate functions in a chart (U, φ) around p. The tensor field J does not depend on the chosen coordinates by virtue of the following result: A map f of an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ preserves the standard almost complex structures of $\mathbb{C}^n$ and $\mathbb{C}^m$ (i.e. $f_* \circ J = J \circ f_*$) if and only if f is holomorphic. The tensor field J is called *the almost complex structure* of the complex manifold M.

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$ and let g be a Riemannian metric on M as a differentiable manifold. If g and the almost complex structure J of M satisfy

$$g_p(J_p v, J_p w) = g_p(v, w), \quad p \in M, \ v, w \in T_p M,$$

then g is said to be a *Hermitian metric* and (M, J, g) is called a *Hermitian manifold*.

The tensor field F on such a manifold defined at any $p \in M$ by

$$F_p(v, w) = g_p(v, J_p w), \quad v, w \in T_p M,$$

is called the *fundamental* (or *Kähler*) *form* of the Hermitian metric g. A *Kähler manifold* is a Hermitian manifold whose Kähler form is closed: $\mathrm{d}F = 0$. It can be proved that this is equivalent to $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g.

Definitions 5.11 Let M be a connected complex manifold of complex dimension n. Given $p \in M$, three definitions are usually considered of tangent space to M at p, of real dimension $2n$:

$T_p M$: *The real tangent space at* p. M has the underlying structure of a $2n$-dimensional differentiable manifold, and $T_p M$ refers to the tangent space of this underlying real structure, that is, to the space of real derivations of $C_p^\infty M$.

A basis of $T_p M$ can be exhibited as follows: Let $z^1, \ldots, z^n$ be local complex coordinates near p and let $z^k = x^k + \mathrm{i}y^k$, $k = 1, \ldots, n$; then $x^1, \ldots, x^n, y^1, \ldots, y^n$, are real coordinates near p and

$$\left\{ \left.\frac{\partial}{\partial x^k}\right|_p, \left.\frac{\partial}{\partial y^k}\right|_p : k = 1, \ldots, n \right\}$$

is a basis of $T_p M$ over $\mathbb{R}$.

The linear map J_p converts $T_p M$ into a complex space with $\dim_{\mathbb{C}} T_p M = n$ by the definition

$$(a + \mathrm{i}b)X = aX + bJ_p X, \quad X \in T_p M, \ a + \mathrm{i}b \in \mathbb{C}.$$

$T_p^h M$: *The holomorphic tangent space at* p, which is the complex vector space of all complex derivations of the local algebra $\mathscr{O}_p M$ of germs of holomorphic functions at p, that is, the $\mathbb{C}$-complex functions $Z\colon \mathscr{O}_p M \to \mathbb{C}$ such that

$$Z(fg) = (Zf)g(p) + f(p)Zg, \quad f, g \in \mathscr{O}_p M.$$

With $z^1, \dots, z^n$ as above,

$$\left\{ \frac{\partial}{\partial z^k}\bigg|_{\mathrm{p}} : k = 1, \dots, n \right\}$$

is a basis of $T_p^h M$ over $\mathbb{C}$, where by definition

$$\frac{\partial}{\partial z^k}\bigg|_p (f) = \frac{\partial f}{\partial z^k}(p),$$

for any holomorphic function f defined near p.

$T_p^{1,0} M$: *The space of vectors of type* $(1,0)$, which is the complex subspace of the complexification $T_p^c M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$ defined by the $(+\mathrm{i})$-eigenspace of the complexification of J. Then, $T_p^{1,0} M$ is spanned by the elements of the form $X - \mathrm{i}JX$, where $X \in T_p M$. That is, with z^k and x^k, y^k as above, since

$$J_p \big(\partial/\partial x^k\big)_p = \big(\partial/\partial y^k\big)_p, \qquad J_p \big(\partial/\partial y^k\big)_p = -\big(\partial/\partial x^k\big)_p,$$

a basis of $T_p^{1,0} M$ is given by

$$\left\{ \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k} \right)_p ;\ k = 1, \dots, n \right\}.$$

Note that every element $Z \in T_p^c M$ can be written as

$$Z = X + \mathrm{i}Y \cong X \otimes 1 + Y \otimes \mathrm{i}, \quad X, Y \in T_p M.$$

Let

$$TM = \bigcup_{p \in M} T_p M, \qquad T^h M = \bigcup_{p \in M} T_p^h M, \qquad T^{1,0} M = \bigcup_{p \in M} T_p^{1,0} M,$$

be the bundles defined fibrewise. $T^h M$ has the obvious structure of a holomorphic vector bundle and it is called the *holomorphic vector bundle* of M.

Definitions 5.12 Let T^*M be the cotangent bundle over a differentiable manifold M of dimension n and let $\pi : T^*M \to M$ be the natural projection. The *canonical* 1-*form* ϑ on T^*M is defined by

$$\vartheta_\omega(X) = \omega(\pi_* X), \quad \omega \in T^*M,\ X \in T_\omega T^*M.$$

An *almost symplectic manifold* is a differentiable manifold M endowed with a non-degenerate differential 2-form Ω. In this case, $\dim M = 2n$, and

$$v = \frac{(-1)^n}{n!}\Omega \wedge \overset{(n)}{\cdots} \wedge \Omega$$

is a volume form on M, called the *standard volume form* associated with Ω. A *symplectic manifold* is an almost symplectic manifold whose corresponding 2-form is closed: $\mathrm{d}\Omega = 0$.

Definitions 5.13 Let (M, Ω) be a symplectic manifold and let H be a smooth function on M. Since the form Ω is non-degenerate, there exists a unique smooth vector field $X_H \in \mathfrak{X}(M)$ such that

$$i_{X_H}\Omega = -\mathrm{d}H.$$

This vector field is called the *Hamiltonian vector field* of the function H. A vector field $X \in \mathfrak{X}(M)$ is called *locally Hamiltonian* if the 1-form $i_X\Omega$ is closed. Since each closed form is locally exact (by the Poincaré lemma) for any point $x \in M$ there exists a neighbourhood $U \subset M$ of x and a local function $f \in C^\infty U$ such that $i_X\Omega = -\mathrm{d}f$.

For any two functions $H, F \in C^\infty M$ the smooth function

$$\{H, F\} = -X_H(F) = -\mathrm{d}F(X_H) = \Omega(X_F, X_H)$$

is called the *Poisson bracket* of the functions H and F.

Theorem 5.14 (Darboux's Theorem) *If (M, Ω) is a symplectic manifold of dimension $2n$, then for every $p \in M$ there exists a chart $(U, x^1, \dots, x^n, y^1, \dots, y^n)$ centred at p such that*

$$\Omega|_U = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i.$$

5.2 Principal Bundles

Problem 5.15 Let $\pi\colon P \to M$ be a surjective submersion and let $P \times G \to P$ be an action of G on P satisfying the following two properties:

(i) G acts freely, i.e. $u \cdot g = u$, $u \in P$, $g \in G$, implies $g = e$.
(ii) The fibres of π coincide with the orbits of G, i.e.

$$\pi^{-1}(x) = u \cdot G, \quad u \in \pi^{-1}(x),\ x \in M.$$

Prove:

1. If π admits a global section (i.e. there exists a smooth map $s\colon M \to P$ such that $\pi \circ s = \mathrm{id}_M$), then π is trivialisable, i.e. there exists a G-equivariant diffeomorphism $\Phi\colon M \times G \to P$.

2. Every $x \in M$ admits an open neighbourhood U such that $\pi^{-1}(U)$ is G-equivariantly diffeomorphic to $U \times G$.

Solution

1. The smooth map

$$\Phi \colon M \times G \to P, \qquad \Phi(x,g) = s(x) \cdot g, \quad x \in M,\ g \in G,$$

is bijective. In fact, if $\Phi(x,g) = \Phi(x',g')$, then

$$\begin{aligned} x &= \pi\big(s(x) \cdot g\big) && \text{(by (ii))} \\ &= \pi\big(\Phi(x,g)\big) = \pi\big(\Phi\big(x',g'\big)\big) = \pi\big(s\big(x'\big) \cdot g'\big) \\ &= x' && \text{(by (ii)).} \end{aligned}$$

Hence, $s(x) \cdot g = s(x) \cdot g'$, or equivalently, $s(x) = s(x) \cdot (g'g^{-1})$, and we conclude $g = g'$ by virtue of (i). Furthermore, we have

$$x = \pi(u) = \pi\big(s(x)\big), \quad u \in P.$$

From (ii) we deduce that an element $g \in G$ exists, such that $s(x) \cdot g = u$, or equivalently, $\Phi(x,g) = u$. The map Φ is also G-equivariant as

$$\Phi\big(x, gg'\big) = s(x) \cdot \big(gg'\big) = \big(s(x) \cdot g\big) \cdot g' = \Phi(x,g) \cdot g', \quad x \in M,\ g, g' \in G.$$

In order to prove that Φ_* is an isomorphism at every $(x_0, g_0) \in M \times G$, it suffices to prove that Φ_* is surjective, as $\dim P = \dim M + \dim G$. In fact,

$$\dim P = \dim T_u P = \dim T_x M + \dim \pi^{-1}(x) = \dim M + \dim \pi^{-1}(x),$$

for all $u \in P$, $x = \pi(u)$, because π is a submersion; but $\pi^{-1}(x) \cong G$ by virtue of (i) and (ii).

If R_g denotes the right translation by $g \in G$ on P as well as on $M \times G$, then $\Phi = R_g \circ \Phi \circ R_{g^{-1}}$, as it is readily checked. Taking differentials, we obtain

$$\Phi_{*(x_0,g_0)} = (R_{g_0})_{*s(x_0)} \circ \Phi_{*(x_0,e)} \circ (R_{g_0^{-1}})_{*(x_0,g_0)}.$$

Hence $\Phi_{*(x_0,g_0)}$ is surjective if and only if $\Phi_{*(x_0,e)}$ is. By using the natural identification $T_{(x_0,e)} M \times G = T_{x_0} M \oplus T_e G$, we have

$$\Phi_{*(x_0,e)}(X_1, X_2) = \big(\Phi^e_*\big)_{x_0}(X_1) + \big(\Phi^{x_0}_*\big)_e(X_2), \quad X_1 \in T_{x_0} M,\ X_2 \in T_e G,$$

where $\Phi^{x_0}(g) = \Phi(x_0, g)$, $\Phi^e(x) = \Phi(x,e) = s(x)$, are the partial mappings. As π is a submersion, the following exact sequence holds:

$$0 \to T_{s(x_0)}\big(s(x_0) \cdot G\big) \to T_{s(x_0)} P \xrightarrow{\pi_*} T_{x_0} M \to 0,$$

which splits via s_{*x_0}, i.e. every $X \in T_{s(x_0)}P$ can be written as

$$X = s_{*x_0}(X_1) + \left(\Phi_*^{x_0}\right)_e (X_2)$$

for certain X_1, X_2, where the identity $T_{s(x_0)}(s(x_0) \cdot G) = (\Phi_*^{x_0})_e(T_e G)$ is used.

2. As π is submersive, every $x \in M$ admits a section $s \colon U \to P$ defined on an open neighbourhood U and we conclude by simply applying the previous result to $\pi \colon \pi^{-1}(U) \to U$.

Problem 5.16 A morphism of principal bundles $\pi \colon P \to M$, $\pi' \colon P' \to M'$, with structure groups G, G', respectively, is a smooth map $\Phi \colon P \to P'$ and a Lie group homomorphism $\gamma \colon G \to G'$ such that

$$\Phi(u \cdot g) = \Phi(u) \cdot \gamma(g), \quad u \in P, \ g \in G.$$

(i) If $\Phi \colon P \to P'$ is a morphism, then prove that there exists a unique smooth map $\phi \colon M \to M'$ making the following diagram commutative:

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\phi} & M' \end{array}$$

(ii) Suppose now that $M' = M$, $G' = G$, $\pi = \pi'$, and $\gamma = \mathrm{id}_G$. Then a principal-bundle morphism $\Phi \colon P \to P$, is said to be an automorphism if Φ is a diffeomorphism.

Prove that all the automorphisms constitute a group with respect to composition. This group is denoted by $\operatorname{Aut} P$. Prove that the map $\operatorname{Aut} P \to \operatorname{Diff} M$, $\Phi \mapsto \phi$, with ϕ given as in (i) above, is a group homomorphism.

Remark The kernel of the last homomorphism is called the *gauge group of P* and it is denoted by $\operatorname{Gau} P$.

Solution

(i) As π is surjective, if ϕ exists, then it is unique. In order to prove the existence of ϕ, it suffices to prove that if $u, v \in P$ are two points in the fibre of $x \in M$, then $\pi'(\Phi(u)) = \pi'(\Phi(v))$. As u and v belong to the same fibre, by virtue of the condition (ii) of Problem 5.15, there exists $g \in G$ such that $v = u \cdot g$. Hence

$$\pi'\big(\Phi(v)\big) = \pi'\big(\Phi(u \cdot g)\big) = \pi'\big(\Phi(u) \cdot \gamma(g)\big) = \pi'\big(\Phi(u)\big).$$

The continuity and smoothness of ϕ follow from the characteristic property of submersions, according to which a map $\phi \colon M \to M'$ is continuous (resp., differentiable) if and only if the composite map $\phi \circ \pi$ is.

(ii) If $\psi \in \operatorname{Diff} M$ and $\Psi \in \operatorname{Aut} P$ are such that $\pi \circ \Psi = \psi \circ \pi$, then

$$\begin{aligned}(\Phi \circ \Psi)(u \cdot g) &= \Phi\big(\Psi(u \cdot g)\big) = \Phi\big(\Psi(u) \cdot g\big) = \Phi\big(\Psi(u)\big) \cdot g \\ &= (\Phi \circ \Psi)(u) \cdot g, \\ \pi \circ (\Phi \circ \Psi) &= (\pi \circ \Phi) \circ \Psi = (\phi \circ \pi) \circ \Psi = \phi \circ (\pi \circ \Psi) = \phi \circ (\psi \circ \pi) \\ &= (\phi \circ \psi) \circ \pi,\end{aligned}$$

for all $u \in P$, $g \in G$, thus proving that $\Phi \circ \Psi$ belongs to $\operatorname{Aut} P$ and its associated diffeomorphism on the base manifold is $\phi \circ \psi$, and we obtain an exact sequence of groups,

$$1 \to \operatorname{Gau} P \to \operatorname{Aut} P \to \operatorname{Diff} M.$$

Problem 5.17 (Hopf Bundles) Set

$$\begin{aligned}S^1 &= \big\{x \in \mathbb{C} : |x| = 1\big\}, \\ S^2 &= \big\{(x, t) \in \mathbb{C} \times \mathbb{R} : |x|^2 + t^2 = 1\big\}, \\ S^3 &= \big\{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\big\}, \\ S^7 &= \big\{(x, y) \in \mathbb{H}^2 : |x|^2 + |y|^2 = 1\big\}.\end{aligned}$$

The spheres S^1 and S^3 are Lie groups with respect to the multiplication induced from $\mathbb{C}$ and $\mathbb{H}$, respectively (see Problem 4.103). Let S^1 act on S^3 (resp., S^3 on S^7) by the formula

$$(x, y) \cdot z = (xz, yz), \quad (x, y) \in S^3,\ z \in S^1 \ \big(\text{resp. } (x, y) \in S^7,\ z \in S^3\big).$$

Let

$$\pi_{\mathbb{C}} \colon S^3 \to \mathbb{C} \times \mathbb{R}, \qquad \pi_{\mathbb{H}} \colon S^7 \to \mathbb{H} \times \mathbb{R},$$

be the maps given by

$$\begin{aligned}\pi_{\mathbb{C}}(x, y) &= \big(2y\bar{x}, |x|^2 - |y|^2\big), \quad (x, y) \in S^3, \\ \pi_{\mathbb{H}}(x, y) &= \big(2y\bar{x}, |x|^2 - |y|^2\big), \quad (x, y) \in S^7.\end{aligned}$$

Prove:

(i) $\pi_{\mathbb{C}}(S^3) = S^2$.
(ii) $\pi_{\mathbb{H}}(S^7) = S^4$.
(iii) The induced map $\pi_{\mathbb{C}} \colon S^3 \to S^2$ is a principal S^1-bundle with respect to the action of S^1 on S^3 defined above.
(iv) The induced map $\pi_{\mathbb{H}} \colon S^7 \to S^4$ is a principal S^3-bundle with respect to the action of S^3 on S^7 defined above.
(v) $\mathbb{C}\mathrm{P}^1 \cong S^2$.
(vi) $\mathbb{H}\mathrm{P}^1 \cong S^4$.

Solution We solve the quaternionic case (ii), (iv), and (vi). The same formulae solve the complex case (i), (iii), and (v), too.

(ii) First we check that $\pi_{\mathbb{H}}(S^7) \subseteq S^4$. In fact, if $(x, y) \in S^7$ one has $|x|^2 + |y|^2 = 1$ and then, since $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ for every $q_1, q_2 \in \mathbb{H}$, we have

$$\left|\pi_{\mathbb{H}}(x, y)\right|^2 = 4|y|^2|x|^2 + \left(|x|^2 - |y|^2\right)^2 = \left(|x|^2 + |y|^2\right)^2 = 1.$$

Let $(u, t) \in \mathbb{H} \times \mathbb{R}$ be a point in S^4, such that $|u|^2 + t^2 = 1$. If $u = 0$, then $t = \pm 1$, and we have $\pi_{\mathbb{H}}(1, 0) = (0, 1)$, $\pi_{\mathbb{H}}(0, 1) = (0, -1)$. Hence we can assume $u \neq 0$. In this case $-1 < t < 1$, and one has

$$\pi_{\mathbb{H}}\left(\sqrt{\frac{1+t}{2}}\,\frac{\bar{u}}{|u|}, \sqrt{\frac{1-t}{2}}\right) = (u, t)$$

and

$$\left(\sqrt{\frac{1+t}{2}}\,\frac{\bar{u}}{|u|}, \sqrt{\frac{1-t}{2}}\right) \in S^7.$$

(iv) First we have

$$\begin{aligned}\pi_{\mathbb{H}}\big((x, y) \cdot z\big) = \pi_{\mathbb{H}}(xz, yz) &= \big(2yz\overline{xz}, |xz|^2 - |yz|^2\big)\\ &= \big(2yz\bar{z}\bar{x}, |x|^2|z|^2 - |y|^2|z|^2\big)\\ &= \big(2y\bar{x}, |x|^2 - |y|^2\big) = \pi_{\mathbb{H}}(x, y),\end{aligned}$$

as $z\bar{z} = |z|^2 = 1$ for $z \in S^3$. Hence the orbit $(x, y) \cdot S^3$ is contained in the fibre $\pi_{\mathbb{H}}^{-1}(\pi_{\mathbb{H}}(x, y))$.

Conversely, if $\pi_{\mathbb{H}}(x_1, y_1) = \pi_{\mathbb{H}}(x_2, y_2)$, then

$$y_1 \bar{x}_1 = y_2 \bar{x}_2, \tag{$\star$}$$

$$|x_1|^2 - |y_1|^2 = |x_2|^2 - |y_2|^2. \tag{$\star\star$}$$

As $|x_1|^2 + |y_1|^2 = 1$, either $x_1 \neq 0$ or $y_1 \neq 0$. Hence we can assume $x_1 \neq 0$. Set $z = x_1^{-1} x_2 \in \mathbb{H}$. Hence ($\star$) implies, since $|q_1||q_2| = |q_1 q_2|$ for every $q_1, q_2 \in \mathbb{H}$,

$$\bar{y}_1 = z\bar{y}_2, \quad \text{i.e.} \quad y_1 = y_2 \bar{z}. \tag{$\star\star\star$}$$

As $|x_1|^2 + |y_1|^2 = |x_2|^2 + |y_2|^2 = 1$, we have $|y_1|^2 = |y_2|^2$ by ($\star\star$), from which $|y_2|^2|z|^2 = |y_1|^2$. If $y_2 = 0$ we should have from ($\star$) that $y_1 = 0$ and so $|x_1|^2 = |x_2|^2 = 1 = |z|^2$. If $y_2 \neq 0$, then $|z| = 1$. That is, in both cases we have $|z| = 1$. Therefore, $z^{-1} = \bar{z}$ and from ($\star\star\star$) we deduce $y_2 = y_1 z$. In other words, $z \in S^3$ and $(x_1, y_1) \cdot z = (x_2, y_2)$, thus concluding.

(vi) By definition, $\mathbb{HP}^1$ is the quotient space $(\mathbb{H}^2 \setminus \{(0, 0)\})/\sim$, where $(x, y) \sim (x', y')$ if and only if there exists $\lambda \in \mathbb{H}^*$ such that $x' = \lambda x$, $y' = \lambda y$. Moreover, the restriction to S^7 of the quotient map

$$q\colon \mathbb{H}^2 \setminus \big\{(0, 0)\big\} \to \mathbb{HP}^1$$

is surjective as $q(x, y) = q(x/r, y/r)$, with $r = \sqrt{|x|^2 + |y|^2}$, and its fibres are the orbits of S^3. Hence $\mathbb{HP}^1 \cong S^7/S^3$ and since we have a principal S^3-bundle $\pi_{\mathbb{H}}\colon S^7 \to S^4$, we conclude that $\mathbb{HP}^1 \cong S^7/S^3 \cong S^4$.

Problem 5.18 Parametrise S^3 (see Remark 1.4) by

$$z_1 = \cos\frac{1}{2}\theta \mathrm{e}^{\psi_1 \mathrm{i}}, \qquad z_2 = \sin\frac{1}{2}\theta \mathrm{e}^{\psi_2 \mathrm{i}}, \quad 0 < \theta < \pi,\ 0 < \psi_1, \psi_2 < 2\pi.$$

(i) Find the expression of $\pi_{\mathbb{C}}(z_1, z_2)$ under the projection map of the Hopf bundle $\pi_{\mathbb{C}}\colon S^3 \to S^2$ given in Problem 5.17, in terms of that parametrisation.
(ii) Take as trivialising neighbourhoods $U_1 = S^2 \backslash \{S\}$ and $U_2 = S^2 \backslash \{N\}$, where N, S stand for the north and south pole. Determine $\pi_{\mathbb{C}}^{-1}(U_k)$, $k = 1, 2$.
(iii) Define bundle trivialisations

$$f_k\colon \pi_{\mathbb{C}}^{-1}(U_k) \to U_k \times \mathrm{U}(1), \qquad f_k(z_1, z_2) = \left(\pi_{\mathbb{C}}(z_1, z_2), \frac{z_k}{|z_k|}\right), \quad k = 1, 2,$$

and put $f_{k,p} = f_k|_{\pi^{-1}(p)}$. Find the transition function $g_{21}\colon U_1 \cap U_2 \to \mathrm{U}(1)$ of the bundle with respect to the given trivialisations.

The relevant theory is developed, for instance, in Göckeler and Schücker [4].

Solution

(i)

$$\begin{aligned}\pi_{\mathbb{C}}(z_1, z_2) &= \left(2\,\mathrm{Re}(z_2\bar{z}_1), 2\,\mathrm{Im}(z_2\bar{z}_1),\ |z_1|^2 - |z_2|^2\right)\\ &= \left(\sin\theta\cos(\psi_2 - \psi_1), \sin\theta\sin(\psi_2 - \psi_1), \cos\theta\right).\end{aligned}$$

(ii) It is easily seen from the definitions of the trivialising neighbourhoods that

$$\pi_{\mathbb{C}}^{-1}(U_k) = \left\{(z_1, z_2) \in S^3 : z_k \neq 0\right\}, \quad k = 1, 2.$$

(iii) Given

$$p = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in U_1 \cap U_2, \qquad \mathrm{e}^{\alpha\mathrm{i}} \in \mathrm{U}(1),$$

with $0 < \varphi < 2\pi$, we obtain, on account of the parametrisation of S^3 and the expression for the projection map π, that

$$f_k(z_1, z_2) = \left(\sin\theta\cos(\psi_2 - \psi_1), \sin\theta\sin(\psi_2 - \psi_1), \cos\theta, \mathrm{e}^{\psi_k\mathrm{i}}\right),$$

so

$$f_{1,p}^{-1}\left(\mathrm{e}^{\alpha\mathrm{i}}\right) = \left(\cos\frac{1}{2}\theta\mathrm{e}^{\alpha\mathrm{i}}, \sin\frac{1}{2}\theta\mathrm{e}^{(\varphi+\alpha)\mathrm{i}}\right),$$

hence

$$\left(f_{2,p} \circ f_{1,p}^{-1}\right)\left(e^{\alpha i}\right) = e^{(\varphi+\alpha)i}.$$

That is, the transition function for the given trivialisations is

$$g_{21} \colon U_1 \cap U_2 \to \mathrm{U}(1), \qquad (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \mapsto e^{\varphi i}.$$

Problem 5.19 Let (P, π, M, G) be a principal fibre bundle and let $\mathfrak{g}$ be the Lie algebra of G. Every $A \in \mathfrak{g}$ induces a vector field $A^* \in \mathfrak{X}(P)$ (called the *fundamental vector field associated with* A), with flow

$$\psi_t(u) = u \exp tA, \quad u \in P.$$

The map

$$\varphi \colon \mathfrak{g} \to \mathfrak{X}(P), \qquad \varphi(A) = A^*,$$

is $\mathbb{R}$-linear, injective and satisfies $[A, B]^* = [A^*, B^*]$, for all $A, B \in \mathfrak{g}$.

(i) Prove that $R_g \cdot A^* = (\mathrm{Ad}_{g^{-1}} A)^*$, where $g \in G$, $A \in \mathfrak{g}$.

(ii) Calculate the expression for $\varphi(aX_1 + bX_2)$, where X_1 and X_2 are the left-invariant vector fields on $\mathbb{C}^*$ given by

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

and φ is the isomorphism associated to the principal fibre bundle

$$\left(\mathbb{C}^{n+1} \setminus \{0\}, \pi, \mathbb{C}\mathrm{P}^n, \mathbb{C}^*\right),$$

where $\mathbb{C}\mathrm{P}^n$ stands for the complex projective space of real dimension $2n$.

The relevant theory is developed, for instance, in Bishop and Crittenden [1].

Solution

(i) For $u \in P$, denote by j_u the injection of G into P given by

$$j_u \colon G \to \pi^{-1}\big(\pi(u)\big), \qquad g \mapsto ug.$$

Let e be the identity element of G. It is clear that

$$A_u^* = \psi_t(u)'(0) = j_{u*}\big((\exp tA)'(0)\big) = j_{u*}A_e.$$

Let $\iota_g \colon G \to G$ be the automorphism of G defined by $\iota_g(h) = ghg^{-1}$, and consider the composition map

$$\begin{array}{ccccc} G & \xrightarrow{j_{ug^{-1}}} & P & \xrightarrow{R_g} & P \\ h & \longmapsto & ug^{-1}h & \longmapsto & ug^{-1}hg = u\iota_{g^{-1}}(h) \end{array}$$

whose differential at e is

$$R_{g*} j_{ug^{-1}*} A_e = (j_u \iota_{g^{-1}})_* A_e = \big(\varphi\big(\mathrm{Ad}_{g^{-1}}(A_e)\big)\big)_u = (\mathrm{Ad}_{g^{-1}} A)^*_u.$$

Hence, the vector field image $R_g \cdot A^*$ is given at u by

$$\big(R_g \cdot A^*\big)_u = R_{g*} A^*_{ug^{-1}} = R_{g*} j_{ug^{-1}*} A_e = (\mathrm{Ad}_{g^{-1}} A)^*_u,$$

so $R_g \cdot A^* = (\mathrm{Ad}_{g^{-1}} A)^*$.

(ii) Let $u^1, \dots, u^{2n+2}$ be the real coordinates on $\mathbb{C}^{n+1} \setminus \{0\}$ (that is, $\{z^j = u^{2j-1} + \mathrm{i}u^{2j}\}$ is the dual basis to the usual complex basis of $\mathbb{C}^{n+1}$). For $u = (u^1 + \mathrm{i}u^2, \dots, u^{2n+1} + \mathrm{i}u^{2n+2}) \in \mathbb{C}^{n+1} \setminus \{0\}$, the map j_u above is now given by $\mathbb{C}^* \to \pi^{-1}(\pi(u))$,

$$\begin{aligned} x + \mathrm{i}y \mapsto{}& \big(u^1 + \mathrm{i}u^2, \dots, u^{2n+1} + \mathrm{i}u^{2n+2}\big)(x + \mathrm{i}y) \\ ={}& \big(u^1 x - u^2 y + \mathrm{i}\big(u^2 x + u^1 y\big), \dots\big) \\ \equiv{}& \big(u^1 x - u^2 y, u^2 x + u^1 y, \dots\big). \end{aligned}$$

Therefore,

$$\varphi = j_{u*} = \begin{pmatrix} \frac{\partial(u^1x-u^2y)}{\partial x} & \frac{\partial(u^1x-u^2y)}{\partial y} \\ \frac{\partial(u^2x+u^1y)}{\partial x} & \frac{\partial(u^2x+u^1y)}{\partial y} \end{pmatrix} = \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \\ \vdots & \vdots \\ u^{2n+1} & -u^{2n+2} \\ u^{2n+2} & u^{2n+1} \end{pmatrix}.$$

Hence

$$\big(\varphi(sX_1 + tX_2)\big)_u = j_{u*}\big((sX_1 + tX_2)_e\big) = j_{u*}\left(s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\right), \quad s, t \in \mathbb{R},$$

as $e \equiv (x = 1, y = 0)$. So,

$$\begin{aligned} \varphi(sX_1 + tX_2) \equiv{}& \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u^1 s - u^2 t \\ u^2 s + u^1 t \\ \vdots \end{pmatrix} \\ \equiv{}& \big(u^1 s - u^2 t\big)\frac{\partial}{\partial u^1} + \big(u^2 s + u^1 t\big)\frac{\partial}{\partial u^2} + \cdots \\ &+ \big(u^{2n+1} s - u^{2n+2} t\big)\frac{\partial}{\partial u^{2n+1}} + \big(u^{2n+2} s + u^{2n+1} t\big)\frac{\partial}{\partial u^{2n+2}}. \end{aligned}$$

Problem 5.20 Let (FM, π, M) be the bundle of linear frames over the C^∞ n-manifold M. If $p \in M$ and $(x^1, \dots, x^n)$ is a coordinate system on a neighbourhood

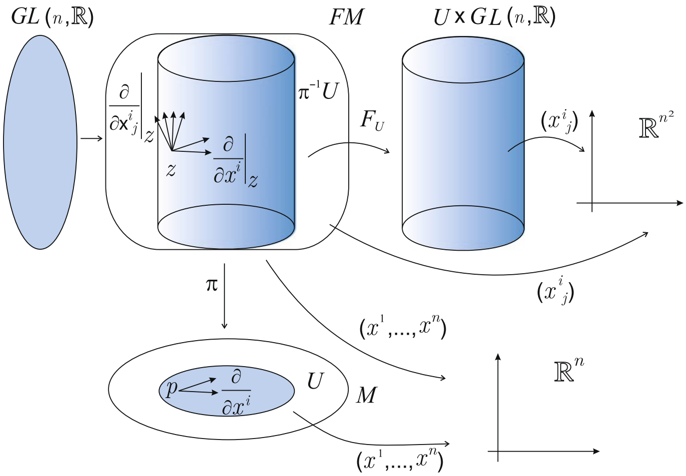

Fig. 5.1 The bundle of linear frames (FM, π, M) over M

U of p, we can define the map

$$F_U : \pi^{-1}(U) \to \mathrm{GL}(n, \mathbb{R}), \qquad z = (q, e_1, \ldots, e_n) \mapsto \big(\mathrm{d}x^i(e_j)\big).$$

The functions $x^i = x^i \circ \pi$ and $x^i_j = x^i_j \circ F_U$, where x^i_j denote the standard coordinates on $\mathrm{GL}(n, \mathbb{R})$, are a coordinate system on $\pi^{-1}(U)$ (see Fig. 5.1). If $z \in \pi^{-1}(U)$, prove that

$$\pi_*\left(\frac{\partial}{\partial x^i}\bigg|_z\right) = \sum_{j=1}^n Y^j_i(z) e_j,$$

where $(Y^i_j(z))$ stands for the inverse matrix of $(x^i_j(z))$.

Solution We have

$$\pi_*\left(\frac{\partial}{\partial x^i}\bigg|_z\right) = \sum_{j=1}^n \frac{\partial(x^j \circ \pi)}{\partial x^i} \frac{\partial}{\partial x^j}\bigg|_q = \frac{\partial}{\partial x^i}\bigg|_q, \tag{$\star$}$$

but the coordinates of $\{e_j\}$ with respect to the canonical basis $\{(\partial/\partial x^i)_q\}$ are precisely $(x^i_j(z))$, that is,

$$(e_1, \ldots, e_n) = \left(\frac{\partial}{\partial x^i}\bigg|_q, \ldots, \frac{\partial}{\partial x^n}\bigg|_q\right) \begin{pmatrix} x^1_1(z) & \cdots & x^1_n(z) \\ \vdots & & \vdots \\ x^n_1(z) & \cdots & x^n_n(z) \end{pmatrix},$$

or equivalently, $e_i = \sum_{j=1}^{n} x_i^j(z) \frac{\partial}{\partial x^j}|_q$. Thus

$$\left.\frac{\partial}{\partial x^i}\right|_q = \sum_{j=1}^{n} Y_i^j(z) e_j. \qquad (\star\star)$$

From ($\star$) and ($\star\star$), it follows that $\pi_*(\frac{\partial}{\partial x^i}|_z) = \sum_j Y_i^j(z) e_j$.

Problem 5.21 Find the fundamental vector fields on the bundle of linear frames FM over a C^∞ n-manifold M.

Solution If X is an element of the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ of $\mathrm{GL}(n, \mathbb{R})$ then its value at the identity element e of $\mathrm{GL}(n, \mathbb{R})$ is the tangent vector at e to the curve e^{tX}, and it has corresponding fundamental field X^* on FM, whose value at $z \in FM$ is X_z^*, the tangent vector to the curve $z\mathrm{e}^{tX}$ in FM at z. Let $\{x^i\}$ be local coordinates on M with domain U, and let $\{x_j^i\}$ be the canonical coordinates on $\mathrm{GL}(n, \mathbb{R})$.

Then the coordinates of z are $x^i(z) = x^i(\pi(z))$, $x_j^i(z)$, as in Problem 5.20. Therefore,

$$\left(X^* x^i\right)_z = \lim_{t \to 0} \frac{x^i(z\mathrm{e}^{tX}) - x^i(z)}{t} = 0,$$

because $\pi(z\mathrm{e}^{tX}) = \pi(z)$ and

$$\begin{aligned}\left(X^* x_j^i\right)_z &= \lim_{t \to 0} \frac{x_j^i(z\mathrm{e}^{tX}) - x_j^i(z)}{t} \\ &= \lim_{t \to 0} \frac{1}{t}\left\{\left(x_j^i(z) + t x_j^i(zX) + \frac{t^2}{2!} x_j^i\left(zX^2\right) + \cdots\right) - x_j^i(z)\right\} \\ &= x_j^i(zX) = x_k^i(z) a_j^k,\end{aligned}$$

where $X = (a_j^i)$. Hence $X_z^* = \sum_{i,j,k=1}^{n} x_k^i(z) a_j^k \frac{\partial}{\partial x_j^i}|_z$ and $X^*|_{\pi^{-1}(U)} = \sum_{i,j,k=1}^{n} a_j^k \times x_k^i \frac{\partial}{\partial x_j^i}$.

Problem 5.22 Prove that a necessary and sufficient condition for a C^∞ $2n$-manifold M to admit an *almost tangent structure*, that is, a G-structure with group

$$G = \left\{\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}) : A \in \mathrm{GL}(n, \mathbb{R})\right\},$$

is that it admits a C^∞ tensor field J of type $(1, 1)$ and rank n such that $J^2 = 0$.

Solution First, suppose that M admits an almost tangent structure. Let (e_j), $j = 1, \ldots, 2n$, be a frame adapted to the G-structure. One has $(e_j) = (e_\alpha, e_{\alpha^*})$, $\alpha =$

$1, \dots, n$, $\alpha^* = n+1, \dots, 2n$, such that if $(e_j) = (e_{\alpha'}, e_{\alpha'^*})$ is another adapted frame, it is related to the previous one by the formulas

$$e_{\alpha'} = \sum_\beta \big(A^\beta_{\alpha'} e_\beta + B^{\beta^*}_{\alpha'} e_{\beta^*}\big), \qquad e_{\alpha'^*} = \sum_\beta A^\beta_{\alpha'} e_{\beta^*}.$$

Hence, we can define a linear operator J of rank n,

$$J_p \colon T_p M \to T_p M, \qquad J_p \sum_\alpha \big(\lambda^\alpha e_\alpha + \lambda^{\alpha^*} e_{\alpha^*}\big) = \sum_\alpha \lambda^\alpha e_{\alpha^*},$$

where (e_α, e_{α^*}) denotes an adapted frame, which is well-defined because

$$J_p e_{\alpha'} = J_p \sum_\beta \big(A^\beta_{\alpha'} e_\beta + B^{\beta^*}_{\alpha'} e_{\beta^*}\big) = \sum_\beta A^\beta_{\alpha'} e_{\beta^*} = e_{\alpha'^*},$$

$$J_p e_{\alpha'^*} = \sum_\beta J_p \big(A^\beta_{\alpha'} e_{\beta^*}\big) = 0.$$

Thus $J_p^2 = 0$. Furthermore, since $J_p e_{\alpha^*} = 0$, (e_{α^*}) is a basis of $\ker J_p$, and (e_α) is a basis of a vector subspace supplementary to $\ker J_p$. That is, J_p is written in the adapted frames as $\left(\begin{smallmatrix} 0 & 0 \\ I_n & 0 \end{smallmatrix}\right)$.

Conversely, if there exists a C^∞ $(1,1)$ tensor field J of rank n on M such that $J^2 = 0$, then (e_α, e_{α^*}) is an adapted frame if (e_{α^*}) is a basis of $\ker J_p$ and (e_α) is a basis of a vector subspace supplementary to $\ker J_p$ (where J_p denotes the linear operator of rank n induced by J on each tangent space $T_p M$), in such a way that $J_p e_\alpha = e_{\alpha^*}$ and $J_p e_{\alpha^*} = 0$. Consider another adapted frame $e_{\alpha'}, e_{\alpha'^*}$. Then

$$e_{\alpha'} = \sum_\beta \big(M^\beta_{\alpha'} e_\beta + N^{\beta^*}_{\alpha'} e_{\beta^*}\big), \qquad e_{\alpha'^*} = \sum_\beta \big(P^\beta_{\alpha'^*} e_\beta + Q^{\beta^*}_{\alpha'^*} e_{\beta^*}\big).$$

Since $J_p e_{\alpha'} = e_{\alpha'^*}$, we have

$$J_p \sum_\beta \big(M^\beta_{\alpha'} e_\beta + N^{\beta^*}_{\alpha'} e_{\beta^*}\big) = \sum_\beta M^\beta_{\alpha'} e_{\beta^*} = \sum_\beta \big(P^\beta_{\alpha'^*} e_\beta + Q^{\beta^*}_{\alpha'^*} e_{\beta^*}\big),$$

so $M = Q$, $P = 0$, and the matrix of the change has the form $\left(\begin{smallmatrix} A & 0 \\ B & A \end{smallmatrix}\right)$, $A \in \mathrm{GL}(n, \mathbb{R})$.

5.3 Connections in Bundles

Problem 5.23 Determine all the connections in the frame bundle $F\mathbb{R}$ over $\mathbb{R}$.

Solution Consider $F\mathbb{R} \cong \mathbb{R} \times \mathrm{GL}(1, \mathbb{R}) = \mathbb{R} \times \mathbb{R}^*$ with coordinates (t, a). A connection in $F\mathbb{R}$ is given by a "horizontal subspace" $H_{(t,a)} \subset T_{(t,a)}(\mathbb{R} \times \mathbb{R}^*)$ at each point $(t,a) \in \mathbb{R} \times \mathbb{R}^*$, such that $H_{(t,a)}$ must be 1-dimensional and satisfy $\pi_*(H_{(t,a)}) = T_t\mathbb{R} \equiv \mathbb{R}$, where π stands for the projection map of $F\mathbb{R}$. Thus, we can put

$$H_{(t,a)} = \left\langle \frac{\partial}{\partial t}\bigg|_{(t,a)} + h(t,a)\frac{\partial}{\partial a}\bigg|_{(t,a)} \right\rangle, \quad h \in C^\infty\big(\mathbb{R} \times \mathbb{R}^*\big).$$

Moreover, H must be invariant under right translations, i.e. if $b \in \mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^*$, then

$$R_{b*}(H_{(t,a)}) = H_{R_b(t,a)} = H_{(t,ab)} = \left\langle \frac{\partial}{\partial t}\bigg|_{(t,ab)} + h(t,ab)\frac{\partial}{\partial a}\bigg|_{(t,ab)} \right\rangle.$$

Since $R_b(t,a) = (t,ab)$, it is clear that

$$R_{b*}\left(\frac{\partial}{\partial a}\bigg|_{(t,a)}\right) = b\frac{\partial}{\partial a}\bigg|_{(t,ab)}.$$

Therefore, $h(t,ab) = bh(t,a)$. Hence $h(t,a) = ah(t,1)$. Thus calling $f\colon \mathbb{R} \to \mathbb{R}$ the function given by $f(t) = h(t,1)$, the connection is given by the distribution $\mathscr{H}$ on $F\mathbb{R}$ generated by the vector field $\frac{\partial}{\partial t} + f(t)a\frac{\partial}{\partial a}$, that is,

$$\mathscr{H} = \left\langle \frac{\partial}{\partial t} + f(t)a\frac{\partial}{\partial a} \right\rangle, \quad f \in C^\infty\mathbb{R},\ a \in \mathbb{R}^*.$$

Problem 5.24 Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Consider the trivial principal G-bundle $(P = \mathfrak{g} \times G, \pi, \mathfrak{g}, G)$,

$$\begin{array}{ccccc} G \hookrightarrow & P = \mathfrak{g} \times G & \ni & (X,g) \\ & \downarrow \pi & & \downarrow \\ & \mathfrak{g} & \ni & X. \end{array}$$

Let

$$\begin{array}{rcl} L_g\colon G & \longrightarrow & G \\ h & \longmapsto & gh, \end{array} \qquad \begin{array}{rcl} R_g\colon G & \longrightarrow & G \\ h & \longmapsto & hg, \end{array}$$

be the left and right multiplication by g, respectively. Let e denote the identity element of G and σ_e the identity section of the given bundle,

$$\begin{array}{ccc} P = \mathfrak{g} \times G & \ni & (X,e) \\ \uparrow \sigma_e & & \updownarrow \\ \mathfrak{g} & \ni & X. \end{array}$$

Prove:

(i) The distribution $\mathscr{H}$ on the total space P of the bundle, given at each point $(X,g) \in P$ by the subspace $\mathscr{H}_{(X,g)}$ of $T_{(X,g)}P$ defined by

$$\mathscr{H}_{(X,g)} = \big\{\big(Y, (R_{g*})_e Y\big) : Y \in \mathfrak{g}\big\}, \tag{$\star$}$$

defines a connection Γ in P whose horizontal distribution is $\mathscr{H}$.

(ii) Let θ denote the connection 1-form, that is, the $\mathfrak{g}$-valued differential 1-form on P defining Γ. The curvature 2-form of Γ is the $\mathfrak{g}$-valued differential 2-form on the total space P defined by

$$\mathrm{d}\theta + \frac{1}{2}[\theta, \theta].$$

Pulling θ back to the base space $\mathfrak{g}$ by σ_e gives the local connection 1-form of Γ (with respect to the identity section σ_e), that is, $\omega = \sigma_e^* \theta$. The local curvature 2-form Ω of Γ (with respect to the identity section σ_e) is defined as the pull-back of the curvature 2-form of Γ by σ_e to the base manifold,

$$\Omega = \sigma_e^* \left(\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] \right) = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]. \qquad (\star\star)$$

Prove that the local curvature 2-form of Γ (with respect to the identity section σ_e) is the constant $\mathfrak{g}$-valued 2-form Ω on $\mathfrak{g}$ given by

$$\Omega(X, Y) = [X, Y], \quad X, Y \in \mathfrak{g}.$$

Hint If V is a finite-dimensional real vector space endowed with its natural C^∞-manifold structure, then every $v \in V$ defines a vector field $X_v \in \mathfrak{X}(V)$ given as follows (*directional derivative*):

$$(X_v)_x(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x + tv), \quad x \in V, \ f \in C^\infty V,$$

and the linear mapping $v \mapsto (X_v)_x$ induces an isomorphism $V \cong T_x V$. In practice, the vector field X_v is identified to v itself.

Moreover, recall that the right action of G on P is given by $(X, h) \cdot g = (X, hg)$ and denote it by R_g, that is,

$$\begin{aligned} R_g : \quad P &\longrightarrow P \\ (X, h) &\longmapsto (X, hg). \end{aligned}$$

Also recall that for any principal bundle (P, π, M, G), given $X \in \mathfrak{g} = \mathrm{Lie}(G)$, the corresponding fundamental vector field on P is $X^* \in \mathfrak{X}(P)$ whose value at $u \in P$ is

$$X_u^* = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(u \cdot \exp(tX)\big).$$

The relevant theory is developed, for instance, in Morrison [6].

Solution

(i) We must prove that $\mathscr{H}$ is a smooth right-equivariant distribution on the total space P, complementary to the vertical tangent space to P at (X, g).

It is obvious by its definition that $\mathscr{H}$ is a smooth distribution.

Moreover, since for any $(Z, W) \in T_{(X,g)}P$ we have

$$(Z, W) = \big(Z, W + (R_{g*})_e Z - (R_{g*})_e Z\big) = \big(Z, (R_{g*})_e Z\big) + \big(0, W - (R_{g*})_e Z\big),$$

$\mathscr{H}_{(X,g)}$ is complementary to the vertical tangent space at (X, g).

Finally, as given $(Y, (R_{g*})_e Y) \in \mathscr{H}_{(X,g)}$, one has

$$(R_{h*})_g\big(Y, (R_{g*})_e Y\big) = \big(Y, (R_{h*})_g (R_{g*})_e Y\big) = \big(Y, (R_{gh*})_e Y\big) \in \mathscr{H}_{(X,gh)},$$

it follows that the distribution ($\star$) is right-equivariant.

(ii) In the present particular case, P being the trivial bundle $\mathfrak{g} \times G$, we have that the fundamental vector field on P corresponding to an element $X \in \mathfrak{g}$, is given at $(Y, g) \in P$ by

$$\begin{aligned} X^*_{(Y,g)} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big((Y, g) \cdot \exp(tX)\big) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(Y, g \cdot \exp(tX)\big) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(Y, L_g\{\exp(tX)\}\big) \\ &= \big(0, (L_{g*})_e X\big) \in \mathfrak{g} \times T_g G. \end{aligned} \tag{$\dagger$}$$

The connection 1-form θ is characterised by

$$\mathscr{H}_{(Z,g)} = \ker \theta_{(Z,g)}, \tag{$\dagger\dagger$}$$

$$\theta_{(Z,g)}\big(Y^*_{(Z,g)}\big) = Y, \tag{$\dagger\dagger\dagger$}$$

$$\big(R_g^*\theta\big)(X, Y) = \mathrm{Ad}_{g^{-1}}\big(\theta(X, Y)\big).$$

From ($\dagger$) and ($\dagger\dagger\dagger$) we get

$$\theta_{(Z,g)}(0, Y) = \theta_{(Z,g)}\big(\{(L_{g^{-1}*})_e Y\}^*_{(Z,g)}\big) = (L_{g^{-1}*})_e Y.$$

Hence, given $(X, Y) \in \mathfrak{g} \times T_g G$, one has that

$$\begin{aligned} \theta_{(Z,g)}(X, Y) &= \theta_{(Z,g)}\big(\big(X, (R_{g*})_e X\big) + \big(0, Y - (R_{g*})_e X\big)\big) \\ &= \theta_{(Z,g)}\big(0, Y - (R_{g*})_e X\big) && \text{(by ($\star$) and ($\dagger\dagger$))} \\ &= \theta_{(Z,g)}\big(\{(L_{g^{-1}*})_g\big(Y - (R_{g*})_e X\big)\}^*_{(Z,g)}\big) && \text{(by ($\dagger$))} \\ &= (L_{g^{-1}*})_g\big(Y - (R_{g*})_e X\big) && \text{(by ($\dagger\dagger\dagger$))} \\ &= (L_{g^{-1}*})_g Y - (L_{g^{-1}*})_g (R_{g*})_e X \\ &= (L_{g^{-1}*})_g Y - (L_{g^{-1}*} \circ R_{g*})_e X \\ &= (L_{g^{-1}*})_g Y - \mathrm{Ad}_{g^{-1}} X. && \text{(by ($\star\star\star$))} \end{aligned} \tag{$\diamond$}$$

According to ($\star\star$), the local curvature form Ω is then

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega].$$

Recall now that for any $Y \in \mathfrak{m}$, by using the directional derivative, one has the identification $T_Y\mathfrak{m} \equiv \mathfrak{m}$. Then, we have from ($\diamond$) that

$$\omega_Z(Y) = \theta_{(Z,e)}(Y, 0) = -Y.$$

Thus, the differential form ω is constant, hence closed. We now evaluate $[\omega, \omega]$ on a pair of tangent vectors $X, Y \in T_Z\mathfrak{g} \equiv \mathfrak{g}$, getting

$$\begin{aligned}[\omega, \omega](X, Y) &= \big[\omega(X), \omega(Y)\big] - \big[\omega(Y), \omega(X)\big] \\ &= [-X, -Y] - [-Y, -X] = 2[X, Y].\end{aligned}$$

Therefore, the local curvature form Ω with respect to σ_e is in fact the constant $\mathfrak{g}$-valued differential 2-form Ω such that

$$\Omega(X, Y) = \left(\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]\right)(X, Y) = [X, Y], \quad X, Y \in \mathfrak{g}.$$

Problem 5.25 Let $\pi : P = M \times \mathbb{C}^* \to M$ be the trivial principal $\mathbb{C}^*$-bundle over the C^∞ manifold M. Prove that, in complex notation, every connection form ω_Γ on P can be written as

$$\omega_\Gamma = z^{-1}\mathrm{d}z + \pi^*\omega, \quad z \in \mathbb{C}^*,$$

where ω is a complex-valued differential 1-form on M; that is, $\omega \in \Lambda^1(M, \mathbb{C})$.

Solution Let $\varphi_1, \varphi_2 \colon \mathbb{R} \to \mathbb{C}^*$ be the homomorphisms $\varphi_1(t) = \mathrm{e}^t$, $\varphi_2(t) = \mathrm{e}^t\mathrm{i}$. These homomorphisms induce a basis $\{A_1, A_2\}$ of the Lie algebra of $\mathbb{C}^*$, which can be identified to $\mathbb{C}$ itself by $1 \mapsto A_1$, $\mathrm{i} \to A_2$. The fundamental vector fields attached to these vectors are:

$$A_1^* = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad A_2^* = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

For example, let us compute A_1^*. The flow generating A_1^* is

$$\psi_t(z) = \varphi_1(t)z = \mathrm{e}^t(x + y\mathrm{i}) = \mathrm{e}^t x + \mathrm{e}^t y\mathrm{i} = \widetilde{x}_t + \widetilde{y}_t\mathrm{i}.$$

Hence

$$A_1^*(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{x}_t = x, \qquad A_1^*(y) = \left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{y}_t = y.$$

Thus, by using the previous identification, ω_Γ can be written as

$$\omega_\Gamma = \eta^1 + \eta^2\mathrm{i},$$

where η^1, η^2 are differential 1-forms on P. By imposing that

$$\omega_\Gamma\big(A_k^*\big) = A_k, \quad k = 1, 2,$$

we obtain $\eta^j(A_k^*) = \delta_k^j$. Hence by using a coordinate system (x^h) on M, the forms η^1, η^2 can be written as

$$\eta^1 = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + f_h^1\,\mathrm{d}x^h, \qquad \eta^2 = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2} + f_h^2\,\mathrm{d}x^h,$$

where $f_h^1, f_h^2 \in C^\infty P$. We remark that

$$z^{-1}\,\mathrm{d}z = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}\mathrm{i}.$$

Hence

$$\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}, \qquad \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$$

are left-invariant differential forms on $\mathbb{C}^*$.

Moreover, as $\mathbb{C}^*$ is commutative, the condition $R_z^*\omega_\Gamma = \mathrm{Ad}_{z^{-1}} \circ \omega_\Gamma$ simply means that ω_Γ is right-invariant. Accordingly, this condition holds if and only if the functions f_h^1, f_h^2 are $\mathbb{C}^*$-invariant, that is, if and only if $f_h^1, f_h^2 \in C^\infty M$. Hence $\omega^1 = \sum_h f_h^1\,\mathrm{d}x^h$, $\omega^2 = \sum_h f_h^2\,\mathrm{d}x^h$, are differential forms on M, and by setting $\omega = \omega^1 + \omega^2\mathrm{i}$, we conclude.

Problem 5.26 Consider the trivial bundle $\pi\colon P = M \times \mathrm{U}(1) \to M$. Parametrise the fibre $\mathrm{U}(1)$ (see Remark 1.4) as $\exp(\mathrm{i}\alpha)$, $0 < \alpha < 2\pi$. If (q^j), $j = 1, \dots, n = \dim M$, are local coordinates on M, then (q^j, α) are local coordinates on P. Let $p\colon T^*M \to M$ be the canonical projection of the cotangent bundle. Prove:

1. For every connection form ω_Γ on P there exists a unique differential 1-form ω on M such that
$$\omega_\Gamma = \big(\mathrm{d}\alpha + \pi^*\omega\big) \otimes A,$$
where $A \in \mathfrak{u}(1)$ is the invariant vector field defined by the homomorphism $\mathbb{R} \to \mathrm{U}(1)$, $t \mapsto \exp(\mathrm{i}t)$.
2. Every automorphism $\Phi\colon P \to P$ can be described locally as $\Phi(x, \alpha) = (\phi(x), \alpha + \psi(x))$, where $\phi\colon M \to M$ is a diffeomorphism and $\psi\colon M \to \mathbb{R}$ is a differentiable map.
3. $(\Phi^{-1})^*\omega_\Gamma$ is another connection form $\omega_{\gamma'}$ on P. Set $\omega_{\gamma'} = (\mathrm{d}\alpha + \pi^*\omega') \otimes A$ and compute ω'.
4. There exists a unique diffeomorphism $\widetilde{\Phi}\colon T^*M \to T^*M$ such that:
   (i) $p \circ \widetilde{\Phi} = \phi \circ p$.
   (ii) If the differential forms ω, ω' on M are related as in (3), then $\widetilde{\Phi} \circ \omega = \omega'$. Here, ω, ω' are viewed as sections of the cotangent bundle.
5. If $\Psi\colon P \to P$ is another automorphism, then $(\Psi \circ \Phi)\widetilde{\ } = \widetilde{\Psi} \circ \widetilde{\Phi}$. (This property justifies the exponent -1 in defining γ' in (3).)

Solution

1. As A is a basis of $\mathfrak{u}(1)$, it is clear that every connection form can be written as $\omega_\Gamma = \eta \otimes A$ for some differential 1-form η on P. Moreover, the fundamental vector field associated to A is readily seen to be $A^* = \partial/\partial\alpha$ and from the very definition of a connection form it must hold that $\omega_\Gamma(A^*) = \eta(A^*)A = A$. Hence $\eta(\partial/\partial\alpha) = 1$, and accordingly,
$$\eta = \mathrm{d}\alpha + \sum_j f_j \,\mathrm{d}q^j,$$
for certain functions $f_j \in C^\infty P$. We now impose
$$R_z^*\omega_\Gamma = \mathrm{Ad}_{z^{-1}} \circ \omega_\Gamma, \quad z \in \mathrm{U}(1), \tag{$\star$}$$
that is, the second property of a connection form. As $\mathrm{U}(1)$ is Abelian, the adjoint representation is trivial, and hence ($\star$) simply means that η is invariant under right translations. As the forms $\mathrm{d}\alpha$ and $\mathrm{d}q^j$ are invariant, we conclude that η is invariant if and only if the functions f_j are invariant, that is, if each f_j does not depend on α, thus projecting to a function on M. Hence $\omega = \sum_j f_j \,\mathrm{d}q^j$.
2. A diffeomorphism $\Phi\colon P \to P$ is a principal bundle automorphism if Φ is equivariant, i.e. $\Phi(u \cdot z) = \Phi(u) \cdot z$, for all $u \in P$, for all $z = \exp(\mathrm{i}\alpha) \in \mathrm{U}(1)$. We have
$$\Phi(x, w) = \big(\xi(x, w), \varphi(x, w)\big), \quad (x, w) \in P,$$
where $\xi\colon P \to M$, $\varphi\colon P \to \mathrm{U}(1)$ are the components of Φ. By imposing the condition of equivariance, we obtain $\Phi(x, wz) = \Phi((x, w)\cdot z) = \Phi(x, w)z$, that is,
$$\big(\xi(x, wz), \varphi(x, wz)\big) = \big(\xi(x, w), \varphi(x, w)z\big).$$
Letting $w = 1$, we have $\xi(x, z) = \xi(x, 1)$ and $\varphi(x, z) = \varphi(x, 1)z$. Hence ξ factors through π by means of a differentiable map $\phi\colon M \to M$ as follows: $\xi = \phi \circ \pi$, and, locally, we have $\varphi(x, 1) = \exp(\mathrm{i}\psi(x))$. Then,
$$\varphi(x, z) = \exp\big(\mathrm{i}\psi(x)\big)\exp(\mathrm{i}\alpha) = \exp\big(\mathrm{i}\big(\alpha + \psi(x)\big)\big).$$
3. As a simple computation shows, we have
$$\Phi^{-1}(x, \alpha) = \big(\phi^{-1}(x), \alpha - \big(\psi \circ \phi^{-1}\big)(x)\big).$$
Thus
$$\begin{aligned}\big(\Phi^{-1}\big)^*\omega_\Gamma &= \big(\big(\Phi^{-1}\big)^*\big(\mathrm{d}\alpha + \pi^*\omega\big)\big) \otimes A \\ &= \big(\mathrm{d}\alpha - \mathrm{d}\big(\psi \circ \phi^{-1} \circ \pi\big) + \pi^*\big(\phi^{-1}\big)^*\omega\big) \otimes A.\end{aligned}$$
Hence $\omega' = (\phi^{-1})^*\omega - \mathrm{d}(\psi \circ \phi^{-1})$.

4. Given a covector $w \in T_x^*M$, let ω be a differential 1-form on M such that $\omega(x) = w$. Then, from conditions (i), (ii) we obtain

$$\begin{aligned}\widetilde{\Phi}(w) = \widetilde{\Phi}\big(\omega(x)\big) = (\widetilde{\Phi} \circ \omega)(x) = \omega'(x) &= \big(\phi^{-1}\big)^* \omega(x) - \big(\mathrm{d}\big(\psi \circ \phi^{-1}\big)\big)_{\phi(x)} \\ &= \big(\phi^{-1}\big)^* w - \big(\mathrm{d}\big(\psi \circ \phi^{-1}\big)\big)_{\phi(x)},\end{aligned}$$

   thus proving the existence and uniqueness of $\widetilde{\Phi}$.
5. Set $\omega_{\gamma'} = (\Phi^{-1})^* \omega_\Gamma$, $\omega_{\Gamma''} = (\Psi^{-1})^* \omega_{\gamma'}$. Then,

$$\omega_{\Gamma''} = \big(\Psi^{-1}\big)^* \big(\Phi^{-1}\big)^* \omega_\Gamma = \big((\Psi \circ \Phi)^{-1}\big)^* \omega_\Gamma.$$

   Hence $(\widetilde{\Psi} \circ \widetilde{\Phi}) \circ \omega = \omega''$ and $(\Psi \circ \Phi)\widetilde{} \circ \omega = \omega''$, so that $\widetilde{\Psi} \circ \widetilde{\Phi}$ and $(\Psi \circ \Phi)\widetilde{}$ satisfy the condition (ii) in part 4. Moreover, we have

$$\begin{aligned}p \circ (\widetilde{\Psi} \circ \widetilde{\Phi}) = (p \circ \widetilde{\Psi}) \circ \widetilde{\Phi} = (\psi \circ p) \circ \widetilde{\Phi} &= \psi \circ (p \circ \widetilde{\Phi}) \\ &= \psi \circ (\phi \circ p) = (\psi \circ \phi) \circ p.\end{aligned}$$

   Hence condition (i) in part 4 holds.

Problem 5.27 Let $z^k = x^k + \mathrm{i} y^k$, $0 \leqslant k \leqslant n$, be the standard coordinates on $\mathbb{C}^{n+1}$. Prove that the 1-form

$$\omega = \sum_{k=0}^{n} \big(-y^k \,\mathrm{d}x^k + x^k \,\mathrm{d}y^k\big)\big|_{S^{2n+1}}$$

is a connection form on the principal U(1)-bundle

$$p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n,$$

where we identify the Lie algebra of U(1) with $\mathbb{R}$ via the isomorphism $\lambda \mapsto \lambda(\partial/\partial\theta)$, where θ stands for the angle function on U(1).

Solution According to the definition of a connection form, we must check the following properties:

(i) $\omega((\lambda \frac{\partial}{\partial\theta})^*) = \lambda$, for all $\lambda \in \mathbb{R}$.
(ii) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega$, for all $g \in \mathrm{U}(1)$.

As the coordinates of the point $z \cdot \exp(\mathrm{i}\lambda\theta)$ are

$$\big(x^k + \mathrm{i} y^k\big)\big(z \cdot \exp(\mathrm{i}\lambda\theta)\big) = x^k \cos(\lambda\theta) - y^k \sin(\lambda\theta) + \mathrm{i}\big(x^k \sin(\lambda\theta) + y^k \cos(\lambda\theta)\big),$$

$0 \leqslant k \leqslant n$, we have

$$\left(\lambda \frac{\partial}{\partial\theta}\right)^* = \lambda \sum_{k=0}^{n} \left(-y^k \frac{\partial}{\partial x^k} + x^k \frac{\partial}{\partial y^k}\right).$$

Hence

$$\omega\left(\left(\lambda\frac{\partial}{\partial\theta}\right)^*\right)=\sum_{k=0}^{n}\left(-y^k\,\mathrm{d}x^k+x^k\,\mathrm{d}y^k\right)\left(\lambda\sum_{l=0}^{n}\left(-y^l\frac{\partial}{\partial x^l}+x^l\frac{\partial}{\partial y^l}\right)\right)$$
$$=\lambda\sum_{k=0}^{n}\left(\left(y^k\right)^2+\left(x^k\right)^2\right)=\lambda,$$

at every point of the sphere, thus proving (i). As for (ii), we first remark that the adjoint representation is trivial since U(1) is Abelian, so (ii) simply tells us that ω is invariant under right translations. In order to prove this, we note that

$$-y^k\,\mathrm{d}x^k+x^k\,\mathrm{d}y^k=\left(\left(x^k\right)^2+\left(y^k\right)^2\right)\mathrm{d}\left(\arctan\frac{y^k}{x^k}\right),$$

and that $R_{\exp(\theta\mathrm{i})}$ leaves the quadratic form $\sum_k((x^k)^2+(y^k)^2)$ invariant. Working in polar coordinates, we thus obtain

$$\begin{aligned}R^*_{\exp(\theta\mathrm{i})}\left(-y^k\,\mathrm{d}x^k+x^k\,\mathrm{d}y^k\right)&=\left(\left(x^k\right)^2+\left(y^k\right)^2\right)R^*_{\exp(\theta\,\mathrm{i})}\mathrm{d}\left(\arctan\frac{y^k}{x^k}\right)\\&=\left(\left(x^k\right)^2+\left(y^k\right)^2\right)\mathrm{d}\left(\arctan\frac{y^k}{x^k}+\theta\right)\\&=\left(\left(x^k\right)^2+\left(y^k\right)^2\right)\mathrm{d}\left(\arctan\frac{y^k}{x^k}\right)\\&=-y^k\mathrm{d}x^k+x^k\,\mathrm{d}y^k.\end{aligned}$$

5.4 Characteristic Classes

Problem 5.28 Consider the trivial principal bundle $(\mathbb{R}^3\setminus\{0\})\times\mathrm{U}(1)$ over $\mathbb{R}^3\setminus\{0\}$. Then, for the connection with connection form described (on the open subset $\mathbb{R}^3\setminus\{(0,0,z),z\geqslant 0\}$ of the base manifold) by the $\mathfrak{u}(1)$-valued differential 1-form (cf. Problem 5.26)

$$A_2=\frac{\mathrm{i}}{2r(z-r)}(x\,\mathrm{d}y-y\,\mathrm{d}x),\qquad(\star)$$

where $r^2=x^2+y^2+z^2$:

(i) Calculate the curvature form F of the connection in terms of A_2.
(ii) Write A_2 in spherical coordinates (r,θ,φ), given (see Remark 1.4) by

$$x=r\sin\theta\cos\varphi,\qquad y=r\sin\theta\sin\varphi,\qquad z=r\cos\theta,$$
$$\theta\in(0,\pi),\ \varphi\in(0,2\pi),$$

and calculate

$$A_1 = A_2 + \gamma^{-1}\,\mathrm{d}\gamma,$$

γ being the U(1)-valued function on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ defined by $\gamma(p) = \mathrm{e}^{\varphi(p)\mathrm{i}}$.

(iii) A_1 and A_2 furnish well-defined differential forms on $U_1 = S^2 - S$ and $U_2 = S^2 - N$, respectively, where N, S denote the north and south pole.

Consider the complex Hopf bundle H studied in Problems 5.17, 5.18 and take real coordinates $u^1, \dots, u^4$ on $\mathbb{C}^2 \equiv \mathbb{R}^4$ such that

$$S^3 = \left\{\left(z^1 = u^1 + \mathrm{i}u^2, z^2 = u^3 + \mathrm{i}u^4\right) \in \mathbb{C}^2 : \left|z^1\right|^2 + \left|z^2\right|^2 = 1\right\}.$$

Prove that

$$\omega = \mathrm{i}\left(u^1\,\mathrm{d}u^2 - u^2\,\mathrm{d}u^1 + u^3\,\mathrm{d}u^4 - u^4\,\mathrm{d}u^3\right)$$

is a connection form on the bundle. Show that $\sigma_1^*\omega = A_1$ and $\sigma_2^*\omega = A_2$, where σ_k is the local section associated to the trivialisation on U_k, $k = 1, 2$, (see Problem 5.18) by means of $\sigma_k(p) = f_{k,p}^{-1}(1)$, where $1 \in \mathrm{U}(1)$ is the identity element. That is, A_1 and A_2 are local representatives of the connection in H with connection form ω.

(iv) Compute the (only) Chern number of the bundle H.

Remark The above bundle is a particular case of a construction named in Physics, namely a *Dirac magnetic monopole bundle*. Each of the given differential forms A_1, A_2 is called a *gauge potential* of a magnetic monopole at the origin of $\mathbb{R}^3$, the transformation in (ii) is called a *gauge transformation*, and F is called the *field strength*. The general construction depends on an integer n, and the bundle of the problem corresponds to $n = 1$.

The relevant theory is developed, for instance, in Göckeler and Schücker [4].

Solution

(i)

$$F = \mathrm{d}A_2 + A_2 \wedge A_2 = \mathrm{d}A_2 = \frac{\mathrm{i}}{2r^3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y).$$

(ii) Since $\gamma^{-1}\,\mathrm{d}\gamma = \mathrm{i}\,\mathrm{d}\varphi$, one has that

$$A_1 = \frac{\mathrm{i}}{2}(1 - \cos\theta)\,\mathrm{d}\varphi, \qquad A_2 = \frac{\mathrm{i}}{2}(-1 - \cos\theta)\,\mathrm{d}\varphi.$$

(iii) Since

$$\mathfrak{u}(1) = \left\{X \in \mathfrak{gl}(1, \mathbb{C}) = \mathbb{C} : {}^tX + \bar{X} = 0\right\} = \mathbb{R}\mathrm{i},$$

one can identify $\mathfrak{u}(1)$ with the purely imaginary complex numbers. The fundamental vector field $X^* \in \mathfrak{X}(S^3)$ corresponding to $X \in \mathfrak{u}(1)$ is (see Problem 5.19) $X^*_{(z^1,z^2)} = j_{(z^1,z^2)*}X_1$, $(z^1,z^2) \in S^3$. According to the parametrisation (see Remark 1.4)

$$\left(z^1,z^2\right) = \left(\cos\frac{1}{2}\theta\,\mathrm{e}^{\psi_1 \mathrm{i}}, \sin\frac{1}{2}\theta\,\mathrm{e}^{\psi_2 \mathrm{i}}\right), \quad \theta \in (0,\pi),\ \psi_1,\psi_2 \in (0,2\pi),$$

of S^3 and the fibre action of S^1 by $\mathrm{e}^{\alpha \mathrm{i}}$, this action corresponds to (the same) changes in the parameters ψ_1 and ψ_2. In fact,

$$\begin{aligned} j_{(z^1,z^2)}\left(\mathrm{e}^{\alpha\mathrm{i}}\right) &= R_{\mathrm{e}^{\alpha\mathrm{i}}}\left(z^1,z^2\right) = \left(z^1\mathrm{e}^{\alpha\mathrm{i}}, z^2\mathrm{e}^{\alpha\mathrm{i}}\right) \\ &= \left(\cos\frac{1}{2}\theta\,\mathrm{e}^{(\psi_1+\alpha)\mathrm{i}}, \sin\frac{1}{2}\theta\,\mathrm{e}^{(\psi_2+\alpha)\mathrm{i}}\right). \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial\psi_1} &= \frac{\partial u^1}{\partial\psi_1}\frac{\partial}{\partial u^1} + \frac{\partial u^2}{\partial\psi_1}\frac{\partial}{\partial u^2} = -\cos\frac{1}{2}\theta\sin\psi_1\frac{\partial}{\partial u^1} + \cos\frac{1}{2}\theta\cos\psi_1\frac{\partial}{\partial u^1} \\ &= -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \end{aligned}$$

and, similarly,

$$\frac{\partial}{\partial\psi_2} = -u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}.$$

So, we can take the vector

$$X^*_{(z^1,z^2)} = a\left(\frac{\partial}{\partial\psi_1} + \frac{\partial}{\partial\psi_2}\right) = a\left(-u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}\right), \quad (\diamond)$$

which is clearly tangent to S^3, as the tangent vector to the fibre at a generic point $(z^1,z^2) \in S^3$, image under $j_{(z^1,z^2)*}$ of $X \equiv \mathrm{i}a \in \mathfrak{u}(1)$. The vector field X^* is the fundamental vector field corresponding to X. In fact, since the Jacobian map of the map $\tau_{\mathrm{e}^{\alpha\mathrm{i}}}\colon (z^1,z^2) \mapsto (z^1\mathrm{e}^{\alpha\mathrm{i}}, z^2\mathrm{e}^{\alpha\mathrm{i}})$ is given, in terms of the real coordinates u^1,u^2,u^2,u^4, by

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

we deduce

$$\tau_{\mathrm{e}^{\alpha\mathrm{i}}*}\left(X^*_{(z^1,z^2)}\right) = a\left\{-\left(u^2\left(z^1\right)\cos\alpha + u^1\left(z^1\right)\sin\alpha\right)\frac{\partial}{\partial u^1}\bigg|_{(z^1\mathrm{e}^{\alpha\mathrm{i}},z^2\mathrm{e}^{\alpha\mathrm{i}})}\right.$$

$$
\begin{aligned}
&\quad - \big(u^2(z^1)\sin\alpha - u^1(z^1)\cos\alpha\big)\frac{\partial}{\partial u^2}\bigg|_{(z^1 e^{\alpha i}, z^2 e^{\alpha i})} \\
&\quad - \big(u^4(z^2)\cos\alpha + u^3(z^2)\sin\alpha\big)\frac{\partial}{\partial u^3}\bigg|_{(z^1 e^{\alpha i}, z^2 e^{\alpha i})} \\
&\quad - \big(u^4(z^2)\sin\alpha - u^3(z^2)\cos\alpha\big)\frac{\partial}{\partial u^4}\bigg|_{(z^1 e^{\alpha i}, z^2 e^{\alpha i})}\Bigg\} \\
&= a\left(-u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}\right)_{(z^1 e^{\alpha i}, z^2 e^{\alpha i})} \\
&= X^*_{(z^1 e^{\alpha i}, z^2 e^{\alpha i})}.
\end{aligned}
$$

This vector field is called the *standard Hopf vector field* on S^3 (see Problem 6.149).

Next, we consider the properties of the form ω. It is clearly C^∞ and takes imaginary values on S^3, which can be identified with elements of $\mathfrak{u}(1)$, as we have seen. It is immediate that

$$\omega\big(X^*_{(z^1,z^2)}\big) \equiv a\mathrm{i} \in \mathfrak{u}(1).$$

Moreover, we have

$$
\begin{aligned}
R^*_{e^{\alpha i}}\omega = \mathrm{i}\big(u^1\cos\alpha - u^2\sin\alpha\big)&\big(-\sin\alpha\, du^1 - \cos\alpha\, du^2\big) \\
- \big(u^1\sin\alpha + u^2\cos\alpha\big)&\big(\cos\alpha\, du^1 - \sin\alpha\, du^2\big) \\
+ \big(u^3\cos\alpha - u^4\sin\alpha\big)&\big(\sin\alpha\, du^3 + \cos\alpha\, du^4\big) \\
- \big(u^3\sin\alpha + u^4\cos\alpha\big)&\big(\cos\alpha\, du^3 - \sin\alpha\, du^4\big) = \omega,
\end{aligned}
$$

and also, trivially, $\mathrm{Ad}_{e^{-\alpha i}}\,\omega = \omega$, hence

$$R^*_{e^{\alpha i}}\omega = \mathrm{Ad}_{e^{-\alpha i}}\,\omega.$$

Finally, we prove that the connection form ω has local representatives A_1 on U_1 and A_2 on U_2. In fact, the local sections corresponding to the trivialisations over U_k are, respectively,

$$\sigma_1(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \left(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\, e^{\varphi i}\right), \quad \theta < \pi,$$

$$\sigma_2(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \left(\cos\frac{1}{2}\theta\, e^{-\varphi i}, \sin\frac{1}{2}\theta\right), \quad 0 < \theta.$$

Thus, it is immediate that the section σ_1 is given in terms of the real coordinates $u^1, \dots, u^4$, by

$$\left(u^1, u^2, u^3, u^4\right) = \left(\cos\frac{1}{2}\theta, 0, \sin\frac{1}{2}\theta\cos\varphi, \sin\frac{1}{2}\theta\sin\varphi\right).$$

Substituting in the expression for ω in the statement, we easily obtain $\sigma_1^*\omega = A_1$. One proceeds similarly to obtain $\sigma_2^*\omega = A_2$.

(iv)

$$\begin{aligned} c_{(1)}(H) &= \frac{\mathrm{i}}{2\pi}\int_{S^2} F = -\frac{1}{4\pi}\int_{S^2} x\,\mathrm{d}y\wedge\mathrm{d}z + y\,\mathrm{d}z\wedge\mathrm{d}x + z\,\mathrm{d}x\wedge\mathrm{d}y \\ &= -\frac{1}{4\pi}\int_{S^2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi = -1. \end{aligned}$$

Problem 5.29

(i) Identify SU(2) with the unit sphere S^3 in $\mathbb{H}$ and prove that there is an isomorphism $\mathfrak{su}(2) \cong \operatorname{Im}\mathbb{H}$ of the Lie algebra of SU(2) onto the vector space of purely imaginary quaternions endowed with the Lie algebra structure given by $[a, b] = ab - ba$, for $a, b \in \operatorname{Im}\mathbb{H}$.

(ii) Any connection in the principal SU(2)-bundle $P = \mathbb{R}^4 \times \mathrm{SU}(2)$ over $\mathbb{R}^4$ can be expressed, by (i), in terms of an $(\operatorname{Im}\mathbb{H})$-valued differential 1-form on $\mathbb{R}^4$. Let $q \in \mathbb{H}$ arbitrarily fixed, and let

$$A_{\lambda,q}(x) = \operatorname{Im}\frac{(x-q)\,\mathrm{d}\bar{x}}{\lambda^2 + |x-q|^2}, \qquad x \in \mathbb{H},\ 0 < \lambda \in \mathbb{R},$$

be an $(\operatorname{Im}\mathbb{H})$-valued connection form. Prove that the curvature form of $A_{\lambda,q}$ is given by

$$F_{\lambda,q}(x) = \frac{\lambda^2\,\mathrm{d}x\wedge\mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2}. \tag{$\star$}$$

Solution

(i) We first remark that $(\operatorname{Im}\mathbb{H}, [\cdot\,,\cdot])$ is a Lie algebra as $a \in \operatorname{Im}\mathbb{H}$ if and only if $a + \bar{a} = 0$, and for every $a, b \in \operatorname{Im}\mathbb{H}$ we have

$$\begin{aligned} [a, b] + \overline{[a, b]} &= (ab - ba) + \overline{(ab - ba)} = (ab - ba) + (\bar{b}\bar{a} - \bar{a}\bar{b}) \\ &= (ab - ba) + \big((-b)(-a) - (-a)(-b)\big) = 0. \end{aligned}$$

Any quaternion can be written as

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\mathbf{j} = z_1 + z_2\mathbf{j},$$

with the rule $\mathbf{j}z = \bar{z}\,\mathbf{j}$, and, hence, q can be identified to the matrix

$$A_q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \qquad (\star\star)$$

In fact, for two quaternions q, q', we have

$$A_q A_{q'} = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} z_1' & z_2' \\ -\bar{z}_2' & \bar{z}_1' \end{pmatrix} = \begin{pmatrix} z_1 z_1' - z_2\bar{z}_2' & z_1 z_2' + z_2\bar{z}_1' \\ -\bar{z}_1\bar{z}_2' - \bar{z}_2 z_1' & \bar{z}_1\bar{z}_1' - \bar{z}_2 z_2' \end{pmatrix} = A_{qq'},$$

where the last equality is immediate from the expression for the product of q and q'. Moreover, we have

$$\mathrm{SU}(2) \equiv \left\{ A \in \mathrm{GL}(2,\mathbb{C}) : A = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},\ \det A = |z_1|^2 + |z_2|^2 = 1 \right\},$$

so that SU(2) can be identified to the quaternions of norm 1, which can be viewed as the 3-sphere in $\mathbb{H} \equiv \mathbb{R}^4$. The Lie algebra of the Lie group S^3 (see Problem 4.103) can be identified with the tangent space at the identity $(1,0,0,0) \in S^3$, that is, with the subspace of $\mathbb{R}^4$ orthogonal to the identity $1 \in S^3$, which is the vector space of purely imaginary quaternions $\operatorname{Im}\mathbb{H}$. The associated matrices ($\star\star$) are thus written as

$$\begin{pmatrix} \mathbf{i}a & z_2 \\ -\bar{z}_2 & -\mathbf{i}a \end{pmatrix}, \quad a \in \mathbb{R}.$$

Now, it is easily seen that these are exactly the matrices of $\mathfrak{su}(2)$. Finally, it is easily checked that the matrices

$$B_1 = \frac{1}{2}\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \qquad B_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B_3 = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$$

are a basis of $\mathfrak{su}(2)$ (remark that $-2\mathbf{i}B_r$, $1 \leqslant r \leqslant 3$, are the *Pauli matrices*) such that

$$[B_1, B_2] = B_3, \qquad [B_2, B_3] = B_1, \qquad [B_3, B_1] = B_2.$$

Similarly, $b_1 = \frac{1}{2}\mathbf{i}$, $b_2 = \frac{1}{2}\mathbf{j}$, $b_3 = \frac{1}{2}\mathbf{k}$ is a basis of $\operatorname{Im}\mathbb{H}$ such that

$$[b_1, b_2] = b_3, \qquad [b_2, b_3] = b_1, \qquad [b_3, b_1] = b_2,$$

and we conclude. Notice that this isomorphism permits us to consider the $\mathfrak{su}(2)$-valued differential forms as $\operatorname{Im}\mathbb{H}$-valued differential forms.

(ii) The quaternion differential is defined by

$$\mathrm{d}x = \mathrm{d}\big(x^0 + x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}\big) = \mathrm{d}x^0 + \mathrm{d}x^1\mathbf{i} + \mathrm{d}x^2\mathbf{j} + \mathrm{d}x^3\mathbf{k},$$
$$\mathrm{d}\bar{x} = \mathrm{d}x^0 - \mathrm{d}x^1\mathbf{i} - \mathrm{d}x^2\mathbf{j} - \mathrm{d}x^3\mathbf{k}.$$

We also use the following properties: If ω, η are two $\mathbb{H}$-valued differential forms and f is an $\mathbb{H}$-valued function, then

$$\omega f \wedge \eta = \omega \wedge f\eta. \tag{†}$$

Every $\mathbb{H}$-valued differential form ω can be decomposed as $\omega = \omega^0 + \omega'$, where ω^0 is an ordinary differential form and

$$\omega' = \operatorname{Im}\omega = \frac{1}{2}(\omega - \bar{\omega})$$

is an $\operatorname{Im}\mathbb{H}$-valued differential form. Hence, if the degree of ω is odd, then we have $\omega \wedge \omega = \omega' \wedge \omega'$, as $\omega^0 \wedge \omega^0 = 0$, $\omega^0 \wedge \omega' + \omega' \wedge \omega^0 = 0$. Therefore,

$$\operatorname{Im}(\omega \wedge \omega) = (\operatorname{Im}\omega) \wedge (\operatorname{Im}\omega). \tag{††}$$

Setting

$$f_{\lambda,q}(x) = \frac{x-q}{\lambda^2 + |x-q|^2},$$

we have $A_{\lambda,q}(x) = \operatorname{Im}\{f_{\lambda,q}(x)\,\mathrm{d}\bar{x}\}$, and by using (††), we obtain

$$\begin{aligned} F_{\lambda,q}(x) &= \mathrm{d}A_{\lambda,q}(x) + A_{\lambda,q}(x) \wedge A_{\lambda,q}(x) \\ &= \operatorname{Im}\big\{\mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} + f_{\lambda,q}(x)\,\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\,\mathrm{d}\bar{x}\big\}. \end{aligned} \tag{‡}$$

Moreover, taking into account that $|x-q|^2 = (\bar{x}-\bar{q})(x-q)$, we have

$$\begin{aligned} \mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} &= \left(\frac{\mathrm{d}x}{\lambda^2 + |x-q|^2} - (x-q)\frac{\mathrm{d}\bar{x}(x-q) + (\bar{x}-\bar{q})\,\mathrm{d}x}{(\lambda^2 + |x-q|^2)^2}\right) \wedge \mathrm{d}\bar{x} \\ &= \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{\lambda^2 + |x-q|^2} - \frac{(x-q)\,\mathrm{d}\bar{x}(x-q) \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} \\ &\quad - \frac{(x-q)(\bar{x}-\bar{q})\,\mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2}, \end{aligned}$$

and by using the formula (†), we obtain

$$\begin{aligned} -\frac{(x-q)\,\mathrm{d}\bar{x}(x-q) \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} &= -\frac{(x-q)\,\mathrm{d}\bar{x} \wedge (x-q)\,\mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} \\ &= -f_{\lambda,q}(x)\,\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\,\mathrm{d}\bar{x}. \end{aligned}$$

Hence, substituting into (‡), we obtain

$$\begin{aligned} F_{\lambda,q}(x) &= \operatorname{Im}\big\{\mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} + f_{\lambda,q}(x)\,\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\,\mathrm{d}\bar{x}\big\} \\ &= \operatorname{Im}\left(\frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{\lambda^2 + |x-q|^2} - \frac{|x-q|^2\,\mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2}\right) \end{aligned}$$

$$= \operatorname{Im} \frac{\lambda^2 \, \mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x - q|^2)^2} = \frac{\lambda^2 \, \mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x - q|^2)^2},$$

for it is immediate that $\mathrm{d}x \wedge \mathrm{d}\bar{x}$ is purely imaginary.

Problem 5.30 Consider the *quaternionic Hopf bundle* $\pi_{\mathbb{H}} \colon S^7 \to S^4$ (see Problem 5.17).

(i) Prove that ω defined by

$$\omega_{(a,b)}(X) = \operatorname{Im}(\bar{a} X_1 - \bar{X}_2 b),$$

where

$$(a, b) \in S^7 = \{(x, y) \in \mathbb{H} \times \mathbb{H} : |x|^2 + |y|^2 = 1\}$$

and

$$X = (X_1, X_2) \in T_{(a,b)} S^7 \hookrightarrow T_{(a,b)}(\mathbb{H} \times \mathbb{H}),$$

is a connection in the Hopf bundle.

(ii) Let $N = (0, 0, 0, 0, 1) \in S^4$, $S = (0, 0, 0, 0, -1) \in S^4$. Consider the maps (inverses of the stereographic projections)

$$\varphi_N^{-1} \colon \mathbb{H} \to S^4 \setminus \{N\}, \qquad \varphi_S^{-1} \colon \mathbb{H} \to S^4 \setminus \{S\},$$

given by

$$\varphi_N^{-1}(x) = \left(\frac{2x}{x\bar{x} + 1}, \frac{x\bar{x} - 1}{x\bar{x} + 1} \right), \qquad \varphi_S^{-1}(x) = \left(\frac{2x}{x\bar{x} + 1}, \frac{1 - x\bar{x}}{x\bar{x} + 1} \right),$$

respectively. Denoting $U_N = S^4 \setminus \{N\}$, $U_S = S^4 \setminus \{S\}$, we construct trivialisations of $\pi_{\mathbb{H}} \colon S^7 \to S^4$,

$$\psi_N \colon \pi_{\mathbb{H}}^{-1}(U_N) \to \mathbb{H} \times S^3, \qquad \psi_S \colon \pi_{\mathbb{H}}^{-1}(U_S) \to \mathbb{H} \times S^3,$$

by

$$\psi_N(x, y) = \left(\varphi_N\big(\pi_{\mathbb{H}}(x, y)\big), \frac{y}{|y|} \right), \qquad \psi_S(x, y) = \left(\varphi_S\big(\pi_{\mathbb{H}}(x, y)\big), \frac{x}{|x|} \right).$$

Consider the sections $\sigma_N, \sigma_S \colon \mathbb{H} \to S^7$ given by

$$\sigma_N(x) = \psi_N^{-1}(x, 1), \qquad \sigma_S(x) = \psi_S^{-1}(x, 1).$$

Let $r^2 = x\bar{x}$, $x \in \mathbb{H}$, and $\gamma \colon \mathbb{H} \setminus \{0\} \to S^3$, $\gamma(x) = r^{-1} x$.

Prove that the local expressions of ω in terms of σ_N and σ_S are

$$\sigma_N^* \omega = -\frac{r^2}{1 + r^2} \gamma^{-1} \, \mathrm{d}\gamma, \qquad \sigma_S^* \omega = \frac{r^2}{1 + r^2} \gamma^{-1} \, \mathrm{d}\gamma.$$

Solution

(i) We have

$$\omega_{(a,b)}(X) = \frac{1}{2}\big((\bar{a}X_1 - \bar{X}_2 b) - \overline{(\bar{a}X_1 - \bar{X}_2 b)}\big)$$
$$= \frac{1}{2}(\bar{a}X_1 - \bar{X}_2 b - \bar{X}_1 a + \bar{b}X_2) \in \mathbb{H}.$$

Since the Lie algebra of S^3 is identified to the purely imaginary quaternions, it follows that $\omega_{(a,b)}(X) \in T_1 S^3$. That is, ω takes its values in the Lie algebra of the Lie group $S^3 \hookrightarrow \mathbb{H}$.

The action of S^3 on S^7 is given by $R_z(a,b) = (az, bz)$ (see Problem 5.17). Then

$$\big(R_z^*\omega\big)_{(a,b)}(X) = \omega_{(az,bz)}(R_{z*}X) = \omega_{(az,bz)}(X_1 z, X_2 z)$$
$$= \frac{1}{2}(\bar{z}\bar{a}X_1 z - \bar{z}\bar{X}_2 bz - \bar{z}\bar{X}_1 az + \bar{z}\bar{b}X_2 z)$$
$$= \bar{z}\omega_{(a,b)}(X)z = z^{-1}\omega_{(a,b)}(X)z = (\mathrm{Ad}_{z^{-1}} \circ \omega)(X).$$

On the other hand, the fundamental vector field A^* corresponding to $A \in T_1 S^3$ is given by $A^*_{(a,b)} = j_{(a,b)*}A$, where

$$j_{(a,b)} \colon S^3 \to S^7, \qquad z \mapsto R_z(a,b) = (az, bz).$$

Hence

$$A^*_{(a,b)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(a(1+At), b(1+At)\big) = (aA, bA).$$

We thus have

$$\omega_{(a,b)}\big(A^*_{(a,b)}\big) = \omega_{(a,b)}(aA, bA) = \frac{1}{2}(\bar{a}aA - \bar{A}\bar{b}b - \bar{A}\bar{a}a + \bar{b}bA)$$
$$= (\bar{a}a + \bar{b}b)A = A,$$

since $\bar{A} = -A$. We have thus proved that ω is, in fact, a connection in the bundle $\pi_{\mathbb{H}} \colon S^7 \to S^4$.

(ii) In order to obtain the explicit expressions of σ_N and σ_S, we first suppose that $(u, v) \in S^7$ satisfies $\psi_S(u,v) = (x, 1)$. We then have

$$\psi_S(u,v) = \left(\varphi_S\big(2v\bar{u}, |u|^2 - |v|^2\big), \frac{u}{|u|}\right) = \left(\frac{2v\bar{u}}{1+|u|^2-|v|^2}, \frac{u}{|u|}\right) = (x, 1).$$

Then $u = k \in \mathbb{R}^+$, and after some computations one has

$$\sigma_S(x) = \left(\frac{1}{\sqrt{1+r^2}}, \frac{x}{\sqrt{1+r^2}}\right).$$

Denote by s the coordinate on $\mathbb{H}$ such that $s(x) = x$, $x \in \mathbb{H}$, and by r_1, r_2 the coordinates in $\mathbb{H} \times \mathbb{H}$ such that $r_1(x, y) = x$, $r_2(x, y) = y$. Then we can write

$$\omega = \frac{1}{2}\big(\bar{r}_1 \,\mathrm{d}r_1 - (\mathrm{d}\bar{r}_2)r_2 - (\mathrm{d}\bar{r}_1)r_1 + \bar{r}_2 \,\mathrm{d}r_2\big).$$

To compute $\sigma_S^*\omega$, we substitute

$$r_1 = \frac{1}{\sqrt{1+r^2}}, \qquad r_2 = \frac{s}{\sqrt{1+r^2}},$$

and after a calculation we obtain

$$\sigma_S^*\omega = \frac{1}{2(1+r^2)}(\bar{s}\,\mathrm{d}s - \mathrm{d}\bar{s}\cdot s),$$

which is well-defined in all of $\mathbb{H}$.

Excluding the origin, we can write $s = r\gamma$ so that $\bar{s} = r\gamma^{-1}$, and we obtain by computation

$$\sigma_S^*\omega = \frac{r^2}{1+r^2}\gamma^{-1}\,\mathrm{d}\gamma.$$

A similar calculation shows that we have the formula for $\sigma_N^*\omega$ in the statement.

Problem 5.31 Let $[w]$ denote the standard generator of the group $H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z}) \cong \mathbb{Z}$, that is, $\int_{\mathbb{C}\mathrm{P}^1} w = 1$, where the canonical orientation as a complex manifold of $\mathbb{C}\mathrm{P}^1$ is considered. Prove that the Chern class of the tautological line bundle E over $\mathbb{C}\mathrm{P}^1$ is equal to $-[w]$.

Solution Let $(P = \mathbb{C}^2 \setminus \{0\}, p, \mathbb{C}\mathrm{P}^1, \mathbb{C}^*)$ be the principal bundle over $\mathbb{C}\mathrm{P}^1$ with group $\mathbb{C}^*$ corresponding to the tautological line bundle E. The differential 1-form ω on P defined by

$$\omega_{z=(z^0,z^1)} = \frac{\bar{z}^0\,\mathrm{d}z^0 + \bar{z}^1\,\mathrm{d}z^1}{\bar{z}^0 z^0 + \bar{z}^1 z^1}$$

is a connection form on P. In fact, it takes values on the Lie algebra $\mathbb{C}$ of $\mathbb{C}^*$. Moreover, consider $\omega_U = \omega_{\sigma_U}$ and $\omega_V = \omega_{\sigma_V}$ for two sections σ_U, σ_V on two intersecting open subsets U, V of $\mathbb{C}\mathrm{P}^1$. Then, if $\sigma_V = \lambda_{UV}\sigma_U$ on $U \cap V$, that is, $\lambda_{UV} \in \mathbb{C}^*$ is the transition function, we have

$$\omega_V = \omega_{\sigma_U \lambda_{UV}} = \omega_U + \frac{\mathrm{d}\lambda_{UV}}{\lambda_{UV}},$$

on $p^{-1}(U \cap V)$, that is, as λ_{UV} takes values in $\mathbb{C}^*$,

$$\omega_V = \lambda_{UV}^{-1}\,\mathrm{d}\lambda_{UV} + \lambda_{UV}^{-1}\omega_U\lambda_{UV} = \lambda_{UV}^{-1}\,\mathrm{d}\lambda_{UV} + \mathrm{Ad}_{\lambda_{UV}^{-1}} \circ \omega_U.$$

The curvature form of ω is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = \frac{1}{(\bar{z}^0 z^0 + \bar{z}^1 z^1)^2}\Big\{\big(\bar{z}^0 z^0 + \bar{z}^1 z^1\big)\big(\mathrm{d}\bar{z}^0 \wedge \mathrm{d}z^0 + \mathrm{d}\bar{z}^1 \wedge \mathrm{d}z^1\big) \\ - \big(\bar{z}^0\,\mathrm{d}z^0 + \bar{z}^1\,\mathrm{d}z^1\big) \wedge \big(z^0\,\mathrm{d}\bar{z}^0 + z^1\,\mathrm{d}\bar{z}^1\big)\Big\}.$$

Denote by U the open subset of $\mathbb{C}\mathrm{P}^1$ defined by $z^0 \neq 0$, and set $w = z^1/z^0$. Then w can be taken as a local coordinate on U. Substituting $z^1 = z^0 w$ into the expression for the curvature form above, we have that

$$\Omega = \frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{1 + w\bar{w}^2}.$$

The first Chern form $c_1(E, \omega)$ can thus be written on U as

$$c_1(E, \omega) = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{(1 + w\bar{w})^2}.$$

Taking polar coordinates, $w = r\,\mathrm{e}^{2\pi \mathrm{i} t}$, one obtains

$$\int_{\mathbb{C}\mathrm{P}^1} c_1(E, \omega) = \int_0^1 \left(\int_0^\infty \frac{2r\,\mathrm{d}r}{(1 + r^2)^2}\right)\mathrm{d}t = -1,$$

as wanted.

Problem 5.32 (Godbillon–Vey Exotic Class for Codimension 1 Foliations) Let M be a C^∞ n-manifold, and let $\mathscr{F}$ be a foliation of codimension 1 (that is, the leaves have dimension $n-1$) on M, defined by a nowhere-vanishing global differential 1-form ω, which is integrable, that is, $\omega \wedge \mathrm{d}\omega = 0$. As $\omega(p) \neq 0$, for all $p \in M$, this condition can be written as

$$\mathrm{d}\omega = \omega \wedge \omega_1, \tag{$\star$}$$

for a certain ω_1. Consider the differential 3-form

$$\Xi = -\omega_1 \wedge \mathrm{d}\omega_1.$$

Prove:

(i) The form Ξ defines a cohomology class $[\Xi] \in H^3_{dR}(M, \mathbb{R})$.
(ii) $[\Xi]$ is an invariant of the foliation, that is, it does not change if either $\mathscr{F}$ is defined by $\omega' = f\omega$, with $f \in C^\infty M$ nowhere-vanishing or if we take another form ω_1' satisfying ($\star$).

The relevant theory is developed, for instance, in Godbillon and Vey [3].

Solution

(i) Taking the exterior derivative of both members of ($\star$), we obtain $0 = -\omega \wedge \mathrm{d}\omega_1$, from which

$$\mathrm{d}\omega_1 = \omega \wedge \omega_2,$$

and thus $\mathrm{d}\Xi = -\omega \wedge \omega_2 \wedge \omega \wedge \omega_2 = 0$.

(ii) If $\mathscr{F}$ is defined by $\omega' = f\omega$, with $f \in C^\infty M$ nowhere-vanishing, we have

$$\mathrm{d}\omega' = \mathrm{d}f \wedge \omega + f\omega \wedge \omega_1 = \omega' \wedge \left(\omega_1 - \frac{\mathrm{d}f}{f}\right).$$

Hence

$$\Xi' = -\omega_1' \wedge \mathrm{d}\omega_1' = -\omega_1 \wedge \mathrm{d}\omega_1 - \frac{\mathrm{d}f}{f} \wedge \mathrm{d}\omega_1 = \Xi - \mathrm{d}\big(\log|f|\,\mathrm{d}\omega_1\big),$$

from which $[\Xi'] = [\Xi]$.

If we choose another form, say ω_1', satisfying ($\star$), then from this equation and $\mathrm{d}\omega = \omega \wedge \omega_1'$ we have $\omega \wedge (\omega_1 - \omega_1') = 0$, that is, $\omega_1 - \omega_1'$ belongs to the ideal generated by ω. Hence, the general expression for such forms ω_1 is $\omega_1'' = \omega_1 + h\omega$, $h \in C^\infty M$. Now, we have

$$\omega_1'' \wedge \mathrm{d}\omega_1'' = (\omega_1 + h\omega) \wedge (\mathrm{d}\omega_1 + \mathrm{d}h \wedge \omega + h\omega \wedge \omega_1) = \omega_1 \wedge \mathrm{d}\omega_1 + \mathrm{d}(h\,\mathrm{d}\omega),$$

hence $[\Xi''] = [\Xi]$.

Problem 5.33 (Roussarie's Example of a Foliation with Non-zero Godbillon–Vey Class) Consider the Lie group

$$G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \Big/ \left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\},$$

its subgroup

$$H = \left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix},\ a > 0\right\},$$

a discrete co-compact subgroup Γ of G, and the (hence compact) quotient $M = \Gamma\backslash G$.

Consider the basis

$$\left\{X = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ Y = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ Z = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}$$

of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. Then $\{X, Y\}$ is a basis of the Lie algebra $\mathfrak{h}$ of H and it is immediate that

$$[X,Y] = 2Y, \qquad [X,Z] = -2Z, \qquad [Y,Z] = X.$$

Let $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$ be the basis of left-invariant differential 1-forms dual to $\{X, Y, Z\}$.

Since X and Y span $\mathfrak{h}$, the form $\tilde{\gamma}$ defines a foliation $\widetilde{\mathscr{F}}$ of G by the left cosets of H, which passes to a foliation $\mathscr{F}$ on M.

Prove that the 1-form $\widetilde{\omega}_1 = -2\tilde{\alpha}$ induces a non-zero Godbillon–Vey class of $\mathscr{F}$.

Hint Since both the given vector fields and 1-forms are invariant under left translations by elements of H, we have

$$\mathrm{d}\tilde{\alpha}(U,V) = U\big(\tilde{\alpha}(V)\big) - V\big(\tilde{\alpha}(U)\big) - \tilde{\alpha}\big([U,V]\big) = -\tilde{\alpha}\big([U,V]\big), \quad U, V \in \mathfrak{g},$$

and similarly for $\tilde{\beta}$ and $\tilde{\gamma}$. Compute $\mathrm{d}\tilde{\gamma}$ and then $\mathrm{d}\widetilde{\omega}_1$.

The relevant theory is developed, for instance, in Bott [2].

Solution We have

$$\mathrm{d}\tilde{\gamma}(X,Y) = 0, \qquad \mathrm{d}\tilde{\gamma}(X,Z) = 2, \qquad \mathrm{d}\tilde{\gamma}(Y,Z) = 0,$$

So (using the same notation as in Problem 5.32),

$$\mathrm{d}\tilde{\gamma} = 2\tilde{\alpha} \wedge \tilde{\gamma} = \tilde{\gamma} \wedge \widetilde{\omega}_1,$$

where $\widetilde{\omega}_1 = -2\tilde{\alpha}$.

In turn, one has

$$\mathrm{d}\widetilde{\omega}_1(X,Y) = 0, \qquad \mathrm{d}\widetilde{\omega}_1(X,Z) = 0, \qquad \mathrm{d}\widetilde{\omega}_1(Y,Z) = 2,$$

hence $\mathrm{d}\widetilde{\omega}_1 = 2\tilde{\beta} \wedge \tilde{\gamma}$, and thus

$$\widetilde{\omega}_1 \wedge \mathrm{d}\widetilde{\omega}_1 = -4\tilde{\alpha} \wedge \tilde{\beta} \wedge \tilde{\gamma}$$

is a nowhere-vanishing 3-form on G.

Since $\tilde{\gamma}$ and $\widetilde{\omega}_1$ are invariant under left translations by elements of H, they descend to well-defined forms γ and ω_1 on M.

The differential 1-form γ defines the foliation $\mathscr{F}$. Moreover, $\mathrm{d}\gamma = \gamma \wedge \omega_1$ and $\omega_1 \wedge \mathrm{d}\omega_1$ is a nowhere-vanishing 3-form on M. But M is a compact, orientable 3-manifold without boundary, so that

$$\int_M \omega_1 \wedge \mathrm{d}\omega_1 \neq 0,$$

implying that

$$[\omega_1 \wedge \mathrm{d}\omega_1] \in H^3(M, \mathbb{R})$$

is not zero.

Problem 5.34 Consider on $\mathbb{C}^2 \setminus \{(0,0)\} = \{(z,w) \neq (0,0)\}$ the differential 1-form given, for fixed $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, by

$$\omega = \lambda_1 w\,\mathrm{d}z + \lambda_2 z\,\mathrm{d}w.$$

Since it depends only on z, w and their differentials, it is completely integrable, so defining a codimension 1 (see Definition 2.16) complex foliation $\mathcal{F}$ on the manifold $\mathbb{C}^2 \setminus \{0\}$.

(i) Find a differential 1-form ω_1 (with the notation in Problem 5.32) such that

$$\mathrm{d}\omega = \omega \wedge \omega_1.$$

(ii) Prove that, for suitable λ_1, λ_2, the form

$$\omega_1 \wedge \mathrm{d}\omega_1$$

defines a non-zero Godbillon–Vey class of the corresponding foliation $\mathcal{F}$ (Bott example of non-zero Godbillon–Vey class of a foliation).

Hint Take

$$\omega_1 = L_X \omega,$$

where X stands for the differentiable (although not holomorphic) vector field

$$X = \frac{1}{\lambda_1} \frac{\bar{w}}{|z|^2 + |w|^2} \frac{\partial}{\partial z} + \frac{1}{\lambda_2} \frac{\bar{z}}{|z|^2 + |w|^2} \frac{\partial}{\partial w},$$

which satisfies $\omega(X) = 1$.

The relevant theory is developed, for instance, in Bott [2].

Solution

(i) According to the hint, we choose

$$\omega_1 = L_X \omega = (\mathrm{d}\iota_X + \iota_X \,\mathrm{d})\omega = \mathrm{d}(1) + \iota_X \,\mathrm{d}\omega = \iota_X \,\mathrm{d}\omega.$$

Then, since

$$\mathrm{d}\omega = (\lambda_2 - \lambda_1)\,\mathrm{d}z \wedge \mathrm{d}w,$$

we have

$$\omega_1 = \frac{\lambda_2 - \lambda_1}{|z|^2 + |w|^2} \left(\frac{1}{\lambda_1} \bar{w}\,\mathrm{d}w - \frac{1}{\lambda_2} \bar{z}\,\mathrm{d}z \right),$$

and

$$\begin{aligned} \omega \wedge \omega_1 &= \frac{\lambda_2 - \lambda_1}{|z|^2 + |w|^2} (\lambda_1 w\,\mathrm{d}z + \lambda_2 z\,\mathrm{d}w) \wedge \left(\frac{1}{\lambda_1} \bar{w}\,\mathrm{d}w - \frac{1}{\lambda_2} \bar{z}\,\mathrm{d}z \right) \\ &= \frac{\lambda_2 - \lambda_1}{|z|^2 + |w|^2} \big(|w|^2 + |z|^2 \big)\,\mathrm{d}z \wedge \mathrm{d}w = \mathrm{d}\omega. \end{aligned}$$

(ii) Since on S^3 we have $|z|^2 + |w|^2 = 1$, we get

$$\mathrm{d}\omega_1 = (\lambda_2 - \lambda_1)\left(\frac{1}{\lambda_1}\,\mathrm{d}\bar{w} \wedge \mathrm{d}w - \frac{1}{\lambda_2}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z\right)$$

and, also on S^3,

$$\begin{aligned}\omega_1 \wedge \mathrm{d}\omega_1 &= (\lambda_2 - \lambda_1)^2\left(\frac{1}{\lambda_1}\bar{w}\,\mathrm{d}w - \frac{1}{\lambda_2}\bar{z}\,\mathrm{d}z\right) \wedge \left(\frac{1}{\lambda_1}\,\mathrm{d}\bar{w} \wedge \mathrm{d}w - \frac{1}{\lambda_2}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z\right)\\ &= \frac{(\lambda_2 - \lambda_1)^2}{\lambda_1\lambda_2}(\bar{z}\,\mathrm{d}z \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} + \bar{w}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w).\end{aligned}$$

Now, since the form $\bar{z}\,\mathrm{d}z \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} + \bar{w}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w$ depends neither on λ_1 nor on λ_2, for suitable values of λ_1 and λ_2, which is always possible, we obtain

$$\int_{S^3} \omega_1 \wedge \mathrm{d}\omega_1 = \int_{S^3} \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}(\bar{z}\,\mathrm{d}z \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} + \bar{w}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w) \neq 0,$$

hence the cohomology class

$$[\omega_1 \wedge \mathrm{d}\omega_1] \in H^3\big(\mathbb{C}^2 \setminus \{0\}, \mathbb{R}\big) \cong H^3\big(S^3, \mathbb{R}\big)$$

is not zero.

Problem 5.35 Let (P, M, G) be a principal bundle over the C^∞ manifold M and let Γ, $\widetilde{\Gamma}$ be connections in P, whose connection forms (resp., curvature forms) can be described on trivialising open subsets of M by the $\mathfrak{g}$-valued differential 1-forms A, $\tilde{A}$ (resp., 2-forms F, $\widetilde{F}$).

(i) Let $I \in \mathscr{I}^r(G)$ be a G-invariant polynomial, and consider the global $2r$-forms $I(F^r)$ and $I(\widetilde{F}^r)$ (see Definitions 6.17 and [4]). Deduce from the *Chern–Simons Formula* in Theorem 6.18 for the difference $I(F^r) - I(\widetilde{F}^r)$, the formula for the particular case where G is a matrix group, $I(F^2) = \mathrm{tr}(F \wedge F)$, and $\tilde{A} = 0$:

$$\mathrm{tr}(F \wedge F) = \mathrm{d}\,\mathrm{tr}\left((\mathrm{d}A) \wedge A + \frac{2}{3}A \wedge A \wedge A\right). \qquad (\star)$$

(ii) Let ξ be the map $\xi : \mathbb{R}^4 \to M(2, \mathbb{C})$,

$$\xi(x) = \begin{pmatrix} x^4 - ix^3 & -x^2 - ix^1 \\ x^2 - ix^1 & x^4 + \mathrm{i}x^3 \end{pmatrix}.$$

Consider on $\mathbb{R}^4$ with the Euclidean metric the differential forms

$$A_1 = \frac{r^2}{r^2 + c^2}\gamma^{-1}\,\mathrm{d}\gamma, \qquad A_2 = \gamma A_1 \gamma^{-1} + \gamma\,\mathrm{d}\gamma^{-1} = \frac{c^2}{r^2 + c^2}\gamma\,\mathrm{d}\gamma^{-1},$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$, $c \in \mathbb{R}$, and

$$\gamma\colon \mathbb{R}^4 \setminus \{0\} \to \mathrm{SU}(2), \qquad \gamma(x) = r^{-1}\xi(x),$$

identifies $S^3(r)$ with SU(2).

The form A_1 is regular at $x = 0$ as it follows taking the formulas

$$A_1 = -\frac{r\,\mathrm{d}r}{r^2 + c^2} + \frac{r^2}{r^2 + c^2}\xi^{-1}\,\mathrm{d}\xi, \qquad \det\xi = r^2,$$

into account, but A_2 is singular at $x = 0$. Let N, S denote the north and south poles of S^4. Identify $\mathbb{R}^4$ with $U_1 = S^4 \setminus \{S\}$ and on the other hand with $U_2 = S^4 \setminus \{N\}$ under convenient stereographic projections (see [4, 10.7]). Then one can accordingly define A_1 on U_1 and A_2 on U_2, since the singularity of A_2 at the origin manifests as a singularity at the north pole, which does not belong to U_2; in such a way that A_1 and A_2 are local representatives of a connection in a principal SU(2)-bundle P' over S^4, whose transition function g_{21} is γ.

Express the Chern number $c_{(2)}(P')$ in terms of $\gamma^{-1}\mathrm{d}\gamma$, by means of the Chern–Simons formula ($\star$).

Remark In Physics, the differential form A with local representatives as in (ii) is called an *instanton potential*. It solves the *Euclidean Yang–Mills equation*, that is, $D \star F = 0$, where $F = \mathrm{d}A + A \wedge A$ and $\star$ stands for the Hodge star operator (see Problem 6.112).

The relevant theory is developed, for instance, in Göckeler and Schücker [4].

Solution

(i) As G is a matrix group, we can write $F = \mathrm{d}A + \frac{1}{2}[A, A]$ as $F = \mathrm{d}A + A \wedge A$. In the particular case $I(F^2) = \mathrm{tr}(F \wedge F)$, we have

$$\mathrm{tr}(F \wedge F) - \mathrm{tr}(\widetilde{F} \wedge \widetilde{F}) = \mathrm{d}Q(A, \tilde{A}).$$

Putting $\alpha = A - \tilde{A}$, one has

$$\begin{aligned}
Q(A, \tilde{A}) &= 2\int_0^1 \mathrm{tr}\big(\alpha \wedge (\mathrm{d}\tilde{A} + t\mathrm{d}\alpha) + \alpha \wedge (\tilde{A} + t\alpha) \wedge (\tilde{A} + t\alpha)\big)\,\mathrm{d}t \\
&= 2\int_0^1 \mathrm{tr}\big(\alpha \wedge \mathrm{d}\tilde{A} + t\alpha \wedge \mathrm{d}\alpha + \alpha \wedge \tilde{A} \wedge \tilde{A} + t\alpha \wedge \tilde{A} \wedge \alpha \\
&\quad + t\alpha \wedge \alpha \wedge \tilde{A} + t^2\alpha \wedge \alpha \wedge \alpha\big)\,\mathrm{d}t \\
&= 2\,\mathrm{tr}\Big(\alpha \wedge \mathrm{d}\tilde{A} + \frac{1}{2}\alpha \wedge \mathrm{d}\alpha + \alpha \wedge \tilde{A} \wedge \tilde{A} + \frac{1}{2}\alpha \wedge \tilde{A} \wedge \alpha \\
&\quad + \frac{1}{2}\alpha \wedge \alpha \wedge \tilde{A} + \frac{1}{3}\alpha \wedge \alpha \wedge \alpha\Big)
\end{aligned}$$

$$= \mathrm{tr}\left(2\alpha \wedge \widetilde{F} + \alpha \wedge \mathrm{d}\alpha + 2\alpha \wedge \tilde{A} \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha\right).$$

For $\tilde{A} = 0$, this expression reduces to the formula in the statement.

(ii) Let S^4_+ (resp., S^4_-) denote the upper (resp., lower) hemisphere of S^4, that is, the subset with last coordinate $\geqslant 0$ (resp., $\leqslant 0$). Then, on account of

$$A_1 = \gamma^{-1} A_2 \gamma + \gamma^{-1} \mathrm{d}\gamma,$$

$$F_1 = \mathrm{d}A_1 + A_1 \wedge A_1, \qquad F_2 = \mathrm{d}A_2 + A_2 \wedge A_2, \qquad F_1 = \gamma^{-1} F_2 \gamma,$$

we can write

$$\begin{aligned}
c_{(2)}\big(P'\big) &= \frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F \wedge F) = \frac{1}{8\pi^2}\int_{S^4_+} \mathrm{tr}(F_1 \wedge F_1) + \frac{1}{8\pi^2}\int_{S^4_-} \mathrm{tr}(F_2 \wedge F_2) \\
&= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(\mathrm{d}A_1 \wedge A_1 + \frac{2}{3}A_1 \wedge A_1 \wedge A_1\right) \\
&\quad - \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(\mathrm{d}A_2 \wedge A_2 + \frac{2}{3}A_2 \wedge A_2 \wedge A_2\right) \quad \text{(by Stokes' Th.)} \\
&= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(F_1 \wedge A_1 - \frac{1}{3}A_1 \wedge A_1 \wedge A_1 - F_2 \wedge A_2\right. \\
&\quad \left. + \frac{1}{3}A_2 \wedge A_2 \wedge A_2\right) \\
&= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(\gamma^{-1}\mathrm{d}A_2 \wedge A_2\gamma + \gamma^{-1}A_2 \wedge A_2 \wedge A_2\gamma\right. \\
&\quad + \gamma^{-1}\mathrm{d}A_2 \wedge \mathrm{d}\gamma + \gamma^{-1}A_2 \wedge A_2 \wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2 \wedge A_2 \wedge A_2\gamma - \frac{1}{3}\gamma^{-1}A_2 \wedge A_2 \wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2 \wedge \mathrm{d}\gamma \wedge \gamma^{-1}A_2\gamma - \frac{1}{3}\gamma^{-1}A_2 \wedge \mathrm{d}\gamma \wedge \gamma^{-1}\,\mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}A_2 \wedge A_2\gamma - \frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}A_2 \wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}A_2\gamma - \frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}\,\mathrm{d}\gamma \wedge \gamma^{-1}\,\mathrm{d}\gamma \\
&\quad \left. - \mathrm{d}A_2 \wedge A_2 - A_2 \wedge A_2 \wedge A_2 + \frac{1}{3}A_2 \wedge A_2 \wedge A_2\right) \\
&= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(\gamma^{-1}\,\mathrm{d}A_2 \wedge A_2\gamma + \gamma^{-1}A_2 \wedge A_2 \wedge A_2\gamma\right. \\
&\quad + \gamma^{-1}\mathrm{d}A_2 \wedge \mathrm{d}\gamma\gamma^{-1}\gamma + \gamma^{-1}A_2 \wedge A_2 \wedge \mathrm{d}\gamma
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge A_2\gamma-\frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma\\
&-\frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma-\frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\\
&-\frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma-\frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\\
&-\frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma-\frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\\
&-\mathrm{d}A_2\wedge A_2-A_2\wedge A_2\wedge A_2+\frac{1}{3}A_2\wedge A_2\wedge A_2\Big)\\
&=\frac{1}{8\pi^2}\int_{S^3}\mathrm{tr}\Big(\mathrm{d}A_2\wedge \mathrm{d}\gamma\gamma^{-1}-\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\\
&-\frac{1}{3}\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\Big)\\
&=\frac{1}{8\pi^2}\int_{S^3}\Big(\mathrm{d}\big(\mathrm{tr}\big(A_2\wedge \mathrm{d}\gamma\gamma^{-1}\big)\big)-\frac{1}{3}\,\mathrm{tr}\big(\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\big)\Big)\\
&=-\frac{1}{24\pi^2}\int_{S^3}\mathrm{tr}\big(\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\wedge\gamma^{-1}\,\mathrm{d}\gamma\big)\quad\text{(by Stokes' Th.).}
\end{aligned}
$$

Remark The last expression for $c_{(2)}(P')$ is the opposite to a certain winding number (the *topological charge*), which is an element of the homotopy group $\pi_3(\mathrm{SU}(2))\cong\mathbb{Z}$, associated to a map from the equator S^3 in S^4 to $\mathrm{SU}(2)\cong S^3$. It is important in Physics as it corresponds to a minimum of the Yang–Mills action functional.

5.5 Linear Connections

Problem 5.36 Given a linear connection ∇ on the C^∞ manifold M, one defines the *conjugate* or *opposite* connection $\widehat{\nabla}$ on M by

$$\widehat{\nabla}_X Y=\nabla_Y X+[X,Y],\qquad X,Y\in\mathfrak{X}(M).$$

(i) Prove that $\widehat{\nabla}$ is a linear connection.
(ii) Compute the local components $\widehat{\Gamma}^i_{jh}$ of $\widehat{\nabla}$ in terms of the components of ∇.

Solution

(i) Since ∇ is a linear connection and from the expression

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X,\qquad f,g\in C^\infty M,$$

we deduce by some computations that $\widehat{\nabla}$ satisfies the properties:

(a) $\widehat{\nabla}_X(Y+Z)=\widehat{\nabla}_X Y+\widehat{\nabla}_X Z$; (b) $\widehat{\nabla}_{X+Y}Z=\widehat{\nabla}_X Z+\widehat{\nabla}_Y Z$;

(c) $\widehat{\nabla}_{fX}Y=f\widehat{\nabla}_X Y$; (d) $\widehat{\nabla}_X fY=(Xf)Y+f\widehat{\nabla}_X Y$.

That is, $\widehat{\nabla}$ is a linear connection.

(ii) One has $\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=\sum_k \widehat{\Gamma}^k_{ij}\frac{\partial}{\partial x^k}$, in terms of the local coordinates $x^1,\dots,x^n$ and also

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}+\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}=\sum_k \Gamma^k_{ji}\frac{\partial}{\partial x^k}.$$

That is, $\widehat{\Gamma}^k_{ij}=\Gamma^k_{ji}$.

Problem 5.37 Let $\varphi\colon M\to M'$ be a diffeomorphism. Given a linear connection ∇ on M, let $\nabla'=\varphi\cdot\nabla$ be defined by

$$\nabla'_{X'}Y'=\varphi\cdot\big(\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot Y'\big),\quad X',Y'\in\mathfrak{X}\big(M'\big).$$

Prove:

(i) ∇' is a linear connection on M'.
(ii) If φ_t is the flow of a vector field $X\in\mathfrak{X}(M)$ such that $\varphi_t\cdot\nabla=\nabla$, $t\in\mathbb{R}$, then

$$L_X\circ\nabla_Y-\nabla_Y\circ L_X=\nabla_{[X,Y]},\quad X,Y\in\mathfrak{X}(M).\tag{$\star$}$$

Solution

(i) We prove one property only: For any $f\in C^\infty M'$,

$$\begin{aligned}\nabla'_{X'}fY'&=\varphi\cdot\big(\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot\big(fY'\big)\big)=\varphi\cdot\big(\nabla_{\varphi^{-1}\cdot X'}(f\circ\varphi)\varphi^{-1}\cdot Y'\big)\\&=\varphi\cdot\big\{\big(\big(\varphi^{-1}\cdot X'\big)(f\circ\varphi)\big)\varphi^{-1}\cdot Y'+(f\circ\varphi)\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot Y'\big\}\\&=\big(X'f\big)Y'+f\nabla'_{X'}Y'.\end{aligned}$$

(ii) Applying both sides of ($\star$) to a function $f\in C^\infty M$, we obtain

$$X\big(Y(f)\big)-Y\big(X(f)\big)=[X,Y](f),$$

which trivially holds. Applying now both sides of ($\star$) to a vector field Z, one has

$$\begin{aligned}L_X\nabla_Y Z&=\lim_{t\to0}\frac{1}{t}\big(\nabla_Y Z-\varphi_t\cdot(\nabla_Y Z)\big)\\&=\lim_{t\to0}\frac{1}{t}(\nabla_Y Z-\nabla_{\varphi_t\cdot Y}Z)+\lim_{t\to0}\frac{1}{t}\big(\nabla_{\varphi_t\cdot Y}Z-\nabla_{\varphi_t Y}(\varphi_t\cdot Z)\big)\end{aligned}$$

$$
\begin{aligned}
&= \nabla_{\lim_{t\to 0}\frac{1}{t}(Y-\varphi_t\cdot Y)} Z + \nabla_Y \left(\lim_{t\to 0} \frac{1}{t}(Z - \varphi_t \cdot Z) \right) \\
&= \nabla_{L_X Y} Z + \nabla_Y L_X Z = \nabla_{[X,Y]} Z + \nabla_Y L_X Z.
\end{aligned}
$$

As L_X and ∇_Y are type-preserving derivations that commute with contractions, for every tensor field T we have

$$
(L_X \circ \nabla_Y - \nabla_Y \circ L_X)(T) = \nabla_{[X,Y]} T.
$$

Problem 5.38 Let M be a C^∞ n-manifold endowed with a torsionless linear connection. Prove that in a system of normal coordinates with origin p, all the Christoffel symbols at p vanish.

Solution In a system of normal coordinates $\{x^i\}$, $i = 1, \dots, n$, around p, the equations of the geodesics through p are given by $x^i = \lambda^i t$, with λ^i constants. These functions must satisfy the differential equations of the geodesics, i.e.

$$
\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1, \dots, n,
$$

that now reduce to $\sum_{i,j} \Gamma^k_{ij}(p)\lambda^i\lambda^j = 0$, for $k = 1, \dots, n$. As the connection is torsionless, it is immediate that $\Gamma^k_{ij}(p) = 0$, for $i, j, k = 1, \dots, n$.

Problem 5.39 The Lie derivative L is another kind of directional derivative of vector fields on a differentiable manifold M.

(i) Prove that the map

$$
\begin{aligned}
L\colon TM \times TM &\longrightarrow TM \\
(X, Y) &\longmapsto L_X Y
\end{aligned}
$$

is not a connection.

(ii) Show that there are vector fields V and W on $\mathbb{R}^2$ such that

$$
V = W = \frac{\partial}{\partial x}
$$

along the x-axis but with

$$
L_V \frac{\partial}{\partial y} \neq L_W \frac{\partial}{\partial y}
$$

along the x-axis.

Remark This shows that Lie differentiation does not give a well-defined way to take directional derivatives of vector fields along curves.

The relevant theory is developed, for instance, in Lee [5].

Solution

(i) Since

$$L_{fX}Y = [fX, Y] = f[X, Y] - (Yf)X = fL_XY - (Yf)X,$$

for all $f \in C^\infty M$ and $X, Y \in \mathfrak{X}(M)$, imposing $L_{fX}Y = fL_XY$, it follows that $(Yf)X = 0$. As X is arbitrary, this implies $Yf = 0$, and since Y is also arbitrary, one gets that f is a locally constant function.

In conclusion, one has that

$$L_{fX}Y = fL_XY, \quad X, Y \in \mathfrak{X}(M),$$

if and only if f is a constant function. Hence the map is not a connection.

(ii) The vector fields

$$V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad W = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

are equal to $\frac{\partial}{\partial x}$ along $y = 0$, but

$$L_V\frac{\partial}{\partial y} = -\frac{\partial}{\partial y}, \qquad L_W\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

do not coincide.

5.6 Torsion and Curvature

Problem 5.40 Consider a linear connection on a C^∞ manifold, with components

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\delta^i_k\theta_j,$$

where the Γ^i_{jk} are the components of another linear connection (it is said that they are *projectively related connections*), θ is a differential 1-form, and δ^i_j denotes the Kronecker delta. Calculate the difference tensor $\widetilde{R}^i_{hjk} - R^i_{hjk}$ of their respective curvature tensors fields.

Solution Putting $\partial_j = \partial/\partial x^j$, we have

$$\begin{aligned}\widetilde{R}^i_{hjk} &= \partial_j\widetilde{\Gamma}^i_{kh} - \partial_k\widetilde{\Gamma}^i_{jh} + \sum_r \widetilde{\Gamma}^r_{kh}\widetilde{\Gamma}^i_{jr} - \sum_r \widetilde{\Gamma}^r_{jh}\widetilde{\Gamma}^i_{kr} \\ &= \partial_j\big(\Gamma^i_{kh} + 2\delta^i_h\theta_k\big) - \partial_k\big(\Gamma^i_{jh} + 2\delta^i_h\theta_j\big) + \sum_r\big(\Gamma^r_{kh} + 2\delta^r_h\theta_k\big)\big(\Gamma^i_{jr} + 2\delta^i_r\theta_j\big) \\ &\quad - \sum_r\big(\Gamma^r_{jh} + 2\delta^r_h\theta_j\big)\big(\Gamma^i_{kr} + 2\delta^i_r\theta_k\big)\end{aligned}$$

$$\begin{aligned}&= R^i_{hjk} + 2\delta^i_h \partial_j \theta_k - 2\delta^i_h \partial_k \theta_j + 2\Gamma^i_{kh}\theta_j + 2\Gamma^i_{jh}\theta_k + 4\delta^i_h \theta_k \theta_j \\ &\quad - 2\Gamma^i_{jh}\theta_k - 2\Gamma^i_{kh}\theta_j - 4\delta^i_h \theta_j \theta_k \\ &= R^i_{hjk} + 2\delta^i_h(\partial_j \theta_k - \partial_k \theta_j),\end{aligned}$$

from which we obtain

$$\widetilde{R}^i_{hjk} - R^i_{hjk} = 2\delta^i_h(\partial_j \theta_k - \partial_k \theta_j).$$

Problem 5.41 Let M be a C^∞ manifold, with a linear connection having components Γ^i_{jk} with respect to a local coordinate system. Write the formulas for the covariant derivative of the following tensor fields on M:

(i) A vector field with components X^i.
(ii) A differential 1-form with components θ_i.
(iii) A $(1,1)$ tensor field with components J^i_j.
(iv) A $(0,2)$ tensor field with components τ_{ij}.

Moreover, prove:

(v) If the given connection is torsionless, for a vector field with components X^i, one has

$$X^i_{;jk} - X^i_{;kj} = -\sum_r X^r R^i_{rjk},$$

where $X^i_{;jk} = (X^i_{;j})_{;k}$, and R^i_{jkl} are the components of the curvature tensor field of the given connection.
(vi) For a differential 1-form with components θ_i, one has

$$\theta_{i;jk} - \theta_{i;kj} = \sum_r \big(\theta_r R^r_{ijk} - 2\theta_{i;r} T^r_{jk}\big),$$

where T^i_{jk} and R^i_{jkl} are the components of the torsion and curvature tensor fields of the given connection, respectively.

Solution Let $\partial_j = \partial/\partial x^j$, where $\{x^j\}$ stand for local coordinates. Then:

(i)

$$\nabla_{\partial_j}\Big(\sum_i X^i \partial_i\Big) = \sum_i (\partial_j X^i)\partial_i + \sum_{i,r} X^i \Gamma^r_{ji}\partial_r.$$

Hence

$$X^i_{;j} = \partial_j X^i + \sum_r \Gamma^i_{jr} X^r.$$

(ii)

$$\theta_{j;i} = (\nabla_{\partial_i}\theta)\partial_j = \nabla_{\partial_i}(\theta\partial_j) - \theta(\nabla_{\partial_i}\partial_j)$$

$$= \nabla_{\partial_i}\left(\left(\sum_l \theta_l \, \mathrm{d}x^l\right)\partial_j\right) - \theta\left(\sum_r \Gamma^r_{ij}\partial_r\right)$$

$$= \partial_i\theta_j - \sum_r \Gamma^r_{ij}\theta_r.$$

(iii)

$$(\nabla_{\partial_i} J)\partial_j = \nabla_{\partial_i} J\partial_j - J\nabla_{\partial_i}\partial_j = \sum_r \big(\nabla_{\partial_i} J^r_j \partial_r - J\Gamma^r_{ij}\partial_r\big)$$

$$= \sum_r \big(\partial_i J^r_j\big)\partial_r + \sum_{r,s} J^r_j \Gamma^s_{ir}\partial_s - \sum_{r,s} \Gamma^r_{ij} J^s_r \partial_s.$$

Hence

$$J^i_{j;k} = \partial_k J^i_j + \sum_r J^r_j \Gamma^i_{kr} - \sum_r \Gamma^r_{kj} J^i_r.$$

(iv)

$$\tau_{ij;k} = (\nabla_{\partial_i}\tau)(\partial_j, \partial_k) = \nabla_{\partial_i}\tau_{jk} - \sum_r \big(\tau\big(\Gamma^r_{ij}\partial_r, \partial_k\big) + \tau\big(\partial_j, \Gamma^r_{ik}\partial_r\big)\big)$$

$$= \partial_i \tau_{jk} - \sum_r \Gamma^r_{ij}\tau_{rk} - \sum_r \Gamma^r_{ik}\tau_{jr}.$$

(v) We have

$$X^i_{;jk} = \partial_k\left(\partial_j X^i + \sum_r \Gamma^i_{jr}X^r\right) + \sum_r \big(X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\big)$$

$$= \partial_k\partial_j X^i + \sum_r \big(\big(\partial_k\Gamma^i_{jr}\big)X^r + \Gamma^i_{jr}\partial_k X^r + X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\big)$$

$$= \partial_k\partial_j X^i + \sum_r \big(\big(\partial_k\Gamma^i_{jr}\big)X^r + \Gamma^i_{jr}X^r_{;k} - \sum_s X^s\Gamma^i_{jr}\Gamma^r_{ks}$$

$$+ X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\big)$$

$$= \partial_k\partial_j X^i + \sum_r \left(X^r\left(\partial_k\Gamma^i_{jr} - \sum_s \Gamma^i_{js}\Gamma^s_{kr}\right) + \Gamma^i_{jr}X^r_{;k}\right.$$

$$\left. + \Gamma^i_{kr}X^r_{;j} - \Gamma^r_{kj}X^i_{;r}\right),$$

and

$$X^i_{;kj} = \partial_j\partial_k X^i + \sum_r \left(X^r\left(\partial_j\Gamma^i_{kr} - \sum_s \Gamma^i_{ks}\Gamma^s_{jr}\right) + \Gamma^i_{kr}X^r_{;j}\right.$$

$$\left. + \Gamma^i_{jr}X^r_{;k} - \Gamma^r_{jk}X^i_{;r}\right),$$

hence

$$X^i_{;jk} - X^i_{;kj} = X^r \sum_r \Big(\partial_k \Gamma^i_{jr} - \partial_j \Gamma^i_{kr} + \sum_s \big(\Gamma^s_{jr} \Gamma^i_{ks} - \Gamma^s_{kr} \Gamma^i_{js} \big) \Big) = \sum_r X^r R^i_{rkj}.$$

(vi)

$$\begin{aligned} \theta_{i;jk} &= \Big(\partial_j \theta_i - \sum_r \Gamma^r_{ji} \theta_r \Big)_{;k} \\ &= \partial_k \Big(\partial_j \theta_i - \sum_r \Gamma^r_{ji} \theta_r \Big) - \Gamma^r_{kj} \sum_r \Big(\partial_r \theta_i - \sum_s \Gamma^s_{ri} \theta_s \Big) \\ &\quad - \Gamma^r_{ki} \Big(\partial_j \theta_r - \sum_s \Gamma^s_{jr} \theta_s \Big). \end{aligned}$$

Expanding this formula and the similar one for $\theta_{i;kj}$, we obtain

$$\begin{aligned} \theta_{i;jk} - \theta_{i;kj} &= \sum_r \Big(\partial_j \Gamma^r_{ki} - \partial_k \Gamma^r_{ji} + \sum_s \big(\Gamma^s_{ki} \Gamma^r_{js} - \Gamma^s_{ji} \Gamma^r_{ks} \big) \Big) \theta_r \\ &\quad + \sum_r \big(\Gamma^r_{jk} - \Gamma^r_{kj} \big) \Big(\partial_r \theta_i - \sum_s \Gamma^s_{ri} \theta_s \Big) \\ &= \sum_r \big(R^r_{ijk} \theta_r + 2 T^r_{jk} \theta_{i;r} \big). \end{aligned}$$

Problem 5.42 Consider the linear connection on the half-plane $y > 0$ of $\mathbb{R}^2$ defined by the components $\Gamma^i_{jk} = 0$, except $\Gamma^1_{12} = 1$, with respect to the frame $(e_1 = \partial/\partial x, e_2 = \partial/\partial y)$. Consider the frame

$$\Big(\bar{e}_1 = \frac{\partial}{\partial x},\ \bar{e}_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \Big).$$

Compute the components of the connection and the components of the torsion tensor with respect to this frame.

Solution We have $\nabla_{\bar{e}_j} \bar{e}_i = \bar{\Gamma}^k_{ij} \bar{e}_k$, and

$$\begin{aligned} &\nabla_{\bar{e}_1} \bar{e}_1 = \nabla_{e_1} e_1 = 0, \qquad \nabla_{\bar{e}_1} \bar{e}_2 = \nabla_{e_1} (x e_1 + y e_2) = (1 + y) e_1 = (1 + y) \bar{e}_1, \\ &\nabla_{\bar{e}_2} \bar{e}_1 = \nabla_{x e_1 + y e_2} e_1 = 0, \qquad \nabla_{\bar{e}_2} \bar{e}_2 = \nabla_{x e_1 + y e_2} (x e_1 + y e_2) = x y \bar{e}_1 + \bar{e}_2. \end{aligned}$$

Thus the non-vanishing components of ∇ with respect to the frame $(\bar{e}_1, \bar{e}_2)$ are

$$\bar{\Gamma}^1_{12} = 1 + y, \qquad \bar{\Gamma}^1_{22} = x y, \qquad \bar{\Gamma}^2_{22} = 1,$$

and the only non-vanishing component of the torsion tensor is $\bar{T}^1_{12} = y$.

Problem 5.43 Let M and N be C^∞ manifolds with linear connections ∇ and ∇', respectively. A C^∞ map $\varphi\colon M \to N$ is said to be connection-preserving if

$$\varphi_*(\nabla_X Y)_p = \big(\nabla'_{X'} Y'\big)_{\varphi(p)}, \tag{$\star$}$$

for all $p \in M$, where X, Y are φ-related to X', Y', respectively. Prove that if φ is also a diffeomorphism, then:

(i)

$$\varphi \cdot \big(R(X,Y)Z\big) = R'(\varphi \cdot X, \varphi \cdot Y)(\varphi \cdot Z),$$

where R and R' are the curvature tensor fields of ∇ and ∇', respectively.

(ii)

$$\varphi \cdot \big(T(X,Y)\big) = T'(\varphi \cdot X, \varphi \cdot Y),$$

where T and T' stand for the torsion tensors of ∇ and ∇', respectively.

Solution

(i) First, we remark (see also Problem 5.37) that if φ is a diffeomorphism, then the formula ($\star$) means

$$\varphi \cdot (\nabla_X Y) = \nabla'_{\varphi \cdot X} \varphi \cdot Y.$$

Thus,

$$\begin{aligned}
\varphi \cdot \big(R(X,Y)Z\big) &= \varphi \cdot (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
&= \varphi \cdot \nabla_X \nabla_Y Z - \varphi \cdot \nabla_Y \nabla_X Z - \varphi \cdot \nabla_{[X,Y]} Z \\
&= \nabla'_{\varphi \cdot X} (\varphi \cdot \nabla_Y Z) - \nabla'_{\varphi \cdot Y} (\varphi \cdot \nabla_X Z) - \nabla'_{\varphi \cdot [X,Y]} \varphi \cdot Z \\
&= \nabla'_{\varphi \cdot X} \nabla'_{\varphi \cdot Y} \varphi \cdot Z - \nabla'_{\varphi \cdot Y} \nabla'_{\varphi \cdot X} \varphi \cdot Z - \nabla'_{[\varphi \cdot X, \varphi \cdot Y]} \varphi \cdot Z \\
&= R'(\varphi \cdot X, \varphi \cdot Y)(\varphi \cdot Z).
\end{aligned}$$

(ii)

$$\begin{aligned}
\varphi \cdot \big(T(X,Y)\big) &= \varphi \cdot \big(\nabla_X Y - \nabla_Y X - [X,Y]\big) \\
&= \varphi \cdot \nabla_X Y - \varphi \cdot \nabla_Y X - \varphi \cdot [X,Y] \\
&= \nabla'_{\varphi \cdot X} \varphi \cdot Y - \nabla'_{\varphi \cdot Y} \varphi \cdot X - [\varphi \cdot X, \varphi \cdot Y] = T'[\varphi \cdot X, \varphi \cdot Y].
\end{aligned}$$

Problem 5.44 If ω is a differential r-form on a C^∞ manifold M equipped with a torsionless linear connection ∇, prove that

$$\mathrm{d}\omega(X_0, \dots, X_r) = \sum_{i=0}^{r} (-1)^{\mathrm{i}} (\nabla_{X_i} \omega)(X_0, \dots, \widehat{X}_i, \dots, X_r),$$

$X_0, \dots, X_r \in \mathfrak{X}(M)$, where the hat symbol denotes that the corresponding vector field is dropped.

Hint If ω is a differential r-form, the formula relating the bracket product and the exterior differential is

$$
\begin{aligned}
(\mathrm{d}\omega)(X_0,\dots,X_r) = {} & \sum_{i=0}^{r}(-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big) \\
& + \sum_{i<j}(-1)^{i+j}\omega\big([X_i,X_j],X_0,\dots,\hat{X}_j,\dots,\hat{X}_i,\dots,X_r\big).
\end{aligned}
$$

Remark The more used cases are those of differential 1-forms and 2-forms:

$$
\begin{aligned}
\mathrm{d}\omega(X,Y) &= (\nabla_X\omega)Y - (\nabla_Y\omega)X, \quad \omega\in\Lambda^1 M, \\
\mathrm{d}\omega(X,Y,Z) &= (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y), \quad \omega\in\Lambda^2 M.
\end{aligned}
$$

Solution

$$
\begin{aligned}
& \sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\omega)(X_0,\dots,\widehat{X}_i,\dots,X_r) \\
&\quad = \sum_{i=0}^{r}(-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big) \\
&\qquad - \sum_{j<i}(-1)^i\omega(X_0,\dots,\nabla_{X_i}X_j,\dots,\widehat{X}_i,\dots,X_r) \\
&\qquad - \sum_{j>i}(-1)^i\omega(X_0,\dots,\widehat{X}_i,\dots,\nabla_{X_i}X_j,\dots,X_r) \\
&\quad = \sum_{i=0}^{r}(-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big) \\
&\qquad - \sum_{j<i}(-1)^{i+j}\omega(\nabla_{X_i}X_j,X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_r) \\
&\qquad + \sum_{j>i}(-1)^{i+j}\omega(\nabla_{X_i}X_j,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r) \\
&\quad = \sum_{i=0}^{r}(-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big) \\
&\qquad + \sum_{i<j}(-1)^{i+j}\omega(\nabla_{X_i}X_j-\nabla_{X_j}X_i,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r) \\
&\quad = \mathrm{d}\omega(X_0,\dots,X_r).
\end{aligned}
$$

Problem 5.45

(i) Prove that if ∇ is a flat connection on a connected manifold M whose parallel transport is globally independent of curves, then there exists a C^∞ global field of frames on M.
(ii) Prove that if ∇ is a flat connection on a connected manifold M, then its curvature tensor field vanishes.

Solution

(i) Let us fix a point $p_0 \in M$ and a basis $\{v_1, \dots, v_n\}$ of $T_{p_0}M$. Given an arbitrary point $p \in M$, there exists a differentiable arc $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p_0, \gamma(1) = p$. We define $X_i|_p = \tau_\gamma(v_i)$, $i = 1, \dots, n$, where $\tau_\gamma\colon T_{p_0}M \to T_pM$ is the parallel transport along γ. The definition makes sense by virtue of the hypothesis and $(X_1, \dots, X_n)$ is a frame as τ_γ is an isomorphism.
(ii) According to the definition of a flat connection, given a point $p \in M$, there exist an open neighbourhood U of p such that the parallel transport in U is independent of curves. Hence from (i) it follows that U admits a linear frame $(X_1, \dots, X_n)$ invariant under parallel transport, that is, $\nabla_X X_i = 0$, for all $X \in \mathfrak{X}(U)$. Then $R(X_i, X_j)X_k = 0$ and hence $R = 0$.

Problem 5.46 Find the (equivalent) expression of Cartan's equation of structure $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, that is, of $\Omega^i_j = \mathrm{d}\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j$, when one considers transpose matrices, i.e. when the upper index denotes the column and the lower index denotes the row of the corresponding matrix.

Solution When one considers the transpose matrices of ω and Ω, it is immediate to see that

$$\Omega^i_j = \mathrm{d}\omega^i_j + \sum_k \omega^k_j \wedge \omega^i_k = \mathrm{d}\omega^i_j - \sum_k \omega^i_k \wedge \omega^k_j,$$

that is, $\Omega = \mathrm{d}\omega - \omega \wedge \omega$.

Remark Some authors prefer to use this expression of Cartan's equation of structure.

Problem 5.47 Find the holonomy group of:

(i) The Euclidean space $\mathbb{R}^n$.
(ii) The sphere S^2 with its usual connection.

Moreover, prove:

(iii) The holonomy group, at any point, of a connection in the principal bundle

$$\left(S^{2n+1}, \pi, \mathbb{C}\mathrm{P}^n, S^1\right)$$

is S^1.

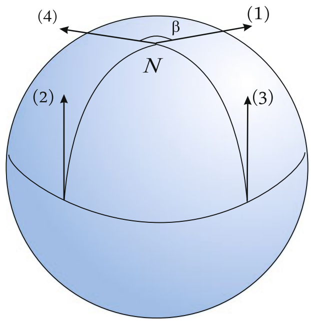

Fig. 5.2 An element β of the holonomy group of S^2

(iv) The holonomy group of a connection in the principal fibre bundle

$$\left(S^{4n+3}, \pi, \mathbb{H}\mathrm{P}^n, S^3\right)$$

is either S^1 or S^3.

The relevant theory is developed, for instance, in Bishop and Crittenden [1].

Solution

(i) Let ∇ be the usual flat connection, then $\mathrm{Hol}(\nabla) = \{0\}$, as the parallel transport along any closed curve is the identity map.

(ii) Let ∇ be the usual connection. As S^2 is orientable, the holonomy group $\mathrm{Hol}(\nabla)$ is a subgroup of $\mathrm{SO}(2)$.

We shall see geometrically that $\mathrm{Hol}(\nabla) = \mathrm{SO}(2)$. Consider, with no loss of generality, any orthonormal basis $\{e_1, e_2\}$ of $T_N S^2$, N being the north pole $(0, 0, 1)$, and do its parallel transport along the piecewise C^∞ curve in S^2, given (see Fig. 5.2 for a certain vector tangent starting as (i) at the north pole) by the half-meridian determined by e_1 until the equator, then the curve along the equator by a rotation of angle β of the equatorial plane, and then the half-meridian of return to N. The net result of the transport is a rotation of angle β. As β can take any value $\beta \in [0, 2\pi]$, in fact $\mathrm{Hol}(\nabla) = \mathrm{SO}(2)$.

(iii) Let Γ be a connection in $(S^{2n+1}, \pi, \mathbb{C}\mathrm{P}^n, S^1)$. Since $\mathbb{C}\mathrm{P}^n$ is simply connected, the holonomy group $G = \mathrm{Hol}(\Gamma)$ at a point $u \in S^{2n+1}$ coincides with the corresponding restricted holonomy group $\mathrm{Hol}^0(\Gamma)$. Hence either $G = S^1$ or $G = \{1\}$. In the latter case, $\pi\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ should admit a G-reduction $\pi\colon P \to \mathbb{C}\mathrm{P}^n$, which should be trivial as $\mathbb{C}\mathrm{P}^n$ is simply connected and the reduction P is a covering. Hence P admits a global section $\sigma\colon \mathbb{C}\mathrm{P}^n \to P$ which induces a section of π, as $P \subset S^{2n+1}$. Consequently, the bundle $\pi\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ should be trivial, that is, $S^{2n+1} \cong \mathbb{C}\mathrm{P}^n \times S^1$. This leads to a contradiction as $H^2(S^{2n+1}, \mathbb{Z}) = 0$ while, by Künneth's Theorem, $H^2(\mathbb{C}\mathrm{P}^n \times S^1, \mathbb{Z}) = \mathbb{Z}$.

(iv) As in the previous case (iii), $G = \mathrm{Hol}(\Gamma) = \mathrm{Hol}^0(\Gamma)$, since the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ is simply connected. Hence G cannot be discrete because in this case, since $H^4(S^{4n+3}, \mathbb{Z}) = 0$, an argument similar to the one above applies. If $\dim G = 1$, then $G = S^1$. The case $\dim G = 2$ cannot occur by virtue of Problem 4.46, and $\dim G = 3$ implies $G = S^3$.

5.7 Geodesics

Problem 5.48 Let $x^1 = x$, $x^2 = y$ be the usual coordinates on $\mathbb{R}^2$. Define a linear connection ∇ of $\mathbb{R}^2$ by $\Gamma^i_{jk} = 0$ except $\Gamma^1_{12} = \Gamma^1_{21} = 1$.

(i) Write and solve the differential equations of the geodesics.
(ii) Is ∇ complete?
(iii) Find the particular geodesic σ with

$$\sigma(0) = (2, 1), \qquad \sigma'(0) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

(iv) Do the geodesics emanating from the origin go through all the points of the plane?
(v) If σ and $\widetilde{\sigma}$ are geodesics with $\sigma(0) = \widetilde{\sigma}(0)$ and $\sigma'(0) = k\widetilde{\sigma}'(0)$, $k \in \mathbb{R}$, prove that $\sigma(t) = \widetilde{\sigma}(kt)$ for all possible t.

Solution

(i) The differential equations are

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\frac{dy}{dt} = 0, \qquad \frac{d^2y}{dt^2} = 0.$$

Now we obtain the equations of the geodesics through a given point (x_0, y_0), that is, such that $\sigma(0) = (x_0, y_0)$.

From the second equation we have $y = At + y_0$.

Let $A = 0$. Then the solutions are

$$x = Bt + x_0, \qquad y = y_0. \tag{$\star$}$$

Let $A \neq 0$. Then from $\frac{d^2x}{dt^2} + 2A\frac{dx}{dt} = 0$, that is, $\frac{d}{dt}(\frac{dx}{dt})/\frac{dx}{dt} = -2A$, one has $\log \frac{dx}{dt} = -2At + C$, so that $\frac{dx}{dt} = De^{-2At}$, $D \neq 0$. Therefore, the equations are

$$x = \frac{D}{2A}\left(1 - e^{-2At}\right) + x_0, \qquad y = At + y_0, \qquad D \neq 0. \tag{$\star\star$}$$

(ii) From equations ($\star$) and ($\star\star$) in (i), we see that ∇ is complete because the geodesics are defined for $t \in (-\infty, +\infty)$.
(iii) Since $dy/dt = 1$, the geodesic is of the type $A \neq 0$, and one has

$$x_0 = 2, \qquad y_0 = 1, \qquad x'(0) = D = 1, \qquad y'(0) = A = 1,$$

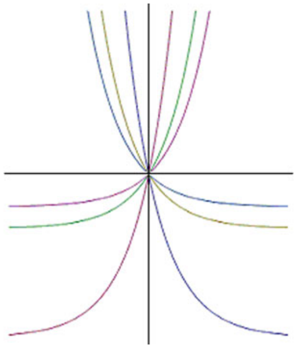

Fig. 5.3 The points $(0, y)$, $y \neq 0$, are never reached from $(0, 0)$

hence

$$x = -\frac{1}{2}e^{-2t} + \frac{5}{2}, \qquad y = t + 1.$$

(iv) Suppose $A = 0$. Then such a geodesic is of the type $x = Bt$, $y = 0$. For $A \neq 0$ one has

$$x = \frac{D}{2A}\big(1 - e^{-2At}\big), \qquad y = At.$$

That is,

$$x = \frac{D}{2A}\big(1 - e^{-2y}\big)$$

is the family of geodesics with $A \neq 0$ emanating from the origin. The points $(0, y)$, $y \neq 0$, are never reached from $(0, 0)$ (see Fig. 5.3).

In fact, if $x = 0$, since $D/2A \neq 0$ we have $e^{-2y} = 1$, thus $y = 0$. Obviously, those points are not reached either from $(0, 0)$ with a geodesic such that $A = 0$.

(v) Suppose $A = 0$. Then from $\sigma(0) = \widetilde{\sigma}(0)$ it follows that

$$(x_0)_\sigma = (x_0)_{\widetilde{\sigma}}, \qquad (y_0)_\sigma = (y_0)_{\widetilde{\sigma}},$$

and from $\sigma'(0) = k\widetilde{\sigma}'(0)$ we deduce $B_\sigma = kB_{\widetilde{\sigma}}$. Hence

$$\sigma(t) = \big(B_\sigma t + (x_0)_\sigma, (y_0)_\sigma\big) = \big(kB_{\widetilde{\sigma}} t + (x_0)_{\widetilde{\sigma}}, (y_0)_{\widetilde{\sigma}}\big) = \widetilde{\sigma}(kt).$$

Suppose now $A \neq 0$. Then from $\sigma(0) = \widetilde{\sigma}(0)$ it follows that

$$(x_0)_\sigma = (x_0)_{\widetilde{\sigma}}, \qquad (y_0)_\sigma = (y_0)_{\widetilde{\sigma}},$$

and from $\sigma'(0) = k\widetilde{\sigma}'(0)$ we deduce $A_\sigma = kA_{\widetilde{\sigma}}$, $D_\sigma = kD_{\widetilde{\sigma}}$. Thus

$$\begin{aligned}\sigma(t) &= \left(\frac{D_\sigma}{2A_\sigma}\big(1 - e^{-2A_\sigma t}\big) + (x_0)_\sigma,\ A_\sigma + (y_0)_\sigma\right) \\ &= \left(\frac{D_{\widetilde{\sigma}}}{2A_{\widetilde{\sigma}}}\big(1 - e^{-2kA_{\widetilde{\sigma}} t}\big) + (x_0)_{\widetilde{\sigma}},\ kA_{\widetilde{\sigma}} t + (y_0)_{\widetilde{\sigma}}\right) = \widetilde{\sigma}(kt).\end{aligned}$$

Problem 5.49 Consider the linear connection ∇ on $\mathbb{R}^2 = \{(x^1, x^2)\}$ with components $\Gamma^k_{ij} = 0$ except $\Gamma^1_{12} = 1$, and the curve

$$\sigma(t) = \bigl(\sigma^1(t), \sigma^2(t)\bigr) = \bigl(-2\mathrm{e}^{-t} + 4, t + 1\bigr).$$

Compute the vector field obtained by parallel transport along σ of its tangent vector at $\sigma(0)$. Is γ a geodesic curve?

Solution The tangent vector to σ at $\sigma(t)$ is

$$\sigma'(t) = 2\mathrm{e}^{-t} \frac{\partial}{\partial x^1}\bigg|_{\sigma(t)} + \frac{\partial}{\partial x^2}\bigg|_{\sigma(t)}.$$

Let

$$Y_{\sigma(t)} = Y^1(t)\frac{\partial}{\partial x^1}\bigg|_{\sigma(t)} + Y^2(t)\frac{\partial}{\partial x^2}\bigg|_{\sigma(t)}$$

be the requested vector field. The parallelism conditions are

$$\frac{\mathrm{d}\sigma^i(t)}{\mathrm{d}t} + \sum_{j,k} \Gamma^i_{jk} \frac{\mathrm{d}\sigma^j(t)}{\mathrm{d}t} Y^k = 0,$$

that is,

$$\frac{\mathrm{d}Y^1(t)}{\mathrm{d}t} + 2\mathrm{e}^{-t}Y^2(t) = 0, \qquad \frac{dY^2(t)}{\mathrm{d}t} = 0.$$

One easily obtains that $Y^1(t) = 2A\mathrm{e}^{-t} + B$. For $t = 0$, the vector $Y_{\sigma(0)}$ is, by hypothesis, $\sigma'(0)$. Thus $A = 1$, $B = 0$, and

$$Y = 2\mathrm{e}^{-t} \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}.$$

Hence the curve is a geodesic.

Problem 5.50 Let M be a C^∞ manifold with two linear connections ∇ and $\widetilde{\nabla}$ with Christoffel symbols Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$, respectively, such that $\Gamma^i_{jk} + \Gamma^i_{kj} = \widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^i_{kj}$.

(i) Have ∇ and $\widetilde{\nabla}$ the same geodesics?
(ii) What intrinsic meaning does the previous condition have?

Hint (to (ii)) Use the difference tensor of ∇ and $\widetilde{\nabla}$.

Solution

(i) The geodesics $\gamma(t)=(x^1(t),\dots,x^n(t))$ for ∇ and $\widetilde{\nabla}$ are given by the systems of differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=0 \quad \text{and}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+\sum_{j,k}\widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=0, \quad i=1,\dots,n,$$

respectively. We have $\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=\sum_{j,k}\Gamma^i_{kj}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}$, from which

$$\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=\frac{1}{2}\sum_{j,k}\left(\Gamma^i_{jk}+\Gamma^i_{kj}\right)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}.$$

If $\Gamma^i_{jk}+\Gamma^i_{kj}=\widetilde{\Gamma}^i_{jk}+\widetilde{\Gamma}^i_{kj}$, then

$$\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=\frac{1}{2}\sum_{j,k}\left(\Gamma^i_{jk}+\Gamma^i_{kj}\right)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=\frac{1}{2}\sum_{j,k}\left(\widetilde{\Gamma}^i_{jk}+\widetilde{\Gamma}^i_{kj}\right)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}$$
$$=\sum_{j,k}\widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t},$$

thus ∇ and $\widetilde{\nabla}$ have the same geodesics.

(ii) It is immediate that the previous condition means that the difference tensor $A=\nabla-\widetilde{\nabla}$ is skew-symmetric.

Problem 5.51 Let x^1, x^2 be the usual coordinates on $\mathbb{R}^2$. Consider the linear connection ∇ of $\mathbb{R}^2$ with components $\Gamma^k_{ij}=0$ except $\Gamma^2_{22}=2$, and the curve $\sigma(t)=(\mathrm{e}^{-4t}+5, 3t+7)$.

(i) Compute the vector field $Y_{\sigma(t)}$ obtained by parallel transport along σ of its tangent vector at $\sigma(2)$.
(ii) Is σ a geodesic curve?

Result

(i) $Y_{\sigma(t)}=-4\mathrm{e}^{-8}\frac{\partial}{\partial x^1}+3\mathrm{e}^{12-6t}\frac{\partial}{\partial x^2}$.
(ii) No.

5.8 Almost Complex Manifolds

Problem 5.52 An *almost complex structure* on a C^∞ manifold M is a differentiable map $J\colon TM\to TM$, such that:

(a) J maps linearly T_pM into T_pM for all $p \in M$.
(b) $J^2 = -I$ on each T_pM, where I stands for the identity map.

Prove:

(i) If M admits an almost complex structure (it is said that M is an *almost complex manifold*), then M has even real dimension $2n$.
(ii) M admits an almost complex structure if and only if the structure group of the bundle of linear frames FM can be reduced to the *real representation of the general linear group* $\mathrm{GL}(n, \mathbb{C})$, given by

$$\begin{aligned} \rho\colon \mathrm{GL}(n, \mathbb{C}) &\to \mathrm{GL}(2n, \mathbb{R}) \\ A + \mathrm{i}B &\mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \end{aligned}$$

Hint Let f be the linear transformation of $\mathbb{R}^{2n}$ with matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Prove that the subset

$$P = \left\{z \in FM : f(\xi) = \left(z^{-1} \circ J \circ z\right)(\xi),\ \xi \in \mathbb{R}^{2n}\right\}$$

of the bundle of linear frames FM over M is a $\mathrm{GL}(n, \mathbb{C})$-structure on M. The reference $z \in FM$ is viewed as an isomorphism $z\colon \mathbb{R}^{2n} \to T_{\pi(z)}M$.

Solution

(i) T_pM admits a structure of complex vector space defining a product by complex numbers by

$$(a + \mathrm{i}b)X = aX + bJ_pX, \quad X \in T_pM,\ a, b \in \mathbb{R}.$$

Thus the real dimension of T_pM is even, and so it is for M.
(ii) We have

$$\rho\big(\mathrm{GL}(n, \mathbb{C})\big) = \left\{\Lambda \in \mathrm{GL}(2n, \mathbb{R}) : \Lambda f = f\Lambda\right\}.$$

In fact, decomposing Λ in $n \times n$ blocks,

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and by imposing $\Lambda f = f\Lambda$, we obtain $C = -B$, $D = A$. And conversely.

An almost complex structure J on M is a $(1, 1)$ tensor field on M such that $J^2 = -I$. The subset P of the bundle of linear frames over M, described in the hint above, determines a $\mathrm{GL}(n, \mathbb{C})$-structure. In fact, a linear frame z at $p \in M$ is an isomorphism $z\colon \mathbb{R}^{2n} \to T_pM$ (see Fig. 5.4). Let us see that the acting group is $\mathrm{GL}(n, \mathbb{C})$. In fact, given $z, z' \in P$, with $\pi(z) = \pi(z')$, we have

$$f = z^{-1} \circ J \circ z \quad \Leftrightarrow \quad z \circ f \circ z^{-1} = J.$$

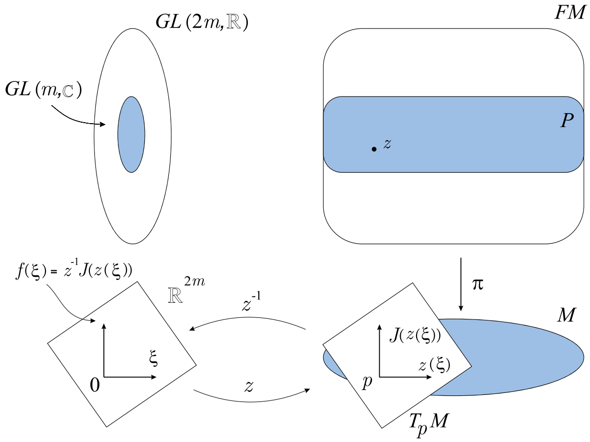

Fig. 5.4 Linear frames adapted to an almost complex structure

Then

$$z^{-1} \circ z' \circ f \circ z'^{-1} \circ z = z^{-1} \circ J \circ z = f.$$

that is, $z^{-1} \circ z' \in \mathrm{GL}(n, \mathbb{C})$.

Conversely, given a $\mathrm{GL}(n, \mathbb{C})$-structure P on M, we consider the operator J_p in T_pM such that

$$J_p X = \left(z^{-1} \circ J \circ z\right)(X), \quad X \in T_pM, \; z \in \pi^{-1}(p) \subset P.$$

By the definition of frame as an isomorphism of $\mathbb{R}^{2n}$ on T_pM, J_pX is an element of T_pM. J_pX does not depend, by the definition of $\mathrm{GL}(n, \mathbb{C})$, on the element $z \in \pi^{-1}(p)$. In fact, if $z, z' \in \pi^{-1}(p)$, then there exists $g \in \mathrm{GL}(n, \mathbb{C})$ such that $z' = zg$, and then

$$J'_p X = \left(z' \circ f \circ z'^{-1}\right)(X) = \left(z \circ g \circ f \circ g^{-1} \circ z^{-1}\right)(X) = \left(z \circ f \circ z^{-1}\right)(X).$$

Moreover, $J^2 = z \circ f \circ z^{-1} \circ z \circ f \circ z^{-1} = -I$.

Problem 5.53

(i) Does the sphere S^2 admit a structure of complex manifold?
(ii) And what about the sphere S^3?

Hint Use the stereographic projections onto the equatorial plane, and identify this one with the complex plane $\mathbb{C}$.

Solution

(i) Let φ_1 be the stereographic projection onto the plane $z = 0$ from the north pole $N = (0, 0, 1) \in S^2$, and φ_2 the stereographic projection onto the plane $z = 0$

from the south pole $S = (0, 0, -1) \in S^2$. We have (see Problem 1.28)

$$\varphi_1(a, b, c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right), \qquad \varphi_2(a, b, c) = \left(\frac{a}{1+c}, \frac{b}{1+c}\right),$$

so the changes of coordinates are

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1} : \mathbb{R}^2 \setminus \{(0, 0)\} &\to \mathbb{R}^2 \setminus \{(0, 0)\} \\ (a, b) &\mapsto \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right). \end{aligned}$$

Identifying the plane $z = 0$ with $\mathbb{C}$, we can write

$$\varphi_1(a, b, c) = \frac{a}{1-c} + \mathrm{i}\frac{b}{1-c}, \qquad \varphi_2(a, b, c) = \frac{a}{1+c} + \mathrm{i}\frac{b}{1+c},$$

so

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} &\to \mathbb{C} \setminus \{0\} \\ z = x + \mathrm{i}y &\mapsto \frac{x}{x^2 + y^2} + \mathrm{i}\frac{y}{x^2 + y^2}. \end{aligned}$$

To see that the changes of coordinates are holomorphic, we have to show that they satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

where $u(x, y) = \frac{x}{x^2+y^2}$, $v(x, y) = \frac{y}{x^2+y^2}$. A computation shows that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

that is, the change of coordinates is anti-holomorphic, instead of holomorphic. This could be expected from the fact that $\varphi_2 \circ \varphi_1^{-1}$ changes the orientation (see Fig. 5.5).

In order for the equations of Cauchy–Riemann to be satisfied, we have to change the sign of one of the (real or imaginary) components of the change of coordinates. Consider, instead of φ_2, the new chart $\psi_2 = \bar{\varphi}_2$ given by

$$\psi_2(a, b, c) = \frac{a}{1+c} - \mathrm{i}\frac{b}{1+c}.$$

The map ψ_2 is a homeomorphism of $S^2 \setminus \{N\}$ on $\mathbb{C} \setminus \{0\}$, as it is the composition map

$$S^2 \setminus \{N\} \quad \overset{\varphi_2}{\to} \quad \mathbb{C} \setminus \{0\} \quad \overset{j}{\to} \quad \mathbb{C} \setminus \{0\},$$

where j denotes the conjugation map. The new changes of coordinates are

$$\left(\psi_2 \circ \psi_1^{-1}\right)(x + \mathrm{i}y) = \frac{x}{x^2 + y^2} - \mathrm{i}\frac{y}{x^2 + y^2},$$

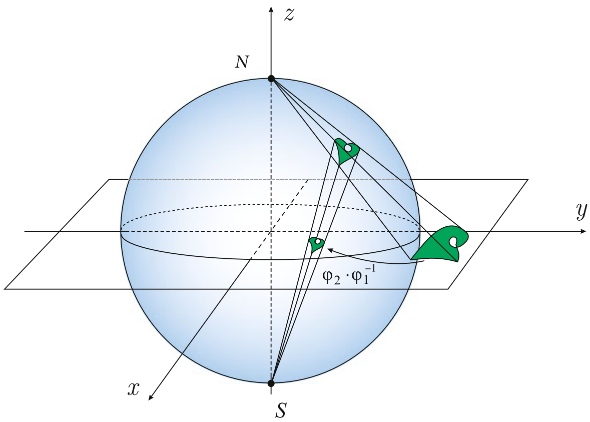

Fig. 5.5 The map $\varphi_2 \circ \varphi_1^{-1}$ changes the orientation

and it is immediate that they satisfy the Cauchy–Riemann equations.

(ii) S^3 does not admit any complex structure because a complex manifold necessarily has even real dimension (see Problem 5.52).

Problem 5.54 Consider the torus $T^2 = S^1 \times S^1$ and let (x, y) be the canonical coordinates $(0 < x < 2\pi,\ 0 < y < 2\pi)$ on T^2. The corresponding coordinate fields define global fields denoted by $\partial/\partial x$, $\partial/\partial y$. Let J be the almost complex structure on T^2 given by

$$J\frac{\partial}{\partial x} = -\left(1+\cos^2 x\right)\frac{\partial}{\partial y}, \qquad J\frac{\partial}{\partial y} = \frac{1}{1+\cos^2 x}\frac{\partial}{\partial x}.$$

(i) Show that J is integrable.

(ii) Find the corresponding chart of complex manifold.

Solution

(i) A necessary and sufficient condition for a complex structure J to be integrable is that its Nijenhuis tensor N_J be identically zero. Since N_J is skew-symmetric in the covariant indices, we only have to show that $N_J(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ vanishes. Substituting, we have

$$N_J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x}\left(1+\cos^2 x\right)\right)\frac{1}{1+\cos^2 x}\frac{\partial}{\partial y} + \left(\frac{\partial}{\partial x}\frac{1}{1+\cos^2 x}\right)\left(1+\cos^2 x\right)\frac{\partial}{\partial y} = 0.$$

(ii) We must find coordinates u, v such that

$$J\frac{\partial}{\partial u} = \frac{\partial}{\partial v}, \qquad J\frac{\partial}{\partial v} = -\frac{\partial}{\partial u}.$$

We must have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}, \tag{$\star$}$$

and applying J in ($\star$),

$$\begin{cases} -(1+\cos^2 x)\dfrac{\partial}{\partial y} = \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial v} - \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial u}, \\ \dfrac{1}{1+\cos^2 x}\dfrac{\partial}{\partial x} = \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial v} - \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial u}. \end{cases} \tag{$\star\star$}$$

From ($\star$) and ($\star\star$) it follows that

$$\begin{cases} \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial v} = (1+\cos^2 x)\dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial v} - (1+\cos^2 x)\dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial u}, \\ \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial v} = -\dfrac{1}{1+\cos^2 x}\dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial v} + \dfrac{1}{1+\cos^2 x}\dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial u}. \end{cases} \tag{$\diamond$}$$

Both equations in ($\diamond$) imply

$$\frac{\partial u}{\partial x} = -\big(1+\cos^2 x\big)\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{1}{1+\cos^2 x}\frac{\partial v}{\partial x}. \tag{$\diamond\diamond$}$$

It suffices to give a particular solution of ($\diamond\diamond$). From the equations

$$v = y, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\big(1+\cos^2 x\big),$$

one has a solution of ($\diamond\diamond$), given by

$$u = A - \frac{3}{2}x - \frac{1}{2}\sin x\cos x, \qquad v = y.$$

Problem 5.55 Let $\pi : M \to N$ be a topological covering.

(i) Prove that if N is a complex manifold, then M also is a complex manifold. Equivalently, M has a unique structure of complex manifold such that π is a local diffeomorphism.
(ii) If M is a complex manifold, is necessarily N another one?

Solution Let $p \in M$. We define the chart (U_p, Φ_p) around p in the following way: Let $x = \pi(p)$ and let U_x be a neighbourhood of x such that U_x is the domain of a chart (U_x, φ_x) and $\pi : U \to \pi(U)$ is a homeomorphism. Then we define

$$\Phi_p = \varphi_x \circ (\pi|_{U_p}),$$

where U_p denotes the neighbourhood of p homeomorphic to U_x by π. Thus we define an atlas on M, and we have to prove that the changes of coordinates in M

are holomorphic. Notice that $\dim M = \dim N$. Let $p, q \in M$ such that $U_p \cap U_q \neq \emptyset$, and let $x = \pi(p)$, $y = \pi(q)$. Then we have to prove that the map

$$\Phi_q \circ \Phi_p^{-1} \colon \Phi_p(U_p \cap U_q) \to \Phi_q(U_p \cap U_q)$$

is holomorphic. But

$$\begin{aligned}\Phi_q \circ \Phi_p^{-1} &= \varphi_y \circ (\pi|_{U_p \cap U_q}) \circ \big(\varphi_x \circ (\pi|_{U_p \cap U_q})\big)^{-1} \\ &= \varphi_y \circ (\pi|_{U_p \cap U_q}) \circ (\pi|_{U_p \cap U_q})^{-1} \circ \varphi_x^{-1} = \varphi_y \circ \varphi_x^{-1},\end{aligned}$$

which is holomorphic because N is a complex manifold.

(ii) It is not true, in general. In fact, the map $\pi : S^2 \to \mathbb{R}\mathrm{P}^2$ is a double covering. S^2 is a complex manifold, as we have seen in Problem 5.53, but $\mathbb{R}\mathrm{P}^2$ is not because it is not orientable (see Problem 3.10), and every complex manifold is orientable.

Problem 5.56 Let (M, J) be an almost complex manifold. If ∇ is a linear connection on M whose torsion tensor T_∇ vanishes, define the linear connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{4}\big((\nabla_{JY} J)X + J(\nabla_Y J)X + 2J(\nabla_X J)Y\big).$$

(i) Prove that $J\nabla_X J = -(\nabla_X J)J$.
(ii) Compute the torsion tensor $T_{\widetilde{\nabla}}$ in terms of the Nijenhuis tensor of J.

Solution

(i)

$$(\nabla_X J)J + J\nabla_X J = \nabla_X J^2 = \nabla_X(-I) = 0.$$

(ii)

$$\begin{aligned}T_{\widetilde{\nabla}}(X, Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] \\ &= T_\nabla(X, Y) - \frac{1}{4}\big((\nabla_{JY} J)X - (\nabla_{JX} J)Y - J(\nabla_Y J)X + J(\nabla_X J)Y\big) \\ &= -\frac{1}{4}\big((\nabla_{JY} J)X - (\nabla_{JX} J)Y + (\nabla_Y J)JX - (\nabla_X J)JY\big) \\ &= -\frac{1}{4}(\nabla_{JY} JX - J\nabla_{JY} X - \nabla_{JX} JY + J\nabla_{JX} Y \\ &\quad - \nabla_Y X - J\nabla_Y JX + \nabla_X Y + J\nabla_X JY) \\ &= -\frac{1}{4}\big([JY, JX] + [X, Y] + J[JX, Y] + J[X, JY]\big) = \frac{1}{4}N(X, Y),\end{aligned}$$

where N denotes the Nijenhuis tensor of J and we have applied (i) above in the third equality.

Problem 5.57 Let (M, J) be an almost complex manifold. Prove that the torsion tensor T and the curvature operator $R(X, Y)$ of an *almost complex linear connection* ∇ (that is, a linear connection such that $(\nabla_X J)Y = 0,\ X, Y \in \mathfrak{X}(M)$), satisfy the following identities:

(i)

$$T(JX, JY) - JT(JX, Y) - JT(X, JY) - T(X, Y) = -N(X, Y),$$

where N denotes the Nijenhuis tensor of J.

(ii) $R(X, Y) \circ J = J \circ R(X, Y)$.

Solution

(i)

$$\begin{aligned} &T(JX, JY) - JT(JX, Y) - JT(X, JY) - T(X, Y) \\ &\quad = J\nabla_{JX}Y - J\nabla_{JY}X - [JX, JY] - J\nabla_{JX}Y - \nabla_Y X + J[JX, Y] \\ &\qquad + \nabla_X Y + J\nabla_{JY}X + J[X, JY] - \nabla_X Y + \nabla_Y X + [X, Y] = -N(X, Y). \end{aligned}$$

(ii)

$$R(X, Y)JZ = J\nabla_X\nabla_Y Z - J\nabla_Y\nabla_X Z - J\nabla_{[X,Y]}Z = JR(X, Y)Z.$$

Problem 5.58 Let M be a complex manifold of complex dimension n. Let $\{z^k\}$, $k = 1, \dots, n$, be a system of complex coordinates around a given $p \in M$. If $z^k = x^k + \mathrm{i}y^k$, let $\{x^k, y^k\}$ be the corresponding system of real coordinates around p. Let T_pM, T_p^hM, and $T_p^{1,0}M$ be the *real tangent space at* p, the *holomorphic tangent space at* p, and the *space of vectors of type* $(1, 0)$ *at* p, respectively (see Definitions 5.11). Prove that there exist unique $\mathbb{C}$-linear isomorphisms

$$\Phi_p \colon T_pM \to T_p^hM, \qquad \Psi_p \colon T_p^hM \to T_x^{1,0}M,$$

with respect to the natural complex structure of each of these spaces given in Definitions 5.11, such that for every system $\{z^k\}$, $k = 1, \dots, n$, we have

$$\begin{aligned} \Phi_p\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) &= \left.\frac{\partial}{\partial z^k}\right|_p, \quad k = 1, \dots, n, \\ \Psi_p\left(\left.\frac{\partial}{\partial z^k}\right|_p\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right)_p, \quad k = 1, \dots, n. \end{aligned} \tag{$\star$}$$

Solution *Uniqueness*. An $\mathbb{R}$-basis of T_pM is $\{\partial/\partial x^k|_p, \partial/\partial y^k|_p\}$. As we have

$$\mathrm{i}\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = J\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = \left.\frac{\partial}{\partial y^k}\right|_p,$$

it is clear that $\{\partial/\partial x^k|_p\}$ is a $\mathbb{C}$-basis of T_pM. Hence Φ_p is unique. Also Ψ_p is unique as $\{\partial/\partial z^k|_p\}$ is a $\mathbb{C}$-basis of T_p^hM and

$$\left\{\frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right)_p\right\}$$

is a $\mathbb{C}$-basis of $T_p^{1,0}M$ (see Definitions 5.11).

Existence. Each $X \in T_pM$ is an $\mathbb{R}$-derivation $X\colon C_p^\infty M \to \mathbb{R}$. Tensoring with $\mathbb{C}$, we obtain a $\mathbb{C}$-derivation

$$X \otimes 1\colon C_p^\infty M \otimes \mathbb{C} \to \mathbb{C}.$$

As $\mathscr{O}_pM \subset C_p^\infty M \otimes \mathbb{C}$, restricting $X \otimes 1$ to $\mathscr{O}_pM$, we obtain

$$\widetilde{X} := (X \otimes 1)|_{\mathscr{O}_pM} \in T_p^hM.$$

We define $\Phi_p\colon T_pM \to T_p^hM$ by $\Phi(X) = \widetilde{X}$. From the very definition of Φ_p, we have

$$\Phi_p(X+Y) = \Phi_p(X) + \Phi_p(Y), \qquad \Phi_p(\lambda X) = \lambda\Phi_p(X),$$

for all $X, Y \in T_pM$, $\lambda \in \mathbb{R}$. Moreover, we have

$$\begin{aligned}(JX \otimes 1)z^k &= JXx^k + \mathrm{i}JXy^k = \left(\mathrm{d}x^k\right)_p JX + \mathrm{i}\left(\mathrm{d}y^k\right)_p JX \\ &= J^*\left(\left(\mathrm{d}x^k\right)_p\right)X + \mathrm{i}J^*\left(\left(\mathrm{d}y^k\right)_p\right)X = -\left(\mathrm{d}y^k\right)_p X + \mathrm{i}\left(\mathrm{d}x^k\right)_p X \\ &= \mathrm{i}\left(\mathrm{d}z^k\right)_p X = \mathrm{i}\left((X \otimes 1)z^k\right).\end{aligned}$$

Hence, Φ_p is $\mathbb{C}$-linear.

Let us compute $\Phi_p(\partial/\partial x^k|_p)$. From the definition, we obtain

$$\mathrm{d}z^\alpha|_p\left(\frac{\partial}{\partial x^k}\bigg|_p\right)^{\sim} = \left(\frac{\partial}{\partial x^k}\bigg|_p\right)^{\sim}(z^\alpha) = \left(\left(\frac{\partial}{\partial x^k}\bigg|_p\right) \otimes 1\right)(z^\alpha) = \delta_\alpha^k.$$

Hence

$$\left(\frac{\partial}{\partial x^k}\bigg|_p\right)^{\sim} = \delta_\alpha^k \frac{\partial}{\partial z^\alpha}\bigg|_p = \frac{\partial}{\partial z^k}\bigg|_p.$$

Let $\Theta_p\colon T_p^{1,0}M \to T_p^hM$ be the map given by

$$\Theta_p(Z) = \widetilde{Z} = Z|_{\mathscr{O}_p(M)}, \quad Z \in T_p^{1,0}M.$$

From its very definition, it follows that Θ_p is $\mathbb{C}$-linear. Let us compute its expression in the standard basis (cf. Definitions 5.11). We have

$$\begin{aligned}\mathrm{d}z^\alpha\big|_p\left(\frac{1}{2}\left(\frac{\partial}{\partial x^k}-\mathrm{i}\frac{\partial}{\partial y^k}\right)_p\right)^{\sim} &= \left(\frac{1}{2}\left(\frac{\partial}{\partial x^k}-\mathrm{i}\frac{\partial}{\partial y^k}\right)_p\right)^{\sim}\left(z^\alpha\right)\\ &= \left(\frac{1}{2}\left(\frac{\partial}{\partial x^k}-\mathrm{i}\frac{\partial}{\partial y^k}\right)_p\right)\left(x^\alpha+\mathrm{i}y^\alpha\right)=\delta^k_\alpha.\end{aligned}$$

Hence

$$\Theta_p\left(\frac{1}{2}\left(\frac{\partial}{\partial x^k}-\mathrm{i}\frac{\partial}{\partial y^k}\right)_p\right)^{\sim}=\delta^k_\alpha\frac{\partial}{\partial z^\alpha}\bigg|_p=\frac{\partial}{\partial z^k}\bigg|_p.$$

Therefore, $\Psi_p=\Theta_p^{-1}$. Moreover, the isomorphisms Φ_p and Ψ_p on each fibre extend naturally to complex vector bundle isomorphisms (see Definitions 5.11)

$$\Phi\colon TM\to T^hM,\qquad \Psi\colon T^hM\to T^{1,0}M.$$

We identify the bundles TM, T^hM and $T^{1,0}M$ via Φ and Ψ. Under the isomorphisms Φ and Ψ, both TM and $T^{1,0}M$ are also holomorphic vector bundles.

Finally, we remark:

(a) The election in ($\star$) is motivated by the fact that if f is a holomorphic function, i.e. $\partial f/\partial\bar{z}^k=0$, then

$$\frac{\partial f}{\partial x^k}=\frac{\partial f}{\partial z^k}=\frac{1}{2}\left(\frac{\partial f}{\partial x^k}-\mathrm{i}\frac{\partial f}{\partial y^k}\right).$$

(b) The identification Φ is always tacitly assumed, i.e. one always writes

$$\frac{\partial}{\partial z^k}\quad\text{for }\frac{1}{2}\left(\frac{\partial}{\partial x^k}-\mathrm{i}\frac{\partial}{\partial y^k}\right)\quad\text{and}\quad\frac{\partial}{\partial\bar{z}^k}\quad\text{for }\frac{1}{2}\left(\frac{\partial}{\partial x^k}+\mathrm{i}\frac{\partial}{\partial y^k}\right).$$

5.9 Almost Symplectic Manifolds

Problem 5.59 Denote by $(q^1,\dots,q^n,p_1,\dots,p_n)$ the usual Cartesian coordinates of the space $\mathbb{R}^{2n}$, on which we consider:

(a) The 2-form $\Omega=\sum_i \mathrm{d}p_i\wedge\mathrm{d}q^i$.
(b) A hypersurface S defined by the implicit equation $H(q,p)=\text{const}$.
(c) The vector field X such that $i_X\Omega=-\mathrm{d}H$.

Prove:

(i) $\mathrm{d}H(X)=0$, that is, X is tangent to S, and $L_X\Omega=0$.

(ii) If ω_{2n-1} is a $(2n-1)$-form such that

$$\Omega \wedge \overset{(n)}{\cdots} \wedge \Omega = \omega_{2n-1} \wedge \mathrm{d}H,$$

then $L_X(\omega_{2n-1}|_S) = 0$.

Solution

(i) We have to prove that $XH = 0$, but

$$XH = (\mathrm{d}H)(X) - (i_X \Omega)(X) = -\Omega(X, X) = 0.$$

(ii) Considering the Lie derivative with respect to X of both sides of the equality $\Omega \wedge \overset{(n)}{\cdots} \wedge \Omega = \omega_{2n-1} \wedge \mathrm{d}H$, we obtain

$$\sum_{i=1}^{n} \Omega \wedge \cdots \wedge \overset{i\text{th place}}{L_X \Omega} \wedge \cdots \wedge \Omega = (L_X \omega_{2n-1}) \wedge \mathrm{d}H + \omega_{2n-1} \wedge L_X \,\mathrm{d}H. \qquad (\star)$$

As

$$L_X \Omega = i_X \,\mathrm{d}\Omega + \mathrm{d}i_X \Omega = \mathrm{d}i_X \Omega = -\mathrm{d}(\mathrm{d}H) = 0$$

and

$$L_X \,\mathrm{d}H = \mathrm{d}L_X H = \mathrm{d}(XH) = 0,$$

equation ($\star$) reduces to

$$(L_X \omega_{2n-1}) \wedge \mathrm{d}H = 0. \qquad (\star\star)$$

Let $(x^1, \dots, x^{2n-1}, H)$ be a local coordinate system adapted to S, that is, such that $(\mathrm{d}H)|_S = 0$. Then, for some $(2n-2)$-form ω_{2n-2}, from ($\star\star$) it follows that

$$L_X \omega_{2n-1} = \omega_{2n-2} \wedge \mathrm{d}H + \lambda \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n-1},$$

so

$$0 = (L_X \omega_{2n-1}) \wedge \mathrm{d}H = \lambda \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n-1} \wedge \mathrm{d}H,$$

and thus $\lambda = 0$. Hence $L_X \omega_{2n-1} = \omega_{2n-2} \wedge \mathrm{d}H$, so one has

$$L_X(\omega_{2n-1}|_S) = \omega_{2n-2}|_S \wedge (\mathrm{d}H)|_S = 0$$

because $(\mathrm{d}H)|_S = 0$.

Problem 5.60 Let $\pi : T^*M \to M$ be the cotangent bundle over a C^∞ n-manifold M. The *canonical* 1*-form* ϑ *on* T^*M is defined by

$$\vartheta_\omega(X) = \omega(\pi_* X), \quad \omega \in T^*M, \; X \in T_\omega T^*M.$$

1. Compute the local expression of ϑ and prove that the 2-form $\Omega = \mathrm{d}\vartheta$ is non-degenerate, that is, that $i_X\Omega = 0$ implies $X = 0$.
2. Show that $\Omega \wedge \overset{(n)}{\cdots} \wedge \Omega \neq 0$ at each point. Hence T^*M is orientable. Ω is called the *canonical symplectic form* on T^*M.

   Let $H \in C^\infty(T^*M)$ and let $\sigma\colon (a,b) \to T^*M$ be a C^∞ curve with tangent vector σ'.
3. Write locally the differential equations

$$i_{\sigma'}(\Omega \circ \sigma) + \mathrm{d}H \circ \sigma = 0 \quad (\textit{Hamilton equations}).$$

4. Show that if σ is a solution, $H \circ \sigma$ is a constant function.
5. Solve the Hamilton equations for the case $M = \mathbb{R}^n$, and

$$H = \frac{1}{2}k\left(q^1\right)^2 + \frac{1}{2}m\sum_{i=1}^{n-1} p_i^2 + \frac{1}{2}p_n^2,$$

   where k and m stand for constants.

Solution

1. Given local coordinates $(q^1,\dots,q^n)$ on M, they induce local coordinates $(q^1,\dots,q^n,p_1,\dots,p_n)$ on T^*M putting $\omega_x = p_i(\omega_x)\,\mathrm{d}q^i|_x$ for $\omega_x \in T^*M$, $x \in M$. If

$$X = \sum_{i=1}^n \left(\lambda^i \frac{\partial}{\partial q^i}\bigg|_{\omega_x} + \mu^i \frac{\partial}{\partial p_i}\bigg|_{\omega_x}\right) \in T_{\omega_x}T^*,$$

   then from the definition of ϑ it follows that

$$\begin{aligned}\vartheta(X) = \omega_x(\pi_* X_{\omega_x}) = \omega_x\left(\sum_{i=1}^n \lambda^i \frac{\partial}{\partial q^i}\bigg|_x\right) &= \sum_j \left(p_j(\omega_x)\,\mathrm{d}q^j|_x\right)\left(\sum_{i=1}^n \lambda^i \frac{\partial}{\partial q^i}\bigg|_x\right)\\ = \lambda^i p_i(\omega_x) = \sum_i p_i(\omega_x)\left(\mathrm{d}q^i\big|_{\omega_x}\right)(X) &= \left(\sum_i p_i\,\mathrm{d}q^i\right)(X),\end{aligned}$$

   and so $\vartheta = \sum_i p_i\,\mathrm{d}q^i$; hence $\Omega = \mathrm{d}\vartheta = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$, which is obviously non-degenerate.
2. From part 1 we have, $\mathfrak{S}_n$ being the group of permutations of order n, and $\operatorname{sgn}\sigma$ the sign of the permutation $\sigma \in \mathfrak{S}_n$:

$$\begin{aligned}\Omega^n &= \left(\mathrm{d}p_1 \wedge \mathrm{d}q^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q^n\right) \wedge \cdots \wedge \left(\mathrm{d}p_1 \wedge \mathrm{d}q^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q^n\right)\\ &= \sum_{\sigma\in\mathfrak{S}_n} \mathrm{d}p_{\sigma(1)} \wedge \mathrm{d}q^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)} \wedge \mathrm{d}q^{\sigma(n)}\end{aligned}$$

$$= (-1)^{1+2+\cdots+n} \sum_{\sigma\in\mathfrak{S}_n} \mathrm{d}q^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}q^{\sigma(n)} \wedge \mathrm{d}p_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)}$$

$$= (-1)^{\frac{n(n+1)}{2}} \sum_{\sigma\in\mathfrak{S}_n} (\operatorname{sgn}\sigma)^2\, \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n$$

$$= n!(-1)^{\frac{n(n+1)}{2}} \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \neq 0.$$

3. Put $\sigma_i = q^i \circ \sigma$, $\widetilde{\sigma}_i = p_i \circ \sigma$. We have $\sigma' = \sum_i (\frac{\mathrm{d}\sigma_i}{\mathrm{d}t}\frac{\partial}{\partial q^i} + \frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t}\frac{\partial}{\partial p_i})$. Therefore,

$$i_{\sigma'}(\Omega\circ\sigma) = \sum_i \big(\sigma'(p_i)\,\mathrm{d}q^i\big|_\sigma - \sigma'(q^i)\,\mathrm{d}p_i\big|_\sigma\big) = \sum_i \left(\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t}\,\mathrm{d}q^i\big|_\sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t}\,\mathrm{d}p_i|_\sigma\right).$$

Hence

$$i_{\sigma'}(\Omega\circ\sigma) + \mathrm{d}H\circ\sigma = \sum_i \left(\left(\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} + \frac{\partial H}{\partial q^i}\circ\sigma\right)\mathrm{d}q^i\big|_\sigma + \left(\frac{\partial H}{\partial p_i}\circ\sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t}\right)\mathrm{d}p_i|_\sigma\right),$$

and the Hamilton equations are

$$\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} + \frac{\partial H}{\partial q^i}\circ\sigma = 0, \qquad \frac{\partial H}{\partial p_i}\circ\sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t} = 0.$$

4. If σ is a solution, then $\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}\circ\sigma$ and $\frac{\mathrm{d}\sigma_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\circ\sigma$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}(H\circ\sigma) = \sum_i \left(\left(\frac{\partial H}{\partial q^i}\circ\sigma\right)\frac{\mathrm{d}\sigma_i}{\mathrm{d}t} + \left(\frac{\partial H}{\partial p_i}\circ\sigma\right)\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t}\right) = 0.$$

Thus $H\circ\sigma$ is a constant function.

5. (a)

$$\frac{\mathrm{d}\widetilde{\sigma}_1}{\mathrm{d}t} = -\frac{\partial H}{\partial q^1}\circ\sigma = -k\sigma_1, \qquad \frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}\circ\sigma = 0,$$

for $i = 2, \ldots, n$, hence:

(b) $\widetilde{\sigma}_i = A_i$, $i = 2, \ldots, n$, with A_i constants;
(c) $\frac{\mathrm{d}\sigma_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\circ\sigma = m\widetilde{\sigma}_i$, for $i = 1, \ldots, n-1$;
(d) $\frac{\mathrm{d}\sigma_n}{\mathrm{d}t} = \frac{\partial H}{\partial p_n}\circ\sigma = \widetilde{\sigma}_n$.

From (b) and (c), it follows that $\sigma_i = mA_i t + B_i$, for $i = 2, \ldots, n-1$, $A_i, B_i \in \mathbb{R}$. From (a) and (c), we deduce that $\frac{\mathrm{d}\sigma_1}{\mathrm{d}t} = m\widetilde{\sigma}_1$ and $\frac{\mathrm{d}\widetilde{\sigma}_1}{\mathrm{d}t} = -k\sigma_1$, hence

$$\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}t^2} + km\sigma_1 = 0,$$

and we have four cases:

(i) $k \neq 0,\ m = 0,\ \sigma_1 = A,\ \widetilde{\sigma}_1 = -kAt + B;$

(ii) $k=0,\ m\neq 0,\ \sigma_1 = mCt + D,\ \widetilde{\sigma}_1 = C$;
(iii) $km = \omega^2 > 0,\ \sigma_1 = E\cos\omega t + F\sin\omega t,\ \widetilde{\sigma}_1 = -\frac{\omega}{m}(E\sin\omega t - F\cos\omega t)$;
(iv) $km = -\omega^2 < 0,\ \sigma_1 = G\cosh\omega t + H\sinh\omega t$,

$$\widetilde{\sigma}_1 = -\frac{\omega}{m}(G\sinh\omega t + H\cosh\omega t).$$

Finally, from (b) and (d), we have $\frac{\mathrm{d}\sigma_n}{\mathrm{d}t} = A_n$, thus $\sigma_n = A_n t + B_n$, for $A_n, B_n \in \mathbb{R}$.

Problem 5.61 Consider the trivial principal bundle $\pi\colon M \times U(1)$ over the C^∞ n-manifold M. We use the same notations as in Problem 5.26.

(i) Let Φ_t be the flow of a vector field $X \in \mathfrak{X}(P)$. Prove that X is $U(1)$-invariant if and only if Φ_t is an automorphism of P, for all $t \in \mathbb{R}$.
(ii) Let $p\colon T^*M \to M$ be the cotangent bundle over M. Each coordinate system $(U, q^1, \dots, q^n)$ on M induces a coordinate system $(p^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$ by setting $w = \sum_i p_i(w)\,\mathrm{d}q^i|_x$ for all covector $w \in T_x^*M$.

If Φ_t is the flow of a $U(1)$-invariant vector field $X \in \mathfrak{X}(P)$, then $\widetilde{\Phi}_t$ is a flow on T^*M, which generates a vector field $\widetilde{X}$. Prove that

$$\widetilde{X} = \sum_i \left(f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \sum_h \frac{\partial f^h}{\partial q^i} p_h \right) \frac{\partial}{\partial p_i} \right),$$

where

$$X = \sum_i f^i\big(q^1, \dots, q^n\big)\frac{\partial}{\partial q^i} + g\big(q^1, \dots, q^n\big)\frac{\partial}{\partial \alpha},$$

and α stands for the local coordinate on $U(1)$.
(iii) Let ϑ be the canonical form on T^*M and let $\Phi(x,\alpha) = (\phi(x), \alpha + \psi(x))$ be an automorphism of P. Compute $\Phi^*\vartheta$.
(iv) Conclude that every automorphism of P leaves the canonical symplectic form $\mathrm{d}\vartheta$ invariant.
(v) Prove that $L_{\widetilde{X}}\,\mathrm{d}\vartheta = 0$, for every $U(1)$-invariant vector field X.

Solution

(i) The vector field X is $U(1)$-invariant if and only if for every $z \in U(1)$ we have $R_z \cdot X = X$. This means that R_z commutes with Φ_t; i.e. $R_z \circ \Phi_t = \Phi_t \circ R_z$, or equivalently, $\Phi_t(u)\cdot z = \Phi_t(u \cdot z)$, thus proving that Φ_t is an automorphism.
(ii) If $\Phi(x,\alpha) = (\phi(x), \alpha + \psi(x))$, from part 4 in Problem 5.26, we have

$$\widetilde{\Phi}(w) = \big(\phi^{-1}\big)^* w - \big(\mathrm{d}\big(\psi \circ \phi^{-1}\big)\big)_{\phi(x)}, \quad w \in T_x^*M.$$

As $p \circ \widetilde{\Phi} = \phi \circ p$, we have

$$q^i \circ \widetilde{\Phi} = q^i \circ \phi. \tag{$\star$}$$

Moreover, from the very definition of the coordinates (p_i), we obtain

$$\begin{aligned}(p_i \circ \widetilde{\Phi})(w) &= p_i\big((\phi^{-1})^* w - \big(\mathrm{d}(\psi \circ \phi^{-1})\big)_{\phi(x)}\big) \\ &= \sum_h p_h(w) p_i\big((\phi^{-1})^* \big(\mathrm{d}q^h\big|_x\big)\big) \\ &\quad - \sum_j \frac{\partial(\psi \circ \phi^{-1})}{\partial q^j}\big(\phi(x)\big) p_i\big(\mathrm{d}q^j\big|_{\phi(x)}\big) \\ &= \sum_h p_h(w) \frac{\partial(q^h \circ \phi^{-1})}{\partial q^i}\big(\phi(x)\big) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i}\big(\phi(x)\big).\end{aligned}$$

Hence

$$p_i \circ \widetilde{\Phi} = \sum_h p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi\right) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i} \circ \phi. \qquad (\star\star)$$

If

$$\Phi_t(x, \alpha) = \big(\phi_t(x), \alpha + \psi_t(x)\big),$$

then substituting $\widetilde{\Phi}_t$ for $\widetilde{\Phi}$ in ($\star$), ($\star\star$), taking derivatives with respect to t, and then $t = 0$, we obtain the formula for $\widetilde{X}$ in the statement.

(iii) We have

$$\begin{aligned}\widetilde{\Phi}^* \vartheta &= \sum_i \left(\sum_h p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi\right) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i} \circ \phi\right) \mathrm{d}\big(q^i \circ \phi\big) \\ &= \sum_{i,h} p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi\right) \mathrm{d}\big(q^i \circ \phi\big) - \phi^* \,\mathrm{d}\big(\psi \circ \phi^{-1}\big) \\ &= \sum_h p_h \,\mathrm{d}q^h - \phi^* \,\mathrm{d}\big(\psi \circ \phi^{-1}\big) = \vartheta - \mathrm{d}\psi.\end{aligned}$$

(iv) From the previous formula, we have

$$\Phi^* \,\mathrm{d}\vartheta = \mathrm{d}\vartheta.$$

(v) It follows taking derivatives in $\Phi_t^* \,\mathrm{d}\vartheta = \mathrm{d}\vartheta$, for all $t \in \mathbb{R}$.

Problem 5.62 Let ϑ be the canonical 1-form on the cotangent bundle T^*M over a C^∞ n-manifold M. Prove that $\mathrm{d}\vartheta$ is the only 2-form Ω on T^*M such that:

(i) The vertical bundle of the natural projection $p \colon T^*M \to M$ is a *Lagrangian foliation*, that is, the fibres of p are totally isotropic submanifolds.

(ii) If η is a differential 1-form on M and we denote by τ_η the translation

$$\tau_\eta\colon T^*M \to T^*M, \qquad \tau_\eta(w) = w + \eta(x), \quad w \in T_x^*M,\ x \in M,$$

then

$$\tau_\eta^* \varOmega = \varOmega + p^*\,\mathrm{d}\eta.$$

(iii) $L_{\widetilde{X}} \varOmega = 0$, for every $U(1)$-invariant vector field $X \in \mathfrak{X}(M \times U(1))$ (see Problem 5.26).

Solution First we prove that $\mathrm{d}\vartheta$ satisfies (i), (i) and (iii). Item (i) follows directly from the local expression $\varOmega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$, as the tangent space to the fibres of p is locally spanned by $\partial/\partial p_i$. As for (iii), it follows from Problem 5.26.

Moreover, if $\eta = \sum_i f_i\,\mathrm{d}q^i$, $f_i \in C^\infty M$, then the equations of τ_η are

$$q^i \circ \tau_\eta = q^i, \qquad p_j \circ \tau_\eta = p_j + f_j.$$

Hence

$$\begin{aligned}\tau_\eta^*\,\mathrm{d}\vartheta = \mathrm{d}\tau_\eta^*\vartheta &= \sum_i \mathrm{d}\big((p_i \circ \tau_\eta)\,\mathrm{d}\big(q^i \circ \tau_\eta\big)\big) = \sum_i \mathrm{d}(p_i + f_i) \wedge \mathrm{d}q^i \\ &= \sum_i \big(\mathrm{d}p_i \wedge \mathrm{d}q^i + \mathrm{d}f_i \wedge \mathrm{d}q^i\big) = \mathrm{d}\vartheta + p^*\,\mathrm{d}\eta.\end{aligned}$$

Conversely, assume $\varOmega$ satisfies (i)–(iii). From (i) we have

$$\begin{aligned}&\varOmega = \sum_i \big(A_{hi}\,\mathrm{d}q^h \wedge \mathrm{d}q^i + B_i^h\,\mathrm{d}p_h \wedge \mathrm{d}q^i\big), \\ &A_{hi} + A_{ih} = 0, \qquad A_{hi}, B_i^h \in C^\infty\big(T^*M\big).\end{aligned}$$

Let us impose condition (ii) on $\varOmega$. We have

$$\begin{aligned}\tau_\eta^*\varOmega &= \sum_{h,i}\big((A_{hi} \circ \tau_\eta)\,\mathrm{d}q^h \wedge \mathrm{d}q^i + \big(B_i^h \circ \tau_\eta\big)(\mathrm{d}p_h + \mathrm{d}f_h) \wedge \mathrm{d}q^i\big) \\ &= \sum_{h,i}\big(A_{hi}\,\mathrm{d}q^h \wedge \mathrm{d}q^i + B_i^h\,\mathrm{d}p_h \wedge \mathrm{d}q^i\big) + \sum_i \mathrm{d}f_i \wedge \mathrm{d}q^i.\end{aligned}$$

Hence

$$\left.\begin{aligned}A_{hi} \circ \tau_\eta + \sum_j \big(B_i^j \circ \tau_\eta\big)\frac{\partial f_j}{\partial q^h} &= A_{hi} + \sum_j \frac{\partial f_j}{\partial q^h}\delta_{ij}, \\ B_i^h \circ \tau_\eta &= B_i^h.\end{aligned}\right\} \qquad (\star)$$

Let $X = \partial/\partial q^l$ in (iii). Then, we obtain

$$L_{\widetilde{X}}\Omega = \sum_{h,i}\left(\frac{\partial A_{hi}}{\partial q^l}\,\mathrm{d}q^h \wedge \mathrm{d}q^i + \frac{\partial B_i^h}{\partial q^l}\,\mathrm{d}p_h \wedge \mathrm{d}q^i\right) = 0.$$

Accordingly,

$$\frac{\partial A_{hi}}{\partial q^l} = \frac{\partial B_i^h}{\partial q^l} = 0,$$

that is, A_{hi} and B_i^h depend only on $(p_1, \dots, p_n)$.

Next, let $X = q^l \partial/\partial\alpha$ in (iii). Then we obtain $\widetilde{X} = -\partial/\partial p_l$, and

$$L_{\widetilde{X}}\Omega = \sum_{h,i}\left(-\frac{\partial A_{hi}}{\partial p_l}\,\mathrm{d}q^h \wedge \mathrm{d}q^i - \frac{\partial B_i^h}{\partial p_l}\,\mathrm{d}p_h \wedge \mathrm{d}q^i\right) = 0.$$

Hence

$$\frac{\partial A_{hi}}{\partial p_l} = \frac{\partial B_i^h}{\partial p_l} = 0.$$

Therefore A_{hi} and B_i^h are constant functions.

Now, let us impose condition (iii) for $X = q^k \partial/\partial q^l$, for two given indices k, l. We have

$$\widetilde{X} = q^k \frac{\partial}{\partial q^l} - p_l \frac{\partial}{\partial p_k}.$$

Hence

$$\begin{aligned}
L_{\widetilde{X}}\Omega &= \sum_{k<i}(A_{li} - A_{il})\,\mathrm{d}q^k \wedge \mathrm{d}q^i + \sum_{i<k}(A_{il} - A_{li})\,\mathrm{d}q^i \wedge \mathrm{d}q^k \\
&\quad - \sum_i \left(B_i^k\,\mathrm{d}p_l \wedge \mathrm{d}q^i - B_l^i\,\mathrm{d}p_i \wedge \mathrm{d}q^k\right) = 0.
\end{aligned}$$

Thus $A_{li} = A_{il}$. As $A_{il} + A_{li} = 0$, we have $A_{li} = 0$. Accordingly, the equation ($\star$) now reads as

$$\sum_j \left(B_i^j - \delta_{ij}\right)\frac{\partial f_j}{\partial q^h} = 0.$$

As the functions $\partial f_j/\partial q^h$ are arbitrary, we have $B_i^j = \delta_{ij}$, thus concluding.

Problem 5.63 Let (M, Ω) be an almost symplectic manifold and let $F_1, \dots, F_k$ be smooth functions on M. Suppose that

$$N = \left\{p \in M\colon\ F_1(p) = 0, \dots, F_k(p) = 0\right\}$$

is a submanifold of the manifold M of codimension k.

(i) Prove that the corresponding Hamiltonian vector fields

$$\{X_{F_i}\}, \qquad i_{X_{F_i}}\Omega = -\mathrm{d}F_i, \quad i = 1, \dots, k,$$

are independent at each point of N.

Denote by $A(p) = (a_{ij}(p))$ the matrix-function on M with

$$a_{ij}(p) = \Omega(X_{F_i}, X_{F_j})(p).$$

(ii) Prove that $(N, \Omega|_N)$ is an almost symplectic manifold (see Definitions 5.12) if and only if the restriction of the function $M \to \mathbb{R}$, $p \mapsto \det A(p)$, to $N \subset M$ is nowhere-vanishing.

Solution Fix a point $p \in N \subset M$. Denote by W_p the subspace of T_pM spanned by the vectors $X_{F_1}(p), \dots, X_{F_k}(p)$. Since

$$T_pN = \left\{Y \in T_pM : \mathrm{d}F_i(p)(Y) = 0\right\}$$

and by definition $-\mathrm{d}F_i = i_{X_{F_i}}\Omega$, we obtain that the tangent space T_pN is the orthogonal complement to the subspace W_p in T_pM with respect to the form Ω. Therefore, the restrictions $\Omega_p|_{T_pN}$ and $\Omega_p|_{W_p}$ of the form Ω_p have the same kernel, which is equal to $T_pN \cap W_p$. Now, the form $\Omega_p|_{W_p}$ is non-degenerate if and only if $\det A(p) \neq 0$.

Problem 5.64 Denote by $(x^1, \dots, x^n, y^1, \dots, y^n)$ the usual Cartesian coordinates of the vector (x, y) of the space $\mathbb{R}^n \times \mathbb{R}^n$, on which we consider:

(a) The 2-form $\Omega = \sum_{i=1}^n \mathrm{d}y^i \wedge \mathrm{d}x^i$.
(b) The subset (the total space of the tangent bundle of the n-sphere)

$$M = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : F_1(x, y) = \sum_{i=1}^n \left(x^i\right)^2 = 1,\ F_2(x, y) = \sum_{i=1}^n x^i y^i = 0\right\}.$$

(c) The function $H(x, y) = \sum_{i=1}^n (y^i)^2$.

Prove:

1. The set M is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ and the restriction $\Omega|_M$ of the 2-form Ω determines a symplectic structure on M.
2. The Hamiltonian vector field X_h of the restriction $h = H|_M$ with respect to the form $\Omega|_M$ has the following form

$$X_h(x, y) = 2\sum_{i=1}^n y^i \frac{\partial}{\partial x^i} - 2\left(\sum_{j=1}^n \left(y^j\right)^2\right) \sum_{i=1}^n x^i \frac{\partial}{\partial y^i}, \quad (x, y) \in M.$$

Solution

(i) To simplify the notation denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^n$ and by $vX+wY$, where $v=(v^1,\dots,v^n)$, $w=(w^1,\dots,w^n)\in\mathbb{R}^n$, the tangent vector $\sum_{i=1}^n v^i\partial/\partial x^i+\sum_{i=1}^n w^i\partial/\partial y^i$. Similarly, we put $v\,\mathrm{d}x+w\,\mathrm{d}y=\sum_{i=1}^n v^i\,\mathrm{d}x^i+\sum_{i=1}^n w^i\mathrm{d}y^i$.

The set M is a submanifold of the manifold $\mathbb{R}^n\times\mathbb{R}^n$ because the differentials $\mathrm{d}F_1=2x\,\mathrm{d}x$ and $\mathrm{d}F_2=y\,\mathrm{d}x+x\,\mathrm{d}y$ are linearly dependent if and only if $x=0$.

Since $\Omega(vX+wY,v'X+w'Y)=\langle w,v'\rangle-\langle v,w'\rangle$, it is easy to check that the Hamiltonian vector fields X_{F_1} and X_{F_2} of the function F_1 and F_2 have the form

$$X_{F_1}(x,y)=-2xY,\qquad X_{F_2}(x,y)=xX-yY,$$

respectively. Because of the relation

$$\Omega(X_{F_1},X_{F_2})=-\mathrm{d}F_1(X_{F_2})=(-2x\,\mathrm{d}x)(xX-yY)=-2\langle x,x\rangle,$$

the pair $(M,\Omega|_M)$ is an almost symplectic manifold (see Problem 5.63). The form $\Omega|_M$ is closed because so is Ω.

(ii) To find the Hamiltonian vector field X_h of the function $h=H|_M$, note that the Hamiltonian vector field X_H of the function H on $\mathbb{R}^n\times\mathbb{R}^n$ is equal to $2yX$ ($\mathrm{d}H=2y\,\mathrm{d}y$). This vector field is not tangent to M because

$$\mathrm{d}F_2(X_H)=(x\,\mathrm{d}y+y\,\mathrm{d}x)(2yX)=2\langle y,y\rangle.$$

Since the vector fields X_{F_1} and X_{F_2} are orthogonal to the tangent space of M with respect to the form Ω, they are independent and transversal to M (at each point of M), the Hamiltonian vector field X_h coincides on M with the vector field

$$Z=X_H+a_1(x,y)X_{F_1}+a_2(x,y)X_{F_2}=2yX-2a_1xY+a_2(xX-yY),$$

where the functions a_1,a_2 are defined uniquely by the condition that Z is tangent to M at each point of M. As

$$x\,\mathrm{d}x(Z)=a_2\langle x,x\rangle\quad\text{and}\quad(x\,\mathrm{d}y+y\,\mathrm{d}x)(Z)=2\langle y,y\rangle-2a_1\langle x,x\rangle,$$

we have $a_2(x,y)=0$ and $a_1(x,y)=\langle y,y\rangle$ on M.

Problem 5.65 Let (M,Ω) be a symplectic manifold.
Prove:

(i) The *Poisson bracket*

$$\{\cdot,\cdot\}\colon C^\infty M\times C^\infty M\to C^\infty M,$$
$$\{H_1,H_2\}=-X_{H_1}(H_2)=-\mathrm{d}H_2(X_{H_1})=\Omega(X_{H_2},X_{H_1}),$$

where $H_j \in C^\infty M$ and

$$i_{X_{H_j}} \Omega = -\mathrm{d}H_j, \quad j = 1, 2,$$

(X_{H_j} is the *Hamiltonian vector field of* H_j) determines a Lie algebra structure on $C^\infty M$ and satisfies the Leibniz rule, i.e.

$$\{H_1, H_2\} = -\{H_2, H_1\} \qquad \text{(skew-symmetry)},$$

$$\big\{\{H_1, H_2\}, H_3\big\} = \big\{H_1, \{H_2, H_3\}\big\} - \big\{H_2, \{H_1, H_3\}\big\} \qquad \text{(Jacobi identity)},$$

$$\{H_1, H_2 H_3\} = H_2\{H_1, H_3\} + H_3\{H_1, H_2\} \qquad \text{(Leibniz rule)}.$$

(ii) The vector field $[X_{H_1}, X_{H_2}]$ is the Hamiltonian vector field of the function $-\{H_1, H_2\}$.
(iii) The Lie bracket of two locally Hamiltonian vector fields is Hamiltonian.

Solution

(i) The Poisson bracket is skew-symmetric because so is the symplectic form Ω. Moreover,

$$L_{X_{H_j}} \Omega = i_{X_{H_j}} \mathrm{d}\Omega + \mathrm{d}i_{X_{H_j}} \Omega = \mathrm{d}i_{X_{H_j}} \Omega = -\mathrm{d}(\mathrm{d}H_j) = 0. \qquad (\star)$$

To prove the Jacobi identity, let us prove that the vector field $[X_{H_1}, X_{H_2}]$ is the Hamiltonian vector field of the function $-\{H_1, H_2\}$. We have

$$\begin{aligned}
i_{[X_{H_1}, X_{H_2}]}\Omega &= L_{X_{H_1}}(i_{X_{H_2}}\Omega) - i_{X_{H_1}}(L_{X_{H_2}}\Omega) \\
&= L_{X_{H_1}}(i_{X_{H_2}}\Omega) && \text{(by } (\star)\text{)} \\
&= -L_{X_{H_1}}(\mathrm{d}H_2) && \text{(by the definition of } X_{H_2}\text{)} \\
&= -(i_{X_{H_1}} \circ \mathrm{d} + \mathrm{d} \circ i_{X_{H_1}})(\mathrm{d}H_2) && \text{(by formula (7.3))} \\
&= -\mathrm{d}(i_{X_{H_1}} \mathrm{d}H_2) \\
&= \mathrm{d}\{H_1, H_2\}.
\end{aligned}$$

Now from the definition of the Poisson bracket we obtain that

$$\begin{aligned}
\big\{\{H_1, H_2\}, H_3\big\} &= -X_{\{H_1, H_2\}} H_3 = [X_{H_1}, X_{H_2}] H_3 \\
&= X_{H_1}(X_{H_2} H_3) - X_{H_2}(X_{H_2} H_3) \\
&= \big\{H_1, \{H_2, H_3\}\big\} - \big\{H_2, \{H_1, H_3\}\big\}.
\end{aligned}$$

To prove the Leibniz rule it is sufficient to note that by definition

$$X_{H_1}(H_2 H_3) = H_2(X_{H_1} H_3) + H_3(X_{H_1} H_2).$$

(ii) Let X_1 and X_2 be two locally Hamiltonian vector fields on M. Then for any point $x \in M$ there exists a connected neighbourhood $U \subset M$ and local functions $f_1^U, f_2^U \in C^\infty U$ such that

$$i_{X_j}\Omega = -\mathrm{d}f_j^U, \quad j = 1, 2.$$

Since the functions f_1^U, f_2^U on U are defined uniquely up to constant summands, the Poisson brackets $\{f_1^U, f_2^U\}$ determine a well-defined smooth function on M (it is clear that constant functions lie in the centre of the Lie algebra $(C^\infty M, \{\cdot,\cdot\})$).

(iii) As we proved above, the Hamiltonian vector field of this function coincides up to sign with the Lie bracket $[X_1, X_2]$.

Problem 5.66 Let G be a Lie group acting on the left on a symplectic manifold (M, Ω) and let $\mathfrak{g}$ be the Lie algebra of G. Suppose that each diffeomorphism $g \in G$ preserves the symplectic form Ω, i.e. $g^*\Omega = \Omega$.

Prove:

(i) For any $X \in \mathfrak{g}$ the vector field $\hat{X}$ generated by one-parameter group $\exp tX \subset G$ is *locally Hamiltonian*, i.e.

$$\mathrm{d}(i_{\hat{X}}\Omega) = 0.$$

Suppose, in addition, that each vector field $\hat{X}$, $X \in \mathfrak{g}$, is Hamiltonian with Hamiltonian function $f_X \colon M \to \mathbb{R}$ (i.e. $-\mathrm{d}f_X = i_{\hat{X}}\Omega$) and the map $X \mapsto f_X$ is linear. Let $\{\cdot,\cdot\}$ denote the standard Poisson bracket on the symplectic manifold (M, Ω). Prove:

(ii) For arbitrary fixed elements $g \in G$ and $X \in \mathfrak{g}$, the difference

$$\left(g^{-1}\right)^* f_X - f_{\mathrm{Ad}_g X}$$

is constant on each connected component of M.

(iii) For arbitrary fixed elements $X, Y \in \mathfrak{g}$, the difference

$$\{f_X, f_Y\} - f_{[X,Y]}$$

is constant on each connected component of M.

(iv) If

$$\left(g^{-1}\right)^* f_X = f_{\mathrm{Ad}_g X}, \quad X \in \mathfrak{g},\ g \in G, \tag{$*$}$$

then

$$\{f_X, f_Y\} = f_{[X,Y]}, \quad X, Y \in \mathfrak{g}. \tag{$**$}$$

(v) If the manifold M and the Lie group G are connected then the two conditions $(*)$ and $(**)$ are equivalent, i.e. the linear map $X \mapsto f_X$ is a homomorphism from the Lie algebra $\mathfrak{g}$ into the Lie algebra $C^\infty M$ with respect to the standard Poisson bracket on (M, Ω) if and only if the map $X \mapsto f_X$ is G-equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the standard action of G on $C^\infty M$.

Solution

(i) It is sufficient to note that the symplectic form Ω is closed, that is, $\mathrm{d}\Omega = 0$, and G-invariant. In particular, $(\exp tX)^*\Omega = \Omega$ and, consequently, $L_{\hat{X}}\Omega = 0$. But

$$L_X = \mathrm{d} \circ i_X + i_X \circ \mathrm{d}$$

(see equation (7.3)). Therefore,

$$\mathrm{d}(i_{\hat{X}}\Omega) = (\mathrm{d} \circ i_{\hat{X}} + i_{\hat{X}} \circ \mathrm{d})\Omega = L_{\hat{X}}\Omega = 0.$$

(ii) It is sufficient to note that

$$\begin{aligned} \mathrm{d}\big(g^* f_X\big) &= g^*(\mathrm{d}f_X) = -g^*(i_{\hat{X}}\Omega) = -i_{(g^{-1}\cdot\hat{X})}\big(g^*\Omega\big) \\ &= -i_{\widehat{\mathrm{Ad}_{g^{-1}} X}}\Omega = \mathrm{d}(f_{\mathrm{Ad}_{g^{-1}} X}) \end{aligned}$$

because $\hat{X}$ is the Hamiltonian vector field of the function f_X.

(iii) Note that the bracket $[\hat{X}, \hat{Y}]$ of the vector fields $\hat{X}$ and $\hat{Y}$ is the Hamiltonian vector field of the function $-\{f_X, f_Y\}$ and

$$[\hat{X}, \hat{Y}] = -\widehat{[X, Y]}$$

(see Problems 4.92 and 5.65). Now (iii) follows immediately from the following chain of expressions

$$-\mathrm{d}\{f_X, f_Y\} = i_{-[\hat{X},\hat{Y}]}\Omega = i_{\widehat{[X,Y]}}\Omega = -\mathrm{d}f_{[X,Y]}.$$

(iv) It is sufficient to note that the condition $(**)$ is an infinitesimal version of $(*)$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big\{\big(\exp(-tY)\big)^* f_X - f_{\mathrm{Ad}_{\exp tY} X}\Big\} \\ &\quad = \mathrm{d}f_X(-\hat{Y}) - f_{[Y,X]} = \Omega(\hat{X}, \hat{Y}) + f_{[X,Y]} \\ &\quad = -\{f_X, f_Y\} + f_{[X,Y]} \end{aligned}$$

because by definition $\{f_X, f_Y\} = -\Omega(\hat{X}, \hat{Y})$ and the map $X \mapsto f_X$ is linear.

Suppose now that the manifold M and the Lie group G are connected. Since the map $X \mapsto f_X$ is linear, by (ii) there exists a unique linear function $Z_g \in \mathfrak{g}^*$ such that

$$\big(g^{-1}\big)^* f_X - f_{\mathrm{Ad}_g X} = Z_g(X). \qquad (\star)$$

Then to prove (iv) consider for a fixed vector $X \in \mathfrak{g}$ the function $Z_g(X)$ defined by ($\star$) as a function on the Lie group G. Then, as we showed above,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Z_{\exp tY}(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\{\left(\exp(-tY)\right)^* f_X - f_{\mathrm{Ad}_{\exp tY} X}\right\}$$
$$= f_{[X,Y]} - \{f_X, f_Y\} \qquad (\star\star)$$

for any vector $Y \in \mathfrak{g}$. Moreover, by definition,

$$\begin{aligned} Z_{gh}(X) &= \left((gh)^{-1}\right)^* f_X - f_{\mathrm{Ad}_{gh} X} \\ &= \left(g^{-1}\right)^*\left[\left(h^{-1}\right)^* f_X - f_{\mathrm{Ad}_h X}\right] + \left[\left(g^{-1}\right)^* f_{\mathrm{Ad}_h X} - f_{\mathrm{Ad}_{gh} X}\right] \\ &= \left(g^{-1}\right)^*\left(Z_h(X)\right) + Z_g(\mathrm{Ad}_h X) = Z_h(X) + Z_g(\mathrm{Ad}_h X) \end{aligned}$$

for all $g, h \in G$, $X \in \mathfrak{g}$. In particular, for any $Y \in \mathfrak{g}$ we have

$$Z_{(\exp tY)h}(X) = Z_h(X) + Z_{\exp tY}(\mathrm{Ad}_h X).$$

In other words, the derivative of the function $Z\colon G \to \mathfrak{g}^*$, $g \mapsto Z_g$, with respect to g at any point $h \in G$ in each direction vanishes if and only if the derivative of this function at the identity e in each direction vanishes. Therefore, the map Z vanishes if and only if

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Z_{\exp tY}(X) = 0, \quad X, Y \in \mathfrak{g},$$

because $Z_e = 0$ by definition and the Lie group G is connected. Now taking into account relations ($\star$) and ($\star\star$) we conclude (v).

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Chapter 6
Riemannian Geometry

Abstract The chapter begins with some definitions and results on Riemannian geometry. Then, it presents a collection of problems covering the following topics: Riemannian connections, geodesics, the exponential map, curvature and Ricci tensors, characteristic classes, isometries, homogeneous Riemannian manifolds and Riemannian symmetric spaces, and left-invariant metrics on Lie groups. Some examples of the topics under study include problems dedicated to operators on Riemannian manifolds: Gradient, divergence, codifferential, curl, Laplacian, and Hodge star operator. Other problems consider affine, Killing, conformal, projective, harmonic or Jacobi vector fields. We focus further on some cases of submanifolds, surfaces in $\mathbb{R}^3$, and pseudo-Riemannian manifolds. Cartan method of moving frames is used in a number of problems as a means of putting the reader in touch with this powerful method. Some problems are related to constant curvature, in order to show whether the Riemannian manifold under study has this property or not, thus granting the reader an approach to the techniques used in this setting. In the present edition, the section concerning Riemannian connections has been enlarged, including six new problems on almost complex structures. The section on Riemannian geodesics also includes four new problems on special metrics. Moreover, a completely new section is devoted to a generalisation of Gauss' Lemma. The section on homogeneous Riemannian and Riemannian symmetric spaces contains two new problems about general properties of homogeneous Riemannian manifolds and two new problems on specific three-dimensional Riemannian spaces. Moreover, a short novel section deals with some properties of the energy of Hopf vector fields. The section on left-invariant metrics on Lie groups contains in particular two new problems: One gives in a detailed way the structure of the Kodaira–Thurston manifold; and the other furnishes the de Rham cohomology of a specific nilmanifold.

> …evolvamus primum conditiones, ut expressio $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ (…), in formam $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, constantibus coefficientibus $\alpha_{\iota,\iota'}$ affectam transformari possit (…) Expressionem $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, si est, id quo supponimus, forma positive ipsarum dx, semper in formam $\sum_\iota dx_\iota^2$ redigi posse constat. Unde si $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ in formam $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ transformari potest, redigi etiam potest in formam $\sum_\iota dx_i^2$ et vice versa. Quaeramus igitur, quando in formam

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_6,

$\sum_\iota dx_i^2$ transformari possit. Sit determinans $\sum \pm b_{1,1} b_{2,2} \cdots b_{n,n} = B$ et determinantes partiales $= \beta_{\iota,\iota'}$; quo pacto erit $\sum_\iota \beta_{\iota,\iota'} b_{\iota,\iota'} = B$ et $\sum_\iota \beta_{\iota,\iota'} b_{\iota,\iota'} = 0$ si $\iota \neq \iota'$. Iam $(\ldots)$ expressionibus eruitur

$$2\sum_\nu \frac{\partial^2 x_\nu}{\partial s_{\iota'} \partial s_{\iota''}} \frac{\partial x_\nu}{\partial s_\iota} = \frac{\partial b_{\iota,\iota'}}{\partial s_{\iota''}} + \frac{\partial b_{\iota,\iota''}}{\partial s_{\iota'}} - \frac{\partial b_{\iota',\iota''}}{\partial s_\iota}$$

et si haec quantitas per $p_{\iota,\iota',\iota''}$ designatur $(\ldots)$ Quantitatibus $p_{\iota,\iota',\iota''}$ iterum differentiatis $(\ldots)$ substitutis valoris $(\ldots)$[1]

$$\begin{aligned}&\frac{\partial^2 b_{\iota,\iota''}}{\partial s_{\iota'} \partial s_{\iota'''}} + \frac{\partial^2 b_{\iota',\iota'''}}{\partial s_\iota \partial s_{\iota''}} - \frac{\partial^2 b_{\iota,\iota'''}}{\partial s_{\iota'} \partial s_{\iota''}} - \frac{\partial^2 b_{\iota',\iota''}}{\partial s_\iota \partial s_{\iota'''}}\\ &\quad + \frac{1}{2}\sum_{\nu,\nu'} (p_{\nu,\iota',\iota'''} p_{\nu',\iota,\iota''} - p_{\nu,\iota,\iota'''} p_{\nu',\iota',\iota''}) \frac{\beta_{\nu,\nu'}}{B} = 0.\end{aligned}$$

BERNHARD RIEMANN, "*Commentatio mathematica, qua respondere tentatur ab Ill^{ma} Academia Parisiense propositae*: $(\ldots)$." Gesammelte Math. Werke, ed. Heinrich Weber, 2nd ed., Teubner 1892, pp. 391–404. (With kind permission from Springer publishers.)

J'ai été conduit à la théorie des espaces symétriques $(\ldots)$ par la considération des espaces riemanniens dont la courbure est conservée par le transport paralèlle $(\ldots)$ Le nom de *espaces riemanniens symétriques*, que je leur ai donné plus tard, tient à ce qu'ils sont encore caractérisés par la condition que la symétrie par rapport a un point $(\ldots)$ soit une transformation isométrique $(\ldots)$ j'ai determiné les espaces riemanniens symétriques $(\ldots)$ ils admettent un group transitif de déplacements G. J'indique alors deux méthodes differents $(\ldots)$ La première consiste a déterminer le *group d'isotropie* $(\ldots)$ q'indique comme les vecteurs issues d'un point O sont transformés par le subgroup of

---

[1] "$(\ldots)$ Let us first set out the conditions under which the expression $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, $(\ldots)$ can be transformed to the form $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, with constant coefficients $\alpha_{\iota,\iota'}$. $(\ldots)$ If the expression $\sum \alpha_{\iota,\iota'}\, ds_\iota ds_{\iota'}$ is, as we supposed, a positive form in the dx themselves, we know that it can always be rewritten in the form $\sum_\iota dx_\iota^2$. Hence, if $\sum \beta_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$ can be transformed to the form $\sum \alpha_{\iota,\iota'}\, ds_\iota\, ds_{\iota'}$, it can also be rewritten in the form $\sum_\iota dx_i^2$ and conversely. Let $\sum \pm b_{1,1} b_{2,2} \cdots b_{n,n} = B$ denote the determinant and let $\beta_{\iota,\iota'}$ denote the partial determinants, with the conditions that $\sum_\iota \beta_{\iota,\iota'} b_{\iota,\iota'} = B$ and $\sum_\iota \beta_{\iota,\iota'} b_{\iota,\iota'} = 0$ if $\iota \neq \iota'$. Now $(\ldots)$ denoting the obtained quantities

$$2\sum_\nu \frac{\partial^2 x_\nu}{\partial s_{\iota'} \partial s_{\iota''}} \frac{\partial x_\nu}{\partial s_\iota} = \frac{\partial b_{\iota,\iota'}}{\partial s_{\iota''}} + \frac{\partial b_{\iota,\iota''}}{\partial s_{\iota'}} - \frac{\partial b_{\iota',\iota''}}{\partial s_\iota}$$

by $p_{\iota,\iota',\iota''}$ $(\ldots)$ differentiating again the quantities $p_{\iota,\iota',\iota''}$ $(\ldots)$ and substituting values $(\ldots)$ then

$$\frac{\partial^2 b_{\iota,\iota''}}{\partial s_{\iota'} \partial s_{\iota'''}} + \frac{\partial^2 b_{\iota',\iota'''}}{\partial s_\iota \partial s_{\iota''}} - \frac{\partial^2 b_{\iota,\iota'''}}{\partial s_{\iota'} \partial s_{\iota''}} - \frac{\partial^2 b_{\iota',\iota''}}{\partial s_\iota \partial s_{\iota'''}} + \frac{1}{2}\sum_{\nu,\nu'} (p_{\nu,\iota',\iota'''} p_{\nu',\iota,\iota''} - p_{\nu,\iota,\iota'''} p_{\nu',\iota',\iota''}) \frac{\beta_{\nu,\nu'}}{B} = 0.$$

> G qui laisse invariant ce point. La seconde méthode consiste á déterminer directment le group G lui-même et conduit en fin de compte à la recherche des formes reelles des groupes simples, problème que j'avait resolu en 1914 (...).[2]
>
> ÉLIE CARTAN, "*Notice sur les travaux scientifiques, XII. Les espaces symétriques*: (...)." Oeuvres Complètes, Publiées avec le concours du C.N.R.S, vol. I, p. 92, Gauthier-Villars, Paris, 1952. (Reproduced with kind permission from Dunod Éditeur, Paris. Not for re-use elsewhere).

6.1 Some Definitions and Theorems on Riemannian Geometry

Definitions 6.1 Let V be a vector space of dimension n with a non-degenerate symmetric bilinear form. It is said that V has *signature* $(k, n-k)$ if, expressing the form as a sum of squares, there are k negative squares and $n-k$ positive squares.

A *metric tensor* g on a differentiable manifold M is a symmetric non-degenerate $(0, 2)$ tensor field on M of constant signature. A (*pseudo*)-*Riemannian manifold* is a pair (M, g) of a differentiable manifold M and a metric tensor g on M. If there is no danger of confusion, one simply writes M.

Let $t^1, \dots, t^n$ be the canonical coordinates on $\mathbb{R}^n$, and let φ be a coordinate map with domain $U \subset M$, such that $x^i = t^i \circ \varphi$. If φ is a conformal map from U onto $\mathbb{R}^n$, with respect to the usual metric of $\mathbb{R}^n$, it is said that the coordinate system $(U, x^1, \dots, x^n)$ is *isothermal* or *conformal* (hence it is also orthogonal).

Given a Riemannian metric g on a connected manifold M, we define the distance function $d_g(p, q)$ on M as follows. The distance $d_g(p, q)$ between two points p and q is, by definition, the infimum of the lengths of all piecewise differentiable curves of class C^1 joining p and q. The function d_g defines the metric on the set M, in particular, $d_g(p, q) = 0$ only if $p = q$. The topology defined by the distance function (metric) d_g is the same as the manifold topology of M.

Definition 6.2 Let (M_1, g_1), (M_2, g_2) be pseudo-Riemannian manifolds, and let f be a C^∞ function on the manifold M_1. The *warped product* $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric

$$g = \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2,$$

where $\pi_i : M \to M_i$, $i = 1, 2$, denote the projection maps.

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[2] "I was led to the theory of symmetric spaces (...) when considering the Riemannian spaces whose curvature is preserved under parallel transport (...) The name *Riemannian symmetric spaces* I gave them later reflects that they are characterised by the condition that the symmetry with respect to a point be an isometric transformation (...) I determined the symmetric spaces (...) they admit a transitive group of motions G (...) I point out two different methods (...) The first one consists in determining the *isotropy group*, which indicates how the vectors from a point O are transformed by the subgroup of G keeping this point invariant. The second method consists in determining directly the group G and actually leads to the research of the real forms of simple groups, problem which I had solved in 1914 (...)."

Definition 6.3 Let (M, g) be a Riemannian manifold with a linear connection ∇, with connection map $\kappa\colon TTM \to TM$ (see Definitions 5.1). The *Sasaki metric* or *connection metric* $\tilde{g}(\cdot\,,\cdot)$ on the manifold TM is defined by

$$\tilde{g}(U, V) = g(\pi_* U, \pi_* V) + g(\kappa U, \kappa V), \quad U, V \in \mathfrak{X}(TM).$$

(Notice that it is a metric on the total space of the bundle TTM over TM, and not on the total space of the bundle TM over M. Note also that although a linear connection on M is considered, the connection map is defined in terms of a connection on the total space of TM (see Definitions 5.1), this connection being defined by the linear connection on M.)

Theorem 6.4 (Koszul Formula for the Levi-Civita Connection) *The only torsionless metric connection ∇ on a (pseudo)-Riemannian manifold (M, g) is given by*

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)\\ &\quad + g\big([X, Y], Z\big) - g\big([Y, Z], X\big) + g\big([Z, X], Y\big).\end{aligned}$$

Definitions 6.5 Let R denote the curvature tensor field of a linear connection ∇ on a differentiable manifold M. Given $X_p, Y_p, Z_p \in T_pM$, one defines $R(X_p, Y_p)Z_p$ by

$$R(X_p, Y_p)Z_p = \nabla_{X_p}\nabla_Y Z - \nabla_{Y_p}\nabla_X Z - \nabla_{[X,Y]_p} Z,$$

where X, Y, Z are vector fields on M whose values at p are respectively X_p, Y_p, Z_p. Similarly, if (M, g) is a (pseudo)-Riemannian manifold, one defines, given the vector fields $X, Y, Z, W \in \mathfrak{X}(M)$,

$$R(X_p, Y_p, Z_p, W_p) = g\big(R(Z_p, W_p)Y_p, X_p\big).$$

Definitions 6.6 A Riemannian manifold M or a Riemannian metric g on M is said to be geodesically complete if its Riemannian connection is complete, that is, if every geodesic of M can be extended for arbitrarily large values of its canonical parameter.

Let N be a submanifold of M, and $\nu(N)$ its normal bundle. The exponential map of M gives, by restriction, a map $\exp\colon \nu(N) \to M$, which is a diffeomorphism on a neighbourhood of the zero section. For $p \in N$, let $\nu_p(N)$ be the fibre of $\nu(N)$ over p. Then $q \in \nu_p(N)$ is a *focal point of* N if $\exp_*$ is singular at q. If ρ is the ray from 0 to q in $\nu_p(N)$, then $\exp q$ is called a *focal point of N along ρ*, which is a geodesic perpendicular to N. When N is a single point, say p, so that $\nu(N) = T_pM$, then a focal point is called a *conjugate point to p*. The *order* of a focal point is the dimension of the linear space annihilated by $\exp_*$.

A *minimal segment* is a geodesic segment which minimizes arc length between its ends. A *minimal point* q of p along a geodesic γ is a point on γ such that the segment of γ from p to q is minimal but no larger segment from p is minimal. The set of all minimal points of p is called the *minimum* (or *cut*) *locus* of p.

Theorem 6.7 *For a connected Riemannian manifold* (M, g) *the following conditions are mutually equivalent*:

(a) (M, g) *is geodesically complete.*
(b) (M, d_g) *is a complete metric space* (*with respect to the distance function* d_g).
(c) *Every bounded subset of* (M, d_g) *is relatively compact* (*has compact closure*).

Moreover, for a connected geodesically complete Riemannian manifold M, *any two points* p *and* q *of* M *can be joined by a minimizing geodesic*, *in particular*, $\exp_p(T_pM) = M$ *for any* $p \in M$.

Proposition 6.8 *Let* N *be the subset of the total space* TM *of the tangent bundle over* M *such that if* $(p, X) \in N$ *then* $\exp_p X$ *is defined*, *and define the map* $\exp\colon N \to M$ *by* $\exp(p, X) = \exp_p X$. *Then* N *is an open set and* $\exp$ *is* C^∞ *on* N. *Let* TM_0 *be the zero section of* TM, *that is*, $TM_0 = \{(p, 0) \in TM : p \in M\} \subset TM$; *then there exists an open subset* $\widehat{N}$ *in* TM *such that* $TM_0 \subset \widehat{N} \subset N$. *Let* $\Phi\colon \widehat{N} \to M \times M$ *be defined by* $\Phi(p, X) = (p, \exp_p X)$. *Then* Φ *is* C^∞ *and* Φ_* *is non-singular and surjective at all points of* TM_0.

Definition 6.9 Let γ be a C^∞ curve in the n-manifold M that is an injective map on the open interval $I \subset \mathbb{R}$. Let $e_1, \dots, e_n$ be vector fields on γ that are independent at each $\gamma(t)$ and with $e_n(t) = \gamma'(t)$ for all $t \in I$. Let $\{\theta^1, \dots, \theta^n\}$ be the basis dual to $\{e_1, \dots, e_n\}$ for each t. By Proposition 6.8, there exists a neighbourhood V of TM_0 such that the map Φ is a diffeomorphism of V onto a neighbourhood U_M of the diagonal in $M \times M$. Let

$$U = \big\{(p, X) \in V : p = \gamma(t),\ \theta^n(X) = 0 \text{ for some } t \in I\big\}.$$

Then $\Psi = \Phi|_U$ is a one-to-one C^∞ map of the submanifold U into $M \times M$. Moreover, Ψ_* is non-singular at each point of U, so that Ψ is an embedding of U into $M \times M$. The map $\Upsilon = \mathrm{pr}_2 \circ \Psi$ then gives a one-to-one C^∞ map of U onto an open neighbourhood W of the image set $\gamma(I)$. Define *Fermi coordinates* x^i on $q \in W$ by letting $\Upsilon^{-1}(q) = (\gamma(t), Y)$ in W and $x^i(q) = \theta^i(Y)$ for $i = 1, \dots, n-1$ and $x^n(q) = t$.

More special types of Fermi coordinates can be defined by taking $e_1, \dots, e_n$ to be parallel along a geodesic, and in the Riemannian case, one can take an orthonormal parallel basis along a geodesic.

Definition 6.10 A Riemannian metric g on a homogeneous space $M = G/H$ is called a G-*invariant metric* (or simply an *invariant metric*) if each $a \in G$ acts on M as an isometry, that is, $g_{ap}(a_*X, a_*Y) = g_p(X, Y)$ for each $p \in M$, $X, Y \in T_pM$, $a \in G$. In this case, $(M = G/H, g)$ is called a *homogeneous Riemannian space*.

Definition 6.11 The *divergence of a* $(0, r)$ *tensor field* α on the Riemannian manifold (M, g) is defined by

$$(\operatorname{div}\alpha)_p(v_1, \dots, v_r) = \sum_i (\nabla_{e_i}\alpha)(e_i, v_1, \dots, v_{r-1}),$$

∇ being the Levi-Civita connection, $v_i \in T_pM$ and $\{e_i\}$ an orthonormal basis of T_pM, $p \in M$.

In the particular case of α being a differential form on (M, g), its *codifferential* is defined by

$$\delta\alpha = -\operatorname{div}\alpha.$$

Definition 6.12 Let (M, g) be an n-dimensional Riemannian manifold. Then the *Clifford multiplication on forms* is defined as follows. If $\theta \in \Lambda^1 M$ and $\omega \in \Lambda^r M$, then

$$\theta \cdot \omega = \theta \wedge \omega - \iota_{\theta^\sharp}\omega, \qquad \omega \cdot \theta = (-1)^r(\theta \wedge \omega + \iota_{\theta^\sharp}\omega).$$

By declaring the product to be bilinear and associative, we can use these properties to define the product of any two forms.

Definition 6.13 Let (M, g) be an n-dimensional Riemannian manifold. Then the *Dirac operator on forms* is defined as

$$D\omega = \sum_{i=1}^{n} \theta^i \cdot \nabla_{e_i}\omega, \quad \omega \in \Lambda^r M,$$

where $\{e_i\}$, $i = 1, \dots, n$, is any local frame and $\{\theta^i\}$ its metrically dual local coframe.

Definition 6.14 Let $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ be the Laplacian on a Riemannian manifold (M, g). The elements of

$$H^r = \left\{\omega \in \Lambda^r M : \Delta\omega = 0\right\}$$

are called the *harmonic r-forms on M*.

Definition 6.15 Let M be a Riemannian $4n$-manifold. The Hodge star operator decomposes the space of harmonic forms H^{2n} on M into subspaces $H^{2n}_{\pm}$ with eigenvalues ± 1. The *Hirzebruch signature* is defined by

$$\tau(M) = \dim H^{2n}_{+} - \dim H^{2n}_{-}.$$

This signature equals the usual topological signature.

Theorem 6.16 (Hirzebruch Signature Formula for Dimension 4) *The signature $\tau(M)$ of a 4-dimensional compact oriented differentiable manifold M is related to its first Pontrjagin form $p_1(M)$ by*

$$\tau(M) = \frac{1}{3}\int_M p_1(M).$$

Definition 6.17 Let (M, g) be a 3-dimensional compact orientable Riemannian manifold. Let $\Omega = (\Omega^i_j)$ denote the curvature form of the Levi-Civita connection ∇, and consider the closed form $TP_1(\Omega)$ on the bundle $\mathscr{O}_+(M)$ of positively-oriented orthonormal frames on M, given by

$$\frac{1}{2}TP_1(\Omega) = \frac{1}{8\pi^2}\sum_{1\leqslant i<j\leqslant 3}\omega^i_j \wedge \Omega^i_j - \frac{1}{8\pi^2}\omega^1_2 \wedge \omega^2_3 \wedge \omega^3_1,$$

where ω^i_j and Ω^i_j denote, respectively, the connection forms and the curvature forms of the linear connection ∇.

The differential form $\frac{1}{2}TP_1(\Omega)$ gives rise to a *Chern–Simons invariant* $J(M, g) \in \mathbb{R}/\mathbb{Z}$ as follows: Since such an M is globally parallelisable, a section $s\colon M \to \mathscr{O}_+(M)$ exists. The integral

$$I(s) = \int_{s(M)} \frac{1}{2}TP_1(\Omega)$$

is a real number, and for another section s' the difference $I(s) - I(s')$ is an integer. The invariant $J(M, g)$ is defined to be $I(s)$ mod 1.

Let Γ, $\widetilde{\Gamma}$ be two connections in a principal bundle $P = (P, M, G)$. On a trivialising neighbourhood, any such Γ can be described by a $\mathfrak{g}$-valued differential 1-form A and the corresponding curvature by

$$F = \mathrm{d}A + \frac{1}{2}[A, A].$$

Then, if $I \in \mathscr{I}^r(G)$ denotes a G-invariant polynomial on $\mathfrak{g}$, it can be proved that the differential $2r$-form $I(F^r)$ does not depend on the particular trivialisation of P. Hence, the various locally defined differential forms $I(F^r)$ fit together to yield a differential 2-form on M, again denoted by $I(F^r)$, which is closed.

Let $\tilde{A}$, $\widetilde{F}$ be the connection form and the curvature form corresponding to $\widetilde{\Gamma}$. Then consider the connection 1-form

$$A_t = \tilde{A} + t(A - \tilde{A}), \quad t \in [0, 1],$$

with corresponding curvature form

$$F_t = \mathrm{d}A_t + \frac{1}{2}[A_t, A_t].$$

One has the following *transgression formula*, sometimes called *Chern–Simons formula*:

Theorem 6.18

$$I\big(F^r\big) - I\big(\widetilde{F}^r\big) = \mathrm{d}Q(A, \widetilde{A}),$$

where $Q(A, \widetilde{A})$ *is defined by*

$$Q(A, \widetilde{A}) = r \int_0^1 I(A - \widetilde{A}, F_t, \dots, F_t)\,\mathrm{d}t.$$

Definitions 6.19 Given a linear connection ∇ on a differentiable manifold M and a geodesic γ on M, a *Jacobi field* along γ is a vector field Y along γ satisfying

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_{\gamma'}\big(T\big(Y, \gamma'\big)\big) + R\big(Y, \gamma'\big)\gamma' = 0,$$

where T denotes the torsion tensor of ∇.

Theorem 6.20 *Let* M, N *be pseudo-Riemannian manifolds, with* N *connected, and let* $\Phi\colon M \to N$ *be a local isometry. Suppose that given any geodesic* $\gamma\colon [0, 1] \to N$ *and a point* $p \in M$ *such that* $\Phi(p) = \gamma(0)$, *there exists a lift* $\widetilde{\gamma}\colon [0, 1] \to M$ *of* γ *through* Φ *such that* $\widetilde{\gamma}(0) = p$. *Then* Φ *is a pseudo-Riemannian covering map.*

Definition 6.21 Let (P, π, M, G) a principal bundle. The *holonomy group* (resp., *restricted holonomy group*) of the connection Γ in P with reference point $p \in M$ is the group $\mathrm{Hol}_M(p)$ (resp., $\mathrm{Hol}^0_M(p)$) consisting of diffeomorphisms of the fiber $\pi^{-1}(p)$ onto itself obtained under parallel transport along closed curves (resp., closed curves homotopic to zero) starting and ending at p. Since the holonomy groups at two points of a manifold are conjugated subgroups of G, we shall write simply $\mathrm{Hol}(\Gamma)$ or $\mathrm{Hol}^0(\Gamma)$ for a given manifold M and Γ as above.

Theorem 6.22 *Let* $\Phi, \Psi\colon M \to N$ *be isometries of pseudo-Riemannian manifolds. If* M *is connected and* $\Phi(p) = \Psi(p)$, $\Phi_{*p} = \Psi_{*p}$, *at some point* $p \in M$, *then* $\Phi = \Psi$.

Definition 6.23 An *affine symmetric space* is a triple (G, H, σ) consisting of a Lie group G, a closed subgroup H of G, and an involutive automorphism σ (that is, $\sigma^2 = \mathrm{id}$) of G such that

$$G_0^\sigma \subset H \subset G^\sigma,$$

where G^σ denotes the closed subgroup of G consisting of all the elements left fixed by σ, and G_0^σ stands for the identity component of G^σ.

Definition 6.24 A (pseudo)-Riemannian manifold (M, g) is said to be an *Einstein manifold* if the Ricci tensor is proportional to the metric, $\mathbf{r} = \lambda g$, for some constant λ.

Definition 6.25 A Riemannian manifold of constant sectional curvature is called a (real) *space form*.

Definition 6.26 The *generalised Heisenberg group* or Heisenberg group $H(p,q)$ is the Lie group of matrices of the form

$$\begin{pmatrix} I_q & A & v \\ 0 & I_p & w \\ 0 & 0 & 1 \end{pmatrix}, \quad A \in M(q \times p, \mathbb{R}),\ v \in \mathbb{R}^q,\ w \in \mathbb{R}^p.$$

Definition 6.27 A vector field X on a Riemannian manifold (M, g) is *harmonic* if the differential form dual with respect to the metric, $X^\flat$, defined at each $p \in M$ by $X^\flat_p(Y) = g_p(X_p, Y_p)$, $Y \in \mathfrak{X}(M)$, is harmonic.

Theorem 6.28 (Hodge Decomposition Theorem) *For each integer r with $0 \leqslant r \leqslant n$, the space H^r defined in Definition 6.14 is finite-dimensional, and we have the following direct sum decompositions of $\Lambda^r M$:*

$$\begin{aligned} \Lambda^r M = \Delta\big(\Lambda^r M\big) \oplus H^r &= \mathrm{d}\delta\big(\Lambda^r M\big) \oplus \delta\mathrm{d}\big(\Lambda^r M\big) \oplus H^p \\ &= \mathrm{d}\big(\Lambda^{r-1} M\big) \oplus \delta\big(\Lambda^{r+1} M\big) \oplus H^p. \end{aligned}$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in \Lambda^r M$ if and only if the differential r-form α is orthogonal to the space of harmonic r-forms.

Corollary 6.29 (Corollary of Green's Theorem) *Let M be a compact Riemannian manifold with volume form υ. Then*

$$\int_M \Delta f \upsilon = 0, \qquad f \in C^\infty M.$$

Theorem 6.30 (Generalised Gauss' Theorema Egregium) *Let M be a hypersurface of a Riemannian manifold $\widetilde{M}$, let P be a subspace of dimension 2 of $T_p M$, $p \in M$, and let $K(P)$, $\widetilde{K}(P)$ be the sectional curvature of P in M and $\widetilde{M}$, respectively; then*

$$\widetilde{K}(P) = K(P) - \det L,$$

where L is the Weingarten map.

Remark 6.31 When $\widetilde{M}$ is 3-dimensional, the above theorem shows that the determinant of L is independent of the embedding (i.e. independent of L) and depends only on the Riemannian structure of $\widetilde{M}$ and M.

Definition 6.32 A C^∞ map $\Phi\colon (M, g) \to (\widetilde{M}, \tilde{g})$ between Riemannian manifolds is said to be a *strictly conformal map of ratio* λ if there exists a strictly positive function $\lambda \in C^\infty M$ such that, for all $p \in M$ and $X, Y \in T_p M$, it satisfies $\tilde{g}(\Phi_* X, \Phi_* Y) = \lambda(p) g(X, Y)$.

Theorem 6.33 *Let (M, g) be a Riemannian manifold of dimension n and let $I(M, g)$ be its isometry group, which is a Lie group. If $n \neq 4$, then $I(M, g)$ does not* *contain any closed subgroup of dimension r with*

$$\frac{1}{2}n(n-1)+1 < r < \frac{1}{2}n(n+1).$$

Moreover, (M, g) *has constant curvature if and only if* $\dim I(M, g) = \frac{1}{2}n(n+1)$.

(See [18, Chap. II, Theorem 3.2].)

Definitions 6.34 Let M be an m-dimensional manifold isometrically immersed in an $(m+n)$-dimensional manifold N. Let ∇' denote covariant differentiation in N. Given $X, Y \in \mathfrak{X}(N)$, since $(\nabla'_X Y)_p$ is defined at each $p \in M$, we decompose it into tangential and normal components,

$$\left(\nabla'_X Y\right)_p = (\nabla_X Y)_p + \alpha_p(X, Y),$$

where $(\nabla_X Y)_p \in T_pM$ and $\alpha_p(X, Y) \in (T_pM)^{\perp}$. Then it is proved [19, II, p. 11] that $\nabla_X Y$ is the covariant differentiation for the Levi-Civita connection of M. One can show that the map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to (\mathfrak{X}(M))^{\perp}$ is a symmetric bilinear map called the *second fundamental form of M* (for the given immersion in N).

We may locally choose $n-m$ fields of unit normal vectors $\xi_1, \dots, \xi_{m-n}$ that are orthogonal at each point and we may then express α by

$$\alpha(X, Y) = \sum_{i=1}^{m-n} h^i(X, Y)\xi_i.$$

Let $X \in \mathfrak{X}(M)$ and $\xi \in (\mathfrak{X}(M))^{\perp}$. Writing the decomposition

$$\left(\nabla'_X \xi\right)_p = -(A_\xi X)_p + (D_X \xi)_p,$$

into tangential (to M) and normal components of $(\nabla'_X)_p$, one can easily prove that both the tangent and the normal vector fields are differentiable. Moreover [19, II, p. 14],

$$g(A_\xi X, Y) = g\big(\alpha(X, Y), \xi\big)$$

and A_ξ is a symmetric linear transformation of T_pM with respect to g_p. Since for any $p \in M$, we have a map

$$\xi \in (T_pM)^{\perp} \mapsto A_\xi,$$

it follows that $\frac{1}{n}\operatorname{tr} A_\xi$ is a linear function on $(T_pM)^{\perp}$. The unique element $H \in (T_pM)^{\perp}$ such that

$$\frac{1}{n}\operatorname{tr} A_\xi = g(\xi, H), \quad \xi \in (T_pM)^{\perp},$$

is called the *mean curvature normal at* p. If $\xi_1, \dots, \xi_{m-n}$ is an orthonormal basis of $(T_pM)^\perp$, then

$$\frac{1}{n}\operatorname{tr} A_{\xi_i} = g(\xi_i, H), \quad i = m+1, \dots, m+n,$$

so that

$$H = \frac{1}{n}\sum_{i=m+1}^{m+n} (\operatorname{tr} A_{\xi_i})\xi_i.$$

The submanifold M is said to be a *minimal submanifold* of N (for the given isometric immersion) if the mean curvature normal H vanishes at each point.

Definitions 6.35 Let $(\widetilde{M}, g, J)$ be an almost Hermitian manifold with metric g and almost complex structure J. An isometrically immersed real submanifold M of $\widetilde{M}$ is said to be a *complex submanifold* (resp., a *totally real submanifold*) of $\widetilde{M}$ if each tangent space to M is mapped into itself (resp., into the subspace normal with respect to g) by the almost complex structure J.

Let Φ be an isometric immersion of the Riemannian manifold M into the Riemannian manifold $\widetilde{M}$. Then M is said to be an *invariant submanifold* of $\widetilde{M}$ if for all $X, Y \in TM$, the map $\widetilde{R}(\Phi_* X, \Phi_* Y)$, where $\widetilde{R}$ denotes the Riemann curvature tensor on $\widetilde{M}$, leaves the tangent space to $\Phi(M)$ invariant.

A Kähler manifold is called a *complex space form* if it has constant holomorphic sectional curvature.

Theorem 6.36 *An invariant submanifold M of a complex manifold $\widetilde{M}$ is either a complex or a totally real submanifold. If M is a complex submanifold, then it is a minimal submanifold.*

Theorem 6.37 *Let $\mathbf{x}_1 \colon U \subset \mathbb{R}^2 \to S_1$ and $\mathbf{x}_2 \colon U \subset \mathbb{R}^2 \to S_2$ be two parametrisations of the surfaces S_1, S_2 in $\mathbb{R}^3$. If the metrics inherited on S_1 and S_2 by the usual metric of $\mathbb{R}^3$ are proportional with constant of proportionality $\lambda > 0$, then the map $\mathbf{x}_2 \circ \mathbf{x}_1^{-1} \colon \mathbf{x}_1(U) \to S_2$ is locally conformal.*

Definition 6.38 A *pseudo-Riemannian submanifold* N of a pseudo-Riemannian manifold (M, g) is a submanifold such that the metric tensor inherited by g on N is non-degenerate.

6.2 Riemannian Manifolds

Problem 6.39 Prove that on any differentiable manifold M there exists some Riemannian metric.

Hint The manifold M is paracompact (see Definitions 1.1).

Solution Consider some atlas on M consisting of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying the following property: For all $\alpha \in A$ the image $\varphi_\alpha(U_\alpha)$ is an open unit ball in $\mathbb{R}^n$ and the closure $\bar{U}_\alpha$ is compact. Since the manifold M is paracompact there exists a locally finite covering $\{V_\beta\}_{\beta \in B}$ which is a refinement of $\{U_\alpha\}_{\alpha \in A}$. Each $\bar{V}_\beta$ is compact as closed subset of some $\bar{U}_\alpha$.

By the axiom of choice there exists a function $c\colon B \to A$ such that for each $\beta \in B$ one has $V_\beta \subset U_{c(\beta)}$. The standard Riemannian structure on $\mathbb{R}^n$ and the map $\varphi_\alpha | V_\beta$, $\alpha = c(\beta)$, determine a Riemannian metric g_β on each open set V_β. By the Partition of Unity Theorem 1.2, one has a partition of unity $\sum_\beta \psi_\beta = 1$, where the support of ψ_β lies in V_β. The formula

$$g = \sum_\beta \psi_\beta g_\beta$$

determines a well-defined metric g on M because each $\psi_\beta \geqslant 0$.

Problem 6.40 Let (M, g) be a Riemannian n-manifold. Prove:

(i) Given $\alpha, \beta \in T_p^*M$ and an orthonormal basis $\{e_i\}$, $i = 1, \ldots, n$, of T_pM, and denoting by g^{-1} the contravariant metric associated to g, one has

$$g^{-1}(\alpha, \beta) = \sum_i \alpha(e_i)\beta(e_i).$$

(ii) For $X \in T_pM$, one has

$$g^{-1}\big(\alpha, X^\flat\big) = \alpha(X) = g\big(\alpha^\sharp, X\big),$$

where

$$\flat\colon T_pM \to T_p^*M, \qquad X^\flat = g(X, \cdot), \qquad \sharp\colon T_p^*M \to T_pM,$$
$$\alpha^\sharp = g^{-1}(\alpha, \cdot),$$

are the *musical isomorphisms* (named “flat” and “sharp”, respectively) associated to g.

Solution

(i) In general, if $(g_{ij}(p))$ is the matrix of g with respect to $\{e_i\}$, then $(g^{ij}(p)) = (g_{ij}(p))^{-1}$ is the matrix of g^{-1} with respect to the dual basis $\{\theta^i\}$ to $\{e_i\}$ in T_p^*M. In this case, $(g_{ij}(p)) = (\delta_{ij})$ with respect to $\{e_i\}$, so

$$g^{-1}(\alpha, \beta) = \sum_{i,j} g^{ij}\alpha_i\beta_j = \sum_{i,j} \delta^{ij}\alpha_i\beta_j = \sum_i \alpha_i\beta_i = \sum_i \alpha(e_i)\beta(e_i).$$

(ii)

$$g^{-1}(\alpha, X^\flat) = \sum_{i,j,k} g^{ij}(p)\alpha_i g_{kj}(p) X^k = \sum_{i,k} \delta^i_k \alpha_i X^k = \sum_i \alpha_i X^i = \alpha(X),$$

$$g(\alpha^\sharp, X) = \sum_{i,j,k} g_{ij}(p) g^{ki}(p)\alpha_k X^j = \sum_{j,k} \delta^k_j \alpha_k X^j = \sum_j \alpha_j X^j = \alpha(X).$$

Problem 6.41 Let X_1 and X_2 be the coordinate vector fields for a set of orthogonal coordinates on a surface. Prove that there are *isothermal coordinates* (also called *conformal coordinates*) with the same domain of definition and the same coordinate curves (as images) if and only if

$$X_2 X_1 \left(\log \frac{g_{11}}{g_{22}} \right) = 0,$$

where $g = \sum_{i,j=1}^2 g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$ is the metric.

Solution We have orthogonal coordinates x^1, x^2 with coordinate fields $X_1 = \partial/\partial x^1$, $X_2 = \partial/\partial x^2$. Since $g(X_1, X_2) = 0$, the metric is

$$g = g_{11}\, \mathrm{d}x^1 \otimes \mathrm{d}x^1 + g_{22}\, \mathrm{d}x^2 \otimes \mathrm{d}x^2.$$

If there exist coordinates y^1, y^2 with the same coordinate curves (as images) it must be that

$$\frac{\partial x^1}{\partial y^2} = \frac{\partial x^2}{\partial y^1} = \frac{\partial y^1}{\partial x^2} = \frac{\partial y^2}{\partial x^1} = 0$$

and thus

$$Y_1 = \frac{\partial}{\partial y^1} = \frac{\partial x^1}{\partial y^1}\frac{\partial}{\partial x^1}, \qquad Y_2 = \frac{\partial}{\partial y^2} = \frac{\partial x^2}{\partial y^2}\frac{\partial}{\partial x^2}. \tag{$\star$}$$

If the coordinates are isothermal, there exists ν such that

$$\widetilde{g} = \nu\big(\mathrm{d}y^1 \otimes \mathrm{d}y^1 + \mathrm{d}y^2 \otimes \mathrm{d}y^2\big).$$

That is, $\widetilde{g}_{11} = \widetilde{g}_{22} = \nu$, where $\widetilde{g}_{ij}$ are the components of g in the new coordinate system; but the change of metric is

$$g_{ij} = \sum_{k,l} \widetilde{g}_{kl} \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j},$$

that is,

$$g_{11} = \widetilde{g}_{11}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + \widetilde{g}_{22}\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} = \left(\frac{\partial y^1}{\partial x^1}\right)^2 \widetilde{g}_{11} = \lambda\big(x^1\big)\widetilde{g}_{11},$$

$$g_{22} = \widetilde{g}_{11}\frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} + \widetilde{g}_{22}\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2} = \left(\frac{\partial y^2}{\partial x^2}\right)^2 \widetilde{g}_{22} = \mu\big(x^2\big)\widetilde{g}_{22}.$$

Since $\widetilde{g}_{11} = \widetilde{g}_{22}$ it follows that $\frac{g_{11}}{g_{22}} = \frac{\lambda(x^1)}{\mu(x^2)}$. Since g is positive definite, g_{11} and $\widetilde{g}_{11}$ are positive, hence $\lambda > 0$, and similarly $\mu > 0$. Thus $\lambda(x^1)/\mu(x^2) > 0$. Taking logarithms, we have

$$\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^1}\log\frac{g_{11}}{g_{22}} = \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^1}\big\{\log\lambda\big(x^1\big) - \log\mu\big(x^2\big)\big\} = 0.$$

Conversely, if $X_2X_1(\log\frac{g_{11}}{g_{22}}) = 0$, then

$$\log\frac{g_{11}}{g_{22}} = \varphi\big(x^1\big) - \psi\big(x^2\big),$$

for some functions φ, ψ, thus $\frac{g_{11}}{g_{22}} = \frac{\mathrm{e}^{\varphi(x^1)}}{\mathrm{e}^{\psi(x^2)}}$. We define

$$Y_1 = \frac{1}{\sqrt{\mathrm{e}^{\varphi(x^1)}}}X_1, \qquad Y_2 = \frac{1}{\sqrt{\mathrm{e}^{\psi(x^2)}}}X_2,$$

and coordinates y^1, y^2 such that $\frac{\partial}{\partial y^1} = Y_1$, $\frac{\partial}{\partial y^2} = Y_2$, or equivalently,

$$\mathrm{d}y^1 = \sqrt{\mathrm{e}^{\varphi(x^1)}}\,\mathrm{d}x^1, \qquad \mathrm{d}y^2 = \sqrt{\mathrm{e}^{\psi(x^2)}}\,\mathrm{d}x^2.$$

The change of coordinates is possible, as the determinant of the Jacobian matrix is

$$\frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} = \frac{1}{\sqrt{\mathrm{e}^{\varphi(x^1)+\psi(x^2)}}} \neq 0.$$

In the new coordinates, the metric $\widetilde{g}$ is given by

$$\begin{aligned}
\widetilde{g}_{11} &= g_{11}\frac{\partial x^1}{\partial y^1}\frac{\partial x^1}{\partial y^1} + g_{22}\frac{\partial x^2}{\partial y^1}\frac{\partial x^2}{\partial y^1} = g_{11}\frac{1}{\mathrm{e}^{\varphi(x^1)}},\\
\widetilde{g}_{22} &= g_{11}\frac{\partial x^1}{\partial y^2}\frac{\partial x^1}{\partial y^2} + g_{22}\frac{\partial x^2}{\partial y^2}\frac{\partial x^2}{\partial y^2} = g_{22}\frac{1}{\mathrm{e}^{\psi(x^2)}},\\
\widetilde{g}_{12} &= \widetilde{g}_{21} = 0.
\end{aligned}$$

Hence

$$\frac{\widetilde{g}_{11}}{\widetilde{g}_{22}} = \frac{g_{11}}{g_{22}}\frac{\mathrm{e}^{\psi(x^2)}}{\mathrm{e}^{\varphi(x^1)}} = 1.$$

Thus y^1, y^2 are isothermal coordinates, with the same coordinate curves (as images) as x^1, x^2.

Problem 6.42 Write the line element of $\mathbb{R}^3 \setminus \{0\}$ in spherical coordinates, and identify $\mathbb{R}^3 \setminus \{0\}$ as a *warped product*.

Solution With the parametrisation (see Remark 1.4) in the spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \qquad r\in\mathbb{R}^+,\ \theta\in(0,\pi),\ \varphi\in(0,2\pi), \\ z &= r\cos\theta, \end{aligned}$$

we get

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 = \big(\mathrm{d}(r\sin\theta\cos\varphi)\big)^2 + \big(\mathrm{d}(r\sin\theta\sin\varphi)\big)^2 + \big(\mathrm{d}(r\cos\theta)\big)^2 \\ &= \mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\big). \end{aligned}$$

Moreover, we have the diffeomorphism

$$\mathbb{R}^3\setminus\{0\}\to\mathbb{R}^+\times S^2, \qquad v\mapsto\left(|v|,\frac{v}{|v|}\right),$$

and since for $r=1$, $\mathrm{d}s^2$ furnishes the line element on S^2, we have, with the notation as in Definition 6.2,

$$\mathbb{R}^3\setminus\{0\}\cong\mathbb{R}^+\times_r S^2,$$

where $\cong$, means "isometric to."

Problem 6.43 (The Round Metric on S^n) Let $\varphi_n\colon[-\frac{\pi}{2},\frac{\pi}{2}]^{n-1}\times[-\pi,\pi]\to\mathbb{R}^{n+1}$ be the map defined by the equations:

$$\begin{cases} x^1 = \sin\theta^1, \\ x^i = (\prod_{j=1}^{i-1}\cos\theta^j)\sin\theta^i, \qquad i=2,\dots,n, \\ x^{n+1} = \prod_{j=1}^{n}\cos\theta^j, \end{cases} \tag{$\star$}$$

with $-\frac{\pi}{2}\leqslant\theta^i\leqslant\frac{\pi}{2}$, $i=2,\dots,n$; $-\pi\leqslant\theta^n\leqslant\pi$.

Prove:

(i) $\operatorname{im}\varphi_n = S^n$.
(ii) The restriction of φ_n to $(-\frac{\pi}{2},\frac{\pi}{2})^n$ is a diffeomorphism onto an open subset of the sphere.
(iii) If $g^{(n)}=\varphi_n^*((\mathrm{d}x^1)^2+\cdots+(\mathrm{d}x^{n+1})^2)$, then

$$g^{(n)} = \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\cos^2\theta^j\right)\big(\mathrm{d}\theta^i\big)^2, \quad \forall n\geqslant 1,$$

with $\prod_{j=1}^{k}\cos^2\theta^j = 1$ for $k<1$.

Solution

(i) Let $\varphi_n = (\varphi_n^1, \dots, \varphi_n^{n+1})$ be the components of φ_n. From the very definition of this map it follows that

$$\begin{cases} \varphi_n^i = \varphi_{n-1}^i, & i = 1, \dots, n-1, \\ \varphi_n^n = \varphi_{n-1}^n \sin\theta^n, \\ \varphi_n^{n+1} = \varphi_{n-1}^n \cos\theta^n. \end{cases} \tag{$\star\star$}$$

These formulas show, by induction on n, that $\operatorname{im}\varphi_n = S^n$, taking into account that for $n = 1$ we have $\varphi_1(\theta^1) = (\sin\theta^1, \cos\theta^1)$ and hence the statement holds obviously in this case.

(ii) From the formulas ($\star$) we obtain

$$\frac{\partial(x^1, \dots, x^n)}{\partial(\theta^1, \dots, \theta^n)} = \begin{vmatrix} \cos\theta^1 & 0 & 0 & \cdots & 0 \\ * & \cos\theta^1\cos\theta^2 & 0 & \cdots & 0 \\ * & * & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \prod_{j=1}^n \cos\theta^j \end{vmatrix}$$

$$= \cos\theta^1 \cdot \left(\cos\theta^1 \cos\theta^2\right) \cdots \left(\prod_{j=1}^n \cos\theta^j\right)$$

$$= \cos^n\theta^1 \cdot \cos^{n-1}\theta^2 \cdot \cdots \cdot \cos\theta^n.$$

Hence on the open subset $(-\frac{\pi}{2}, \frac{\pi}{2})^{n-1} \times ((-\pi, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi))$ we have

$$\frac{\partial(x^1, \dots, x^n)}{\partial(\theta^1, \dots, \theta^n)} \neq 0.$$

Moreover, φ_n is injective on $(-\frac{\pi}{2}, \frac{\pi}{2})^{n-1} \times (-\pi, \pi)$, for $\varphi_n(\theta) = \varphi_n(\theta')$, with $\theta = (\theta^1, \dots, \theta^n)$, $\theta' = (\theta'^1, \dots, \theta'^n)$, means according to ($\star\star$):

$$\varphi_n^i(\theta) = \varphi_n^i\big(\theta'\big), \quad i = 1, \dots, n-1, \tag{$\dagger$}$$

$$\varphi_{n-1}^n(\theta) \sin\theta^n = \varphi_{n-1}^n\big(\theta'\big) \sin\theta'^n, \tag{$\dagger\dagger$}$$

$$\varphi_{n-1}^n(\theta) \cos\theta^n = \varphi_{n-1}^n\big(\theta'\big) \cos\theta'^n. \tag{$\dagger\dagger\dagger$}$$

As $\varphi_{n-1}^n(\theta) > 0$, $\varphi_{n-1}^n(\theta') > 0$, from equations ($\dagger\dagger$)–($\dagger\dagger\dagger$) we obtain $\varphi_{n-1}^n(\theta) = \varphi_{n-1}^n(\theta')$; hence $\theta^n = \theta'^n$, and proceeding by recurrence on n, from equations ($\star$) we conclude that $\theta = \theta'$.

(iii) We have $g^{(1)} = (\mathrm{d}\theta^1)^2$ obviously. Hence the formula in the statement of (iii) holds true in the case $n = 1$. Assume $n \geqslant 2$. We have

$$\begin{aligned}
g^{(n)} &= \left(\mathrm{d}\varphi_n^1\right)^2 + \cdots + \left(\mathrm{d}\varphi_n^{n-1}\right)^2 + \left(\mathrm{d}\varphi_n^n\right)^2 + \left(\mathrm{d}\varphi_n^{n+1}\right)^2 \\
&= \left(\mathrm{d}\varphi_{n-1}^1\right)^2 + \cdots + \left(\mathrm{d}\varphi_{n-1}^{n-1}\right)^2 + \left(\sin\theta^n\,\mathrm{d}\varphi_{n-1}^n + \varphi_{n-1}^n\cos\theta^n\,\mathrm{d}\theta^n\right)^2 \\
&\quad + \left(\cos\theta^n\,\mathrm{d}\varphi_{n-1}^n - \varphi_{n-1}^n\sin\theta^n\,\mathrm{d}\theta^n\right)^2 \\
&= \left(\mathrm{d}\varphi_{n-1}^1\right)^2 + \cdots + \left(\mathrm{d}\varphi_{n-1}^{n-1}\right)^2 + \left(\mathrm{d}\varphi_{n-1}^n\right)^2 + \left(\varphi_{n-1}^n\right)^2\left(\mathrm{d}\theta^n\right)^2 \\
&= \varphi_{n-1}^*\left(\left(\mathrm{d}x^1\right)^2 + \cdots + \left(\mathrm{d}x^n\right)^2\right) + \left(\varphi_{n-1}^n\right)^2\left(\mathrm{d}\theta^n\right)^2 \\
&= g^{(n-1)} + \left(\varphi_{n-1}^n\right)^2\left(\mathrm{d}\theta^n\right)^2 \\
&= \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1}\cos^2\theta^j\right)\left(\mathrm{d}\theta^i\right)^2 + \left(\prod_{j=1}^{n-1}\cos^2\theta^j\right)\left(\mathrm{d}\theta^n\right)^2 \\
&\quad \text{(by the induction hypothesis)} \\
&= \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\cos^2\theta^j\right)\left(\mathrm{d}\theta^i\right)^2.
\end{aligned}$$

6.3 Riemannian Connections

Problem 6.44 Let M be an n-dimensional Riemannian manifold, and Y a vector field defined along a curve $\gamma(t)$ in M. The covariant derivative $DY(t)/\mathrm{d}t$ of $Y(t) = Y_{\gamma(t)}$ is defined by

$$\frac{DY(t)}{\mathrm{d}t} = \nabla_{\mathrm{d}\gamma/\mathrm{d}t}Y,$$

where ∇ denotes the Levi-Civita connection of the metric. If Y is given by $Y(t) = Y^i(t)(\partial/\partial x^i)_{\gamma(t)}$ in local coordinates x^i and $\gamma(t)$ is given by $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, then

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \sum_{i=1}^{n}\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\frac{\partial}{\partial x^i}$$

and

$$\frac{DY(t)}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\mathrm{d}Y^i(t)}{\mathrm{d}t} + \sum_{j,k=1}^{n}\Gamma_{jk}^i\frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t}Y^k(t)\right)\frac{\partial}{\partial x^i}, \tag{$\star$}$$

where Γ_{jk}^i are the Christoffel symbols of ∇ with respect to that local coordinate frame, given by

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial}{\partial x^k}.$$

Let U be an open neighbourhood of (u_0, v_0) in $\mathbb{R}^2$ with coordinates (u, v) and let $f: U \to M$ be a C^∞ map. Consider the two tangent vector fields $\partial f/\partial u$ and $\partial f/\partial v$ to the curves $v = \text{const}$ and $u = \text{const}$, respectively, and let $DX/\partial u$, $DX/\partial v$ be the covariant derivatives of any vector field X along these respective curves.

(i) Using the previous expression ($\star$) for DY/dt, prove by direct computation that

$$\frac{D}{\partial v}\frac{\partial f}{\partial u} = \frac{D}{\partial u}\frac{\partial f}{\partial v}.$$

(ii) Which property of the Levi-Civita connection does the equality in (i) correspond to?

The relevant theory is developed, for instance, in Hicks [16].

Solution

(i) We have

$$\frac{\partial f}{\partial u} = f_* \frac{\partial}{\partial u}, \qquad \frac{\partial f}{\partial v} = f_* \frac{\partial}{\partial v},$$

and

$$\frac{D}{\partial v}\frac{\partial f}{\partial u} = \text{covariant derivative along } t \mapsto f(u, t) \text{ of } \frac{\partial f}{\partial u},$$
$$\frac{D}{\partial u}\frac{\partial f}{\partial v} = \text{covariant derivative along } t \mapsto f(t, v) \text{ of } \frac{\partial f}{\partial v}.$$

Let

$$\frac{\partial f}{\partial u} = \sum_{i=1}^n \lambda^i \left(\frac{\partial}{\partial x^i} \circ f\right), \qquad \frac{\partial f}{\partial v} = \sum_{i=1}^n \mu^i \left(\frac{\partial}{\partial x^i} \circ f\right),$$

where λ^i, μ^i are functions on U.

Then,

$$\frac{D}{\partial v}\frac{\partial f}{\partial u} = \nabla_{\frac{\partial}{\partial v}} \sum_{i=1}^n \lambda^i \left(\frac{\partial}{\partial x^i} \circ f\right) = \sum_{i=1}^n \left(\frac{\partial \lambda^i}{\partial v} + \sum_{j,k=1}^n (\Gamma^i_{jk} \circ f)\mu^j \lambda^k\right)\left(\frac{\partial}{\partial x^i} \circ f\right).$$

Similarly,

$$\frac{D}{\partial u}\frac{\partial f}{\partial v} = \sum_{i=1}^n \left(\frac{\partial \mu^i}{\partial u} + \sum_{j,k=1}^n (\Gamma^i_{jk} \circ f)\lambda^j \mu^k\right)\left(\frac{\partial}{\partial x^i} \circ f\right).$$

But since

$$\frac{\partial f}{\partial u} = \sum_{i=1}^n \lambda^i \left(\frac{\partial}{\partial x^i} \circ f\right) = f_* \circ \frac{\partial}{\partial u} = \sum_{i=1}^n \frac{\partial (x^i \circ f)}{\partial u}\left(\frac{\partial}{\partial x^i} \circ f\right),$$

we have $\lambda^i = \frac{\partial(x^i \circ f)}{\partial u}$. Hence

$$\frac{\partial \lambda^i}{\partial v} = \frac{\partial^2 (x^i \circ f)}{\partial v\, \partial u} = \frac{\partial \mu^i}{\partial u}.$$

Thus, as $\Gamma^i_{jk} = \Gamma^i_{kj}$, the claim proceeds.

(ii) The property used is that ∇ is torsionless. The converse is immediate from the above local expressions of $\frac{D}{\partial v}\frac{\partial f}{\partial u}$ and $\frac{D}{\partial u}\frac{\partial f}{\partial v}$.

Problem 6.45 Let (M, g) be a Riemannian manifold. Prove that for $X \in \mathfrak{X}(M)$ one has

$$|L_X g|^2 = 2|\nabla X|^2 + 2\,\mathrm{tr}(\nabla X \circ \nabla X) \in C^\infty M,$$

with respect to the extension of g to a metric on $T^*M \otimes T^*M$, where:

(a) $|L_X g|$ denotes the length of the Lie derivative $L_X g$.
(b) ∇ denotes the Levi-Civita connection of g.
(c) $|\nabla X|^2 = \sum_i g(\nabla_{e_i} X, \nabla_{e_i} X)$, where (e_i) is a g-orthonormal frame on a neighbourhood of $p \in M$.
(d) $\mathrm{tr}(\nabla X \circ \nabla X) = \sum_i g(\nabla_{\nabla_{e_i} X} X, e_i)$.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution The extension of g to a metric on the fibre bundle $T^*M \otimes T^*M$ is the map

$$\langle \cdot, \cdot \rangle \colon \bigl(\otimes^2 T^*M\bigr) \otimes \bigl(\otimes^2 T^*M\bigr) \to \mathbb{R}$$

defined by

$$\langle \eta_1 \otimes \eta_2, \mu_1 \otimes \mu_2 \rangle = g\bigl(\eta_1^\sharp, \mu_1^\sharp\bigr) g\bigl(\eta_2^\sharp, \mu_2^\sharp\bigr), \qquad \eta_1, \eta_2, \mu_1, \mu_2 \in T^*M.$$

Given a g-orthonormal basis $\{e_i\}$ at $p \in M$, we have $g(X, Y) = \sum_i g(X, e_i) g(Y, e_i)$. Hence for any $\eta_1 \otimes \eta_2 \in T^*M \otimes T^*M$ we have

$$\begin{aligned} \langle \eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2 \rangle &= \sum_{i,j} g\bigl(\eta_1^\sharp, e_i\bigr)^2 g\bigl(\eta_2^\sharp, e_j\bigr)^2 = \sum_{i,j} \eta_1(e_i)^2 \eta_2(e_j)^2 \\ &= \sum_{i,j} \bigl((\eta_1 \otimes \eta_2)(e_i, e_j)\bigr)^2, \end{aligned}$$

so that for any $h \in T^*M \otimes T^*M$ one has $\langle h, h \rangle = \sum_{i,j} (h(e_i, e_j))^2$.

In particular, the length of the Lie derivative of g with respect to a local orthonormal frame (e_i) in a neighbourhood of $p \in M$ is given by

$$|L_X g|^2 = \sum_{i,j} \bigl((L_X g)(e_i, e_j)\bigr)^2.$$

Hence

$$
\begin{aligned}
|L_X g|^2 &= \sum_{i,j} \big((L_X g)(e_i, e_j)\big)^2 = \sum_{i,j} \big(L_X g(e_i, e_j) - g(L_X e_i, e_j) - g(e_i, L_X e_j)\big)^2 \\
&= \sum_{i,j} \big(X\delta_{ij} - g(\nabla_X e_i, e_j) + g(\nabla_{e_i} X, e_j) - g(e_i, \nabla_X e_j) + g(e_i, \nabla_{e_j} X)\big)^2 \\
&= \sum_{i,j} \big(g(\nabla_{e_i} X, e_j) + g(e_i, \nabla_{e_j} X)\big)^2 \\
&= 2\sum_{i,j} \big(g(\nabla_{e_j} X, e_i) g(\nabla_{e_j} X, e_i) + g(\nabla_{e_i} X, e_j) g(\nabla_{e_j} X, e_i)\big) \\
&= 2\sum_i \Big(\sum_j g\big(\nabla_{e_j} X, g(\nabla_{e_j} X, e_i) e_i\big) + g(\nabla_{\sum_j g(\nabla_{e_i} X, e_j) e_j} X, e_i)\Big) \\
&= 2\sum_i \big(g(\nabla_{e_i} X, \nabla_{e_i} X) + g(\nabla_{\nabla_{e_i} X} X, e_i)\big) = 2\big(|\nabla X|\big)^2 + 2\,\mathrm{tr}(\nabla X \circ \nabla X).
\end{aligned}
$$

Problem 6.46 Let g be a *Hermitian metric* on an almost complex manifold (M, J), i.e. a Riemannian metric satisfying

$$g(JX, JY) = g(X, Y), \qquad X, Y \in \mathfrak{X}(M).$$

We define a tensor field F of type $(0, 2)$ on M by

$$F(X, Y) = g(X, JY), \qquad X, Y \in \mathfrak{X}(M).$$

Prove:

(i) F is skew-symmetric (thus it is a 2-form on M, called the *fundamental* 2*-form* of the almost Hermitian manifold (M, g, J)).
(ii) F is invariant by J, that is, $F(JX, JY) = F(X, Y)$.
Suppose, moreover, that ∇ is any linear connection such that $\nabla g = 0$. Then prove:
(iii) $(\nabla_X F)(Y, Z) = g(Y, (\nabla_X J)Z)$.
(iv) $g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) = 0$.

Solution

(i)

$$
\begin{aligned}
F(Y, X) &= g(Y, JX) = g\big(JY, J^2 X\big) = -g(JY, X) = -g(X, JY) \\
&= -F(X, Y).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
F(JX, JY) &= g\big(JX, J^2 Y\big) = -g(JX, Y) = -g\big(J^2 X, JY\big) = g(X, JY) \\
&= F(X, Y).
\end{aligned}
$$

(iii)

$$\begin{aligned}
\nabla_X F(Y,Z) &= (\nabla_X F)(Y,Z) + F(\nabla_X Y, Z) + F(Y, \nabla_X Z),\\
\nabla_X g(Y,JZ) &= (\nabla_X g)(Y,JZ) + g(\nabla_X Y, JZ) + g(Y, \nabla_X JZ)\\
&= g(\nabla_X Y, JZ) + g\big(Y, (\nabla_X J)Z\big) + g(Y, J\nabla_X Z)\\
&= F(\nabla_X Y, Z) + g\big(Y, (\nabla_X J)Z\big) + F(Y, \nabla_X Z).
\end{aligned}$$

Thus, $(\nabla_X F)(Y,Z) = g(Y, (\nabla_X J)Z)$.

(iv) Since F is skew-symmetric, $\nabla_X F$ is also skew-symmetric. In fact,

$$\begin{aligned}
(\nabla_X F)(Y,Z) &= \nabla_X F(Y,Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z)\\
&= -\nabla_X F(Z,Y) + F(Z, \nabla_X Y) + F(\nabla_X Z, Y)\\
&= -(\nabla_X F)(Z,Y).
\end{aligned}$$

Thus, by (iii),

$$\begin{aligned}
g\big((\nabla_X J)Y, Z\big) &= g\big(Z, (\nabla_X J)Y\big) = (\nabla_X F)(Z,Y) = -(\nabla_X F)(Y,Z)\\
&= -g\big(Y, (\nabla_X J)Z\big).
\end{aligned}$$

Problem 6.47 The *complex tangent space* $T_p^c M$ of a manifold M at p is the complexification of the tangent space $T_p M$, i.e.

$$T_p^c M = T_p M \otimes_{\mathbb{R}} \mathbb{C}.$$

The complex conjugation in $T_p^c M$ is the real linear endomorphism defined by

$$Z = X + \mathrm{i}Y \mapsto \bar{Z} = X - \mathrm{i}Y, \quad X, Y \in T_p M.$$

Let J be an almost complex structure on a manifold M of dimension $2n$. Then J can be uniquely extended to a complex linear endomorphism of each complex space $T_p^c M$, $p \in M$, and the extended map, also denoted by J, satisfies the equation

$$J^2 = -I.$$

Set

$$T_p^{1,0} M = \big\{Z \in T_p^c M : JZ = \mathrm{i}Z\big\}, \qquad T_p^{0,1} M = \big\{Z \in T_p^c M : JZ = -\mathrm{i}Z\big\}.$$

Prove that for any $p \in M$:

(i) There exist elements $X_1, \dots, X_n \in T_p M$ such that

$$\{X_1, \dots, X_n, JX_1, \dots, JX_n\}$$

is a basis of $T_p M$.

(ii) $T_p^{1,0}M \oplus T_p^{0,1}M = T_p^c M$ (complex vector space direct sum).

(iii)

$$T_p^{1,0}M = \{X - \mathrm{i}JX, X \in T_pM\}, \qquad T_p^{0,1}M = \{X + \mathrm{i}JX, X \in T_pM\}.$$

(iv) The complex conjugation in $T_p^c M$ defines a real linear isomorphism between $T_p^{1,0}M$ and $T_p^{0,1}M$.

Solution

(i) As we showed above (see the proof of Problem 5.52), the linear map $J_p \colon T_pM \to T_pM$ defines a complex structure in the $2n$-dimensional (real) vector space T_pM. Let $X_1, \dots, X_n$ be a basis of T_pM as a complex vector space. Then the set

$$\{X_1, \dots, X_n, JX_1, \dots, JX_n\}$$

is a basis for T_pM as a real vector space.

(ii) By definition, the set $\{X_1, \dots, X_n, JX_1, \dots, JX_n\}$ is a basis of the complex space $T_p^c M$. Therefore, the two sets of vectors

$$\{X_1 - \mathrm{i}\,JX_1, \dots, X_n - \mathrm{i}\,JX_n\}, \qquad \{X_1 + \mathrm{i}\,JX_1, \dots, X_n + \mathrm{i}\,JX_n\}, \qquad (\star)$$

form a basis of $T_p^c M$.

(iii) Denote by $V^{1,0}$ and $V^{0,1}$ the (complex) subspaces of $T_p^c M$ spanned by the sets in ($\star$). By definition $J_p^2 = -I$. The eigenvalues of J_p are therefore i and $-$i. It is easy to verify that $V^{1,0} \subset T_p^{1,0}M$ and $V^{0,1} \subset T_p^{0,1}M$. But $V^{1,0} \oplus V^{0,1} = T_p^c M$ and by definition $T_p^{1,0}M \cap T_p^{0,1}M = 0$, So that $V^{1,0} = T_p^{1,0}M$ and $V^{0,1} = T_p^{0,1}M$. Then (iii) follows noting that

$$\mathrm{i}(X_j - \mathrm{i}\,JX_j) = JX_j - \mathrm{i}\,J(JX_j), \qquad \mathrm{i}(X_j + \mathrm{i}JX_j) = (-JX_j) + \mathrm{i}J(-JX_j).$$

(iv) It is now evident.

Problem 6.48 Let $\mathscr{F}$ be a smooth complex distribution on a manifold M, i.e. $\mathscr{F}_p \subset T_p^c M$, such that

$$\mathscr{F}_p + \bar{\mathscr{F}}_p = T_p^c M, \qquad \mathscr{F}_p \cap \bar{\mathscr{F}}_p = 0, \quad p \in M.$$

Prove that there exists a unique almost complex structure J on M such that

$$\mathscr{F}_p = T_p^{0,1}M = \{X + \mathrm{i}JX; X \in T_pM\}$$

for each $p \in M$.

Solution From the formula $T_p^c M = \mathscr{F}_p \oplus \bar{\mathscr{F}}_p$ it follows that

$$\dim_{\mathbb{C}} \mathscr{F}_p = \dim_{\mathbb{C}} \bar{\mathscr{F}}_p = \big(\dim_{\mathbb{C}} T_p^c M\big)/2 = n.$$

Let $\{Z_1, \ldots, Z_n\}$ be a basis of the complex space $\mathscr{F}_p \subset T_p^c M$, with $Z_j = X_j + \mathrm{i}Y_j$, $X_j, Y_j \in T_p M$. Since $T_p^c M = \mathscr{F}_p \oplus \bar{\mathscr{F}}_p$, the vectors $\{X_j, Y_j\}$, $j = 1, \ldots, n$, span the complex space $T_p^c M$. These vectors form a basis of $T_p^c M$ because $2n = \dim_{\mathbb{C}} T_p^c M$. It is easy to verify that the map J such that $J(X_j) = Y_j$ and $J(Y_j) = -X_j$ defines an almost complex structure on M with the space $T_p^{0,1} M = \mathscr{F}_p$.

Problem 6.49 Let (M, Ω) be an almost symplectic manifold and let $\mathscr{F}$ be a complex distribution on M. Suppose that for each $p \in M$:

(a) $\mathscr{F}_p + \bar{\mathscr{F}}_p = T_p^c M$.
(b) $\mathscr{F}_p \cap \bar{\mathscr{F}}_p = 0$.
(c) $\Omega(\mathscr{F}_p, \mathscr{F}_p) = 0$.
(d) $-\mathrm{i}\Omega(Z, \bar{Z}) > 0$, $Z \in \mathscr{F}_p \setminus \{0\}$.

Prove:

1. There exists a unique almost Hermitian structure (J, g) on M, where J is an almost complex structure and g is a Riemannian metric, such that
$$\mathscr{F} = T^{0,1} M$$
and Ω is the fundamental form of g.
2. If, in addition, the form Ω is closed and the complex distribution $\mathscr{F}$ is involutive, i.e.
$$[\tilde{X}, \tilde{Y}] \in \mathscr{F}, \quad \tilde{X}, \tilde{Y} \in \mathscr{F},$$
then (J, g) is a Kähler structure on M.

Solution

1. By (a) and (b), there exists an almost complex structure $J\colon TM \to TM$ such that $\mathscr{F}_p = \{X + \mathrm{i}JX, X \in T_p M\}$ (see Problem 6.48). Put by definition
$$g(X, Y) = \Omega(JX, Y)$$
for any $X, Y \in T_p M$. But by (c), $\Omega(X + \mathrm{i}JX, Y + \mathrm{i}JY) = 0$. Therefore,
$$\big[\Omega(X, Y) - \Omega(JX, JY)\big] + \mathrm{i}\big[\Omega(X, JY) + \Omega(JX, Y)\big] = 0, \quad X, Y \in T_p M,$$
and, consequently,
$$g(X, Y) = g(JX, JY) \quad \text{and} \quad g(X, Y) = g(Y, X)$$
because the form Ω is skew-symmetric. Let us prove that the symmetric tensor g is positive definite. Indeed, by (d),
$$-\mathrm{i}\Omega(X + \mathrm{i}\,JX, X - \mathrm{i}\,JX) = 2\Omega(JX, X) = 2g(X, X) > 0 \quad \text{if } X \neq 0.$$
By definition the form Ω is the fundamental form of (J, g).

2. It is sufficient to note that according to Newlander–Nirenberg's Theorem the almost complex structure is a complex structure if and only if the distribution $T^{0,1}M$ of (0, 1)-vectors is involutive. In this case, by definition, (J, g) is a Kähler structure if $\mathrm{d}\Omega = 0$ (see Definitions 5.10). Remark that it can be proved that this is equivalent to $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g.

Problem 6.50 Let (M, J, g) be an almost Hermitian manifold with fundamental form Ω. Let $\mathscr{F}$ be the distribution $T^{0,1}M$ of (0, 1)-vectors of J.

Prove that for each $p \in M$:

(i) $\mathscr{F}_p + \bar{\mathscr{F}}_p = T^c_p M$.
(ii) $\mathscr{F}_p \cap \bar{\mathscr{F}}_p = 0$.
(iii) $\Omega(\mathscr{F}_p, \mathscr{F}_p) = 0$.
(iv) $-\mathrm{i}\Omega(Z, \bar{Z}) > 0$ for all $Z \in \mathscr{F}_p \setminus \{0\}$.

Solution Since J is an almost complex structure, the distribution $\mathscr{F} = T^{0,1}M$ satisfies conditions (i) and (ii) (see Problem 6.47(ii) and (iv)).

(iii) By definition $\Omega(X, Y) = g(X, JY)$, $X, Y \in T_pM$. But g is a Hermitian metric, i.e. $g(X, Y) = g(JX, JY)$. Since $J^2 = -I$, we obtain that $g(JX, Y) = -g(X, JY)$. In particular, $g(JX, X) = 0$ because g is symmetric. Therefore,

$$\begin{aligned}\Omega(X + \mathrm{i}JX, Y + \mathrm{i}JY) &= g(X + \mathrm{i}JX, JY - \mathrm{i}Y)\\ &= \big[g(X, JY) + g(JX, Y)\big] + \mathrm{i}\big[g(JX, JY) - g(X, Y)\big]\\ &= 0.\end{aligned}$$

(iv) Similarly,

$$\begin{aligned}2g(X, X) &= \big[g(X, X) + g(JX, JX)\big] - \mathrm{i}\big[g(X, JX) - g(JX, X)\big]\\ &= -\mathrm{i}g(X + \mathrm{i}\,JX, JX + \mathrm{i}X)\\ &= -\mathrm{i}\Omega(X + \mathrm{i}\,JX, X - \mathrm{i}JX) > 0 \quad \text{if } X \neq 0.\end{aligned}$$

Problem 6.51 Denote by $(x^1, \dots, x^n, y^1, \dots, y^n)$ the usual Cartesian coordinates of the vector (x, y) of the space $M = \mathbb{R}^n \times \mathbb{R}^n$, on which we consider:

(a) The 2-form $\Omega = \sum_{j=1}^n \mathrm{d}y^j \wedge \mathrm{d}x^j$.
(b) Three smooth operator-functions R, S and J given by

$$R\colon M \to \mathrm{Aut}\big(\mathbb{R}^n\big),\ p \mapsto R_p, \qquad S\colon M \to \mathrm{End}\big(\mathbb{R}^n\big),\ p \mapsto S_p,$$

$$J\colon M \to \mathrm{End}\big(\mathbb{R}^n \times \mathbb{R}^n\big),\ p \mapsto J_p, \qquad J_p = \begin{pmatrix} -R_p^{-1}S_p & -R_p^{-1} \\ R_p + S_pR_p^{-1}S_p & S_pR_p^{-1}\end{pmatrix}.$$

Prove:

1. The map $p \mapsto J_p$, $p \in M$, where each endomorphism J_p is considered as an endomorphism of the tangent space $T_pM \cong \mathbb{R}^n \times \mathbb{R}^n$, defines an almost complex structure (denoted again by) J on M.
2. The almost complex structure J is an almost Hermitian structure with fundamental form Ω if and only if for all $p \in M$ the endomorphisms R_p, S_p are symmetric and the endomorphism R_p is positive definite (with respect to the canonical scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$).

Hint Show that the complex distribution generated by the vector fields

$$\sum_{j=1}^n \left(v^j \frac{\partial}{\partial x^j} + \mathrm{i} \sum_{k=1}^n (R_p + \mathrm{i}S_p)_{jk} v^k \frac{\partial}{\partial y^j} \right), \qquad \left(v^1, \dots, v^n\right) \in \mathbb{C}^n,$$

is the distribution of (0, 1)-vectors of J.

Solution

1. It is easy to verify that $J_p^2 = -I$ for each $p \in M$. In fact, with an obvious notation we have:

$$\begin{aligned}
\left(J_p^2\right)_{11} &= R_p^{-1} S_p R_p^{-1} S_p - R_p^{-1}\left(R_p + S_p R_p^{-1} S_p\right) = -I_n, \\
\left(J_p^2\right)_{12} &= R_p^{-1} S_p R_p^{-1} - R_p^{-1} S_p R_p^{-1} = 0, \\
\left(J_p^2\right)_{21} &= -\left(R_p + S_p R_p^{-1} S_p\right) R_p^{-1} S_p + S_p R_p^{-1}\left(R_p + S_p R_p^{-1} S_p\right) = 0, \\
\left(J_p^2\right)_{22} &= -\left(R_p + S_p R_p^{-1} S_p\right) R_p^{-1} + S_p R_p^{-1} S_p R_p^{-1} = -I_n.
\end{aligned}$$

   The structure J is smooth because so is the map $p \mapsto J_p$.
2. Let us describe the distribution of (0, 1)-vectors of the almost complex structure J. To simplify the notation denote by $vX + wY$, where $v = (v^1, \dots, v^n)$, $w = (w^1, \dots, w^n) \in \mathbb{C}^n$, the vector

$$\sum_{i=1}^n \left(v^i \frac{\partial}{\partial x^i} + w^i \frac{\partial}{\partial y^i} \right) \in T_p^c M.$$

   The space $T_p^{0,1}M$ is the set

$$\begin{aligned}
&\big\{(vX + wY) + \mathrm{i}\big[\left(-R_p^{-1} S_p v - R_p^{-1} w\right) X \\
&\quad + \left(R_p v + S_p R_p^{-1} S_p v + S_p R_p^{-1} w\right) Y\big] : v, w \in \mathbb{R}^n\big\}.
\end{aligned}$$

   Since this complex space is n-dimensional, the vectors (with $v = 0$)

$$\begin{aligned}
&\left\{wY + \mathrm{i}\left[-\left(R_p^{-1} w\right) X + \left(S_p R_p^{-1} w\right) Y\right] : w \in \mathbb{C}^n\right\} \\
&= \left\{wX + \mathrm{i}(R_p w + \mathrm{i} S_p w) Y : w \in \mathbb{C}^n\right\}
\end{aligned}$$

span this space. Now to prove part 2, it is sufficient to show that $\Omega(T_p^{0,1}M, T_p^{0,1}M) = 0$ and $-\mathrm{i}\Omega(Z, \bar{Z}) > 0$ for all $Z \in T_p^{0,1}M \setminus \{0\}$ (see Problems 6.49 and 6.50).

Note that by definition

$$\Omega\bigl(vX + wY, v'X + w'Y\bigr) = \langle w, v'\rangle - \langle v, w'\rangle,$$

where for $v, w \in \mathbb{C}^n$ we put $\langle w, v\rangle = \sum_{j=1}^n w^j v^j$. Denote by P_p the operator $R_p + \mathrm{i}S_p$. Then let a be defined by

$$a\bigl(w, w'\bigr) = \Omega\bigl(wX + \mathrm{i}(P_p w)Y, w'X + \mathrm{i}\bigl(P_p w'\bigr)Y\bigr) = \bigl\langle \mathrm{i}(P_p w), w'\bigr\rangle - \bigl\langle w, \mathrm{i}\bigl(P_p w'\bigr)\bigr\rangle.$$

The function a vanishes for all $w, w' \in \mathbb{C}^n$ if and only if the operators R_p and S_p are symmetric. Similarly, if b is the function defined, for arbitrary vectors $w, w' \in \mathbb{R}^n$, by

$$\begin{aligned}
b\bigl(w, w'\bigr) &= -\mathrm{i}\Omega\bigl(\bigl(w + \mathrm{i}w'\bigr)X + \mathrm{i}\bigl(P_p\bigl(w + \mathrm{i}w'\bigr)\bigr)Y, \bigl(w - \mathrm{i}w'\bigr)X \\
&\qquad - \mathrm{i}\bigl(\bar{P}_p\bigl(w - \mathrm{i}w'\bigr)\bigr)Y\bigr) \\
&= \mathrm{i}\bigl\langle w + \mathrm{i}w', -\mathrm{i}\bar{P}_w\bigl(w - \mathrm{i}w'\bigr)\bigr\rangle - \mathrm{i}\bigl\langle w - \mathrm{i}w', \mathrm{i}P_p\bigl(w + \mathrm{i}w'\bigr)\bigr\rangle \\
&= \bigl(\bigl\langle(\bar{P}_p + P_p)w, w\bigr\rangle + \bigl\langle(\bar{P}_p + P_p)w', w'\bigr\rangle\bigr) \\
&\qquad - \mathrm{i}\bigl(\bigl\langle(P_p - \bar{P}_p)w, w'\bigr\rangle - \bigl\langle(P_p - \bar{P}_p)w', w\bigr\rangle\bigr) \\
&= \bigl(\langle 2R_p w, w\rangle + \bigl\langle 2R_p w', w'\bigr\rangle\bigr) - \mathrm{i}\bigl(\bigl\langle 2\mathrm{i}S_p w, w'\bigr\rangle - \bigl\langle 2\mathrm{i}S_p w', w\bigr\rangle\bigr),
\end{aligned}$$

then, since the operator S_p is symmetric, the function $b(w, w')$ is positive if and only if the symmetric operator R_p is positive, i.e. $\langle R_p w, w\rangle > 0$, $w \in \mathbb{R}^n \setminus \{0\}$.

Problem 6.52 Let (M, Ω) be an almost symplectic manifold. An almost complex structure J on the manifold M is called compatible with the almost symplectic structure Ω, if $g(X, Y) = \Omega(JX, Y)$ is an Hermitian metric on M, i.e.

(a) $\Omega(JX, X) > 0$ for any non-zero tangent vector $X \in T_pM$, $p \in M$.
(b) $\Omega(JX, Y) + \Omega(X, JY) = 0$ for any tangent vectors $X, Y \in T_pM$, $p \in M$.

Prove that on any almost symplectic manifold (M, Ω) there exists an almost complex structure J which is compatible with Ω.

The relevant theory is developed, for instance, in Gromov [14] and Aebisher et al. [1].

Solution Let us choose an arbitrary (smooth) Riemannian metric g_0 on the manifold M (see Problem 6.39). Define linear endomorphisms $A_p \colon T_pM \to T_pM$, $p \in M$, by

$$g_0(X, A_pY) = \Omega(X, Y), \quad X, Y \in T_pM.$$

Then each A_p is skew-symmetric with respect to the metric g_0. Therefore, $-A_p^2$ is symmetric (and non-degenerate) and

$$g_0\big(X, -A_p^2 X\big) = \Omega(X, -A_p X) = \Omega(A_p X, X) = g_0(A_p X, A_p X) > 0$$

provided $X \neq 0$. Thus $-A_p^2$ is a symmetric positive definite operator. Define $B_p = \sqrt{-A_p^2}$ to be the symmetric positive definite square-root of $-A_p^2$. Since the domain $D = \mathbb{C} \setminus (-\infty, 0]$ contains the spectra of all operators $-A_p^2$, $p \in M$, and each of these spectra is compact, this can be done canonically and smoothly by using the Cauchy integral. Indeed, choosing the positive determination of F, that is, the branch $F(z)$ of the square root in D with $F(1) = 1$, we can define a real-analytic function $\tilde{F}\colon S^+ \to S^+$ on the set of all symmetric positive definite matrices by

$$\tilde{F}\colon A_p \mapsto \frac{1}{2\pi i} \int_\Gamma F(\zeta)(\zeta \mathrm{Id} - A_p)^{-1}\, \mathrm{d}\zeta,$$

where Γ is a contour in D enclosing the spectrum of A_p. Clearly, each B_p is invertible and commutes with A_p. Put $J_p = B_p^{-1} A_p$. Then $J_p^2 = -\mathrm{Id}_p$, where $\mathrm{Id}_p\colon T_p M \to T_p M$ stands for the identity map. The operator J_p is antisymmetric with respect to the metric g_0:

$$\begin{aligned} g_0(J_p X, Y) &= g_0\big(B_p^{-1} A_p X, Y\big) = g_0\big(A_p X, B_p^{-1} Y\big) = g_0\big(X, -A_p B_p^{-1} Y\big) \\ &= -g_0(X, J_p Y). \end{aligned}$$

Moreover,

$$\Omega(J_p X, X) = \Omega\big(B_p^{-1} A_p X, X\big) = g_0\big(B_p^{-1} A_p X, A_p X\big) > 0, \quad X \in T_p M \setminus \{0\},$$

because B_p^{-1} is also symmetric and positive definite. Now to prove that the $(1, 1)$-tensor field J is compatible with Ω it is sufficient to remark that

$$\begin{aligned} \Omega(J_p X, Y) &= g_0(J_p X, A_p Y) = g_0\big(B_p^{-1} A_p X, A_p Y\big) = -g_0\big(X, A_p B_p^{-1} A_p Y\big) \\ &= -\Omega(X, J_p Y). \end{aligned}$$

Thus the formula $g(X, Y) = \Omega(JX, Y)$ defines an Hermitian metric g on M. Remark also that this construction of such an almost complex structure J on M depends on a choice of an arbitrary start-up Riemannian metric g_0 on M. After this choice the construction of J becomes canonical.

6.4 Geodesics

Problem 6.53 Consider $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ with the usual metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$ and consider the *distance function* d_g given by

$$d_g\colon M\times M\to\mathbb{R}^+$$
$$(p,q)\mapsto d_g(p,q)=\inf\int_0^1\sqrt{g\big(\gamma'(t),\gamma'(t)\big)}\,\mathrm{d}t,$$

where γ denotes a piecewise C^∞ curve with $\gamma(0)=p$ and $\gamma(1)=q$.

(i) Compute the distance between $p=(-1,0)$ and $q=(1,0)$.
(ii) Is there a geodesic minimizing the distance between p and q?
(iii) Is the topological metric space (M,d_g) complete?
(iv) A Riemannian manifold is said to be *geodesically complete* if every geodesic $\gamma(t)$ is defined for every real value of the parameter t. Is in the present case M geodesically complete?
(v) Find an open neighbourhood U_p for each point $p\in M$, such that for all $q\in U_p$, the distance $d_g(p,q)$ be achieved by a geodesic.

Solution

(i) Let γ_a be the piecewise C^∞ curve obtained as the union of the line segment from $(-1,0)$ to $(0,a)$ and the line segment from $(0,a)$ to $(1,0)$. Since

$$d_g\big((-1,0),(0,a)\big)=d_g\big((0,a),(1,0)\big)=\sqrt{1+a^2},$$

we have

$$d_g\big((-1,0),(1,0)\big)\leqslant\inf_{a\to0}\big\{2\sqrt{1+a^2}\big\}=2.$$

On the other hand, as M is an open subset of $\mathbb{R}^2$, if $d_{\mathbb{R}^2}$ stands for the Euclidean distance, we have

$$d_g\big((-1,0),(1,0)\big)\geqslant d_{\mathbb{R}^2}\big((-1,0),(1,0)\big)=2,$$

thus $d_g(p,q)=2$.
(ii) Since M is an open subset of $\mathbb{R}^2$, the geodesics of M are the ones of $\mathbb{R}^2$ intersecting with M. There is only one geodesic $\gamma_{\mathbb{R}^2}$ in $\mathbb{R}^2$ joining p and q, but $\gamma=\gamma_{\mathbb{R}^2}\cap M$ is not connected, and so the distance is not achieved by a geodesic.
(iii) (M,d_g) is not complete. It is enough to give a counterexample: The sequence $\{(1/n,1/n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d_g) which is not convergent.
(iv) (M,d_g) is not geodesically complete, because none of the lines passing (in $\mathbb{R}^2$) through the origin is a complete geodesic for the Levi-Civita connection. In fact, the geodesics $x=at$, $y=bt$ do define, for $t=0$, no point of M.
(v) Given $p\in M$, take as U_p the open ball $B(p,|p|)$.

Problem 6.54

(i) Find an example of a connected Riemannian manifold (M,g) to show that the property "*Any $p,q\in M$ can be joined by a geodesic whose arc length equals the distance $d_g(p,q)$*" (see Problem 6.53) does not imply that M is complete.

(ii) Find an example of a connected Riemannian manifold to show that a minimal geodesic between two points need not be unique; in fact, there may be infinitely many.

Solution

(i) The open ball

$$M = B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\} \subset (\mathbb{R}^n, g),$$

where g denotes the usual flat metric, and M is equipped with the inherited metric.

(ii) The sphere (S^n, g), g being the usual metric. There exist infinitely many minimal geodesics joining two antipodal points.

Problem 6.55 Consider on $\mathbb{R}^3$ the metric

$$g = (1 + x^2)\,\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{e}^z\,\mathrm{d}z^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection of g.
(ii) Write and solve the differential equations of the geodesics.
(iii) Consider the curve $\gamma(t)$ with equations $x = t$, $y = t$, $z = t$. Obtain the parallel transport of the vector $(a, b, c)_{(0,0,0)}$ along γ.
(iv) Is γ a geodesic?
(v) Calculate two parallel vector fields defined on γ, $X(t)$ and $Y(t)$, such that $g(X(t), Y(t))$ is constant.
(vi) Are there two parallel vector fields defined on γ, $Z(t)$ and $W(t)$, such that $g(Z(t),\ W(t))$ is not constant?

Solution

(i) We have

$$g \equiv \begin{pmatrix} 1+x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^z \end{pmatrix}, \qquad g^{-1} \equiv \begin{pmatrix} 1/(1+x^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-z} \end{pmatrix}.$$

Taking $x^1 = x$, $x^2 = y$, $x^3 = z$, the only non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = \frac{x}{1+x^2}, \qquad \Gamma^3_{33} = \frac{1}{2}. \tag{$\star$}$$

(ii) The differential equations of the geodesics are, by ($\star$),

$$\text{(a)}\quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \text{(b)}\quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0,$$

$$\text{(c)}\quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}+\frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2=0.$$

The solutions are:

(a) We can write

$$\frac{x''}{x'}+\frac{xx'}{1+x^2}=0,$$

hence $\log x'+\frac{1}{2}\log(1+x^2)=\log A$, or equivalently, $x'=\frac{A}{\sqrt{1+x^2}}$. We have $\sqrt{1+x^2}\,\mathrm{d}x=A\,\mathrm{d}t$ and

$$\int A\,\mathrm{d}t=At+B=\int\sqrt{1+x^2}\,\mathrm{d}x=\frac{1}{2}\big(x\sqrt{1+x^2}+\log\big(x+\sqrt{1+x^2}\big)\big).$$

(b) $y=Ct+D$.

(c) Let $p=\frac{\mathrm{d}z}{\mathrm{d}t}$. Then we have $\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{p^2}{2}=0$, from which $\frac{1}{p}=\frac{t}{2}+\frac{E}{2}$. Thus $\frac{2}{t+E}=\frac{\mathrm{d}z}{\mathrm{d}t}$, so one has

$$z=2\log(t+E)+2\log F=\log(Ft+G)^2.$$

(iii) The equations of parallel transport of the vector $X=(a^1,a^2,a^3)$ along a curve γ are $\nabla_{\gamma'}X=0$, that is,

$$\frac{\mathrm{d}a^i}{\mathrm{d}t}+\sum_{j,h}\Gamma^i_{jh}\frac{\mathrm{d}x^j}{\mathrm{d}t}a^h=0,\qquad i=1,2,3.$$

In this case, we have the equations:

$$\text{(a)}\quad \frac{\mathrm{d}a^1}{\mathrm{d}t}+\frac{x}{1+x^2}\frac{\mathrm{d}x}{\mathrm{d}t}a^1=0,\qquad \text{(b)}\quad \frac{\mathrm{d}a^2}{\mathrm{d}t}=0,$$

$$\text{(c)}\quad \frac{\mathrm{d}a^3}{\mathrm{d}t}+\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}t}a^3=0,$$

along the curve $x=t$, $y=t$, $z=t$, that is, the previous equations are reduced to:

$$\text{(a)}\quad \frac{\mathrm{d}a^1}{\mathrm{d}t}+\frac{t}{1+t^2}a^1=0,\qquad \text{(b)}\quad \frac{\mathrm{d}a^2}{\mathrm{d}t}=0,\qquad \text{(c)}\quad \frac{\mathrm{d}a^3}{\mathrm{d}t}+\frac{1}{2}a^3=0.$$

Integrating we have:

(a) $\log a^1=-\frac{1}{2}\log(1+t^2)+\log A$, thus one has $a^1=A/\sqrt{1+t^2}$, with $a^1(0)=a$, so $a^1=a/\sqrt{1+t^2}$.

(b) $a^2=A$, with $a^2(0)=A$; thus $a^2=b$.

(c) $a^3=A\mathrm{e}^{-t/2}$, with $a^3(0)=c=A$; thus $a^3=c\,\mathrm{e}^{-t/2}$.

(iv) The curve must verify the equations of the geodesics obtained in (ii). Since $x(t)=t$, $y(t)=t$, $z(t)=t$, we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=\frac{t}{1+t^2}\neq 0$$

unless $t=0$, so it is not a geodesic.

(v) We have obtained in (iii) the vector field obtained by parallel transport from $(a,b,c)_{(0,0,0)}$, that is,

$$a^i(t)=\left(\frac{a}{\sqrt{1+t^2}},\ b,\ c\,\mathrm{e}^{-t/2}\right).$$

Taking $X(0)=(1,0,0)$, $Y(0)=(0,1,0)$, one obtains under parallel transport the vector fields

$$X(t)=\left(\frac{1}{\sqrt{1+t^2}},0,0\right),\qquad Y(t)=(0,1,0),$$

that satisfy $g(X(t),Y(t))=0$.

(vi) No. In fact, consider the vectors $Z(0)=(a,b,c)$, $W(0)=(\lambda,\mu,\nu)$. Then the vector fields $Z(t)$, $W(t)$ obtained by parallel transport of the vectors along γ, satisfy

$$g\big(Z(t),W(t)\big)=\big(1+t^2\big)\frac{a\lambda}{1+t^2}+b\mu+\mathrm{e}^t\frac{c\nu}{\mathrm{e}^t}=a\lambda+b\mu+c\nu,$$

which is a constant function.

This can be obtained directly considering that ∇ is the Levi-Civita connection of g, and for all the Riemannian connections the parallel transport preserves the length and the angle.

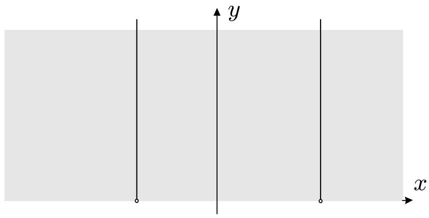
Problem 6.56 Prove with an example that there exist Riemannian manifolds on which the distance between points is bounded, that is, $d(p,q)<a$, for $a>0$ fixed, but on which there is a geodesic with infinite length but that does not intersect itself.

Solution The flat torus T^2 is endowed with the flat metric obtained from the metric of $\mathbb{R}^2$ by the usual identification $T^2=\mathbb{R}^2/\mathbb{Z}^2$. It is thus clear that the maximum distance is $\sqrt{2}/2$.

Nevertheless, the image curve of a straight line through the origin of $\mathbb{R}^2$ with irrational slope is a geodesic of infinite length which does not intersect itself in T^2 (see Problem 4.52).

Problem 6.57 Give an example of a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ but such that none of its geodesics can be indefinitely extended.

Fig. 6.1 The vertical lines of the Poincaré upper half-plane are geodesics



Solution The open cube $(-1,1)^n \subset \mathbb{R}^n$, with center at $(0,\dots,0) \in \mathbb{R}^n$, is diffeomorphic to $\mathbb{R}^n$ by the map

$$\varphi \colon \mathbb{R}^n \to (-1,1)^n, \qquad \bigl(y^1,\dots,y^n\bigr) \mapsto \bigl(\tanh y^1,\dots,\tanh y^n\bigr).$$

In fact, φ is one-to-one and C^∞, and its inverse map on each component is also C^∞.

Take now on $(-1,1)^n$ the flat metric, restriction of $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$ on $\mathbb{R}^n$. It is obvious that none of the geodesics which are the connected open segments of straight lines of $\mathbb{R}^n$ in $(-1,1)^n$ can be indefinitely extended.

Problem 6.58 Prove that the vertical lines $x = \text{const}$ in the Poincaré upper half-plane H^2 are complete geodesics.

Solution We have the Riemannian manifold (M, g), where

$$M = \bigl\{(x,y) \in \mathbb{R}^2 : y > 0\bigr\}, \qquad g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

(see Fig. 6.1). That is, $g_{ij} = (1/y^2)\delta_{ij}$ and $g^{ij} = y^2\delta^{ij}$, $i, j = 1, 2$. Taking $x^1 = x$, $x^2 = y$, the non-vanishing Christoffel symbols are

$$\Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} = -1/y,$$

so the differential equations of the geodesics are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{2}{y}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{1}{y}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0.$$

Suppose $x(0) = x_0$, $y(0) = y_0$, $\mathrm{d}x/\mathrm{d}t = 0$, $\mathrm{d}y/\mathrm{d}t = 1$, that is, one considers the vertical line through (x_0, y_0). The previous equations are satisfied, and one has the equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2.$$

The conditions $x(0) = x_0$, $y(0) = y_0$, $(\mathrm{d}x/\mathrm{d}t)_0 = 0$, $(\mathrm{d}y/\mathrm{d}t)_0 = 1$, determine a unique geodesic. Integrating, we have $x = At + B$; and, from $y''/y' = y'/y$, one has $\log y' = \log y + C$, or equivalently, $y = e^{Ct+D}$. By the previous conditions, it follows that

$$x = x_0, \qquad y = y_0 e^{t/y_0},$$

which proves $t \in (-\infty, +\infty)$, that is, the given geodesic is complete.

Problem 6.59 Consider $\mathbb{R}^2$ with the usual flat metric $g = dx^2 + dy^2$. Is the curve $\gamma(t)$ given by $x = t^3$, $y = t^3$, a geodesic?

Remark The fact that a curve is a geodesic depends both on its shape and its parametrisation, as it is shown by the curve $\sigma(t) = (t, t)$ in $\mathbb{R}^2$ and the curve above.

Solution Write $\gamma(t) = (t^3, t^3)$. Then

$$\frac{d\gamma}{dt} = 3t^2 \frac{\partial}{\partial x} + 3t^2 \frac{\partial}{\partial y}.$$

As $\frac{D}{dt} = \frac{d}{dt}$ on $(\mathbb{R}^2, g)$, we have

$$\frac{D}{dt}\frac{d\gamma}{dt} = \frac{d}{dt}\left(3t^2 \frac{\partial}{\partial x} + 3t^2 \frac{\partial}{\partial y}\right) = 6t \frac{\partial}{\partial x} + 6t \frac{\partial}{\partial y} \neq 0,$$

hence $\gamma(t)$ is not a geodesic.

Another solution is as follows: Since γ is a geodesic curve, one should have $|\gamma'(t)| = \text{const}$, but actually $|\gamma'(t)| = 3\sqrt{2}t^2$.

Problem 6.60 Let (M, g) be a connected Riemannian manifold. Let $H : M \to \mathbb{R}$ be a smooth function on M with gradient vector field $\operatorname{grad} H$, i.e.

$$dH(Z) = g(\operatorname{grad} H, Z), \qquad Z \in \mathfrak{X}(M),$$

and let

$$D = \left\{p \in M : |\operatorname{grad} H|(p) \neq 0\right\},$$

where

$$|\operatorname{grad} H| = \sqrt{g(\operatorname{grad} H, \operatorname{grad} H)}.$$

Suppose that there exists a smooth function $\alpha \in C^\infty(H(D))$ such that

$$|\operatorname{grad} H| = \alpha \circ H$$

on the subset $D \subset M$.

Prove that the unit vector field

$$U = \frac{\operatorname{grad} H}{|\operatorname{grad} H|}$$

is a geodesic vector field on the subset D, that is, its integral curves are geodesics of the metric g.

Solution Put $\beta = 1/\alpha$ on the interior of the set $H(D) \subset \mathbb{R}$ (where $\alpha > 0$). Then $U = \beta(H) \cdot \operatorname{grad} H$ and

$$UH = \mathrm{d}H(U) = g(\operatorname{grad} H, U) = \beta(H)|\operatorname{grad} H|^2 = |\operatorname{grad} H| = \alpha(H).$$

To prove that $\nabla_U U = 0$ consider the Koszul formula (6.4) for the Levi-Civita connection,

$$2g(\nabla_U U, Z) = 2Ug(U, Z) - Zg(U, U) + 2g\big([Z, U], U\big).$$

Taking the definition of the vector field $\operatorname{grad} H$ into account, we obtain that

$$\begin{aligned} Ug(U, Z) + g\big([Z, U], U\big) &= Ug\big(\beta(H) \operatorname{grad} H, Z\big) + g\big(\beta(H) \operatorname{grad} H, [Z, U]\big) \\ &= U\big(\beta(H) \cdot \mathrm{d}H(Z)\big) + \beta(H)\,\mathrm{d}H\big([Z, U]\big) \\ &= \beta'(H) \cdot UH \cdot ZH + \beta(H) \cdot U(ZH) \\ &\quad + \beta(H)\big(Z(UH) - U(ZH)\big) \\ &= \beta'(H)\alpha(H) \cdot ZH + \beta(H)\alpha'(H) \cdot ZH \\ &= (\alpha\beta)'(H) \cdot ZH = 0. \end{aligned}$$

Thus $\nabla_U U = 0$ on $D \subset M$ because $|U| = 1$.

Problem 6.61 We retain the hypotheses and notation of Problem 6.60. Suppose in addition that:

(a) $H(M) = [0, +\infty)$.
(b) $\operatorname{grad} H(p) = 0$ if and only if $H(p) = 0$, and
(c) For each number $c \in \mathbb{R}$ the subset $\{p \in M : H(p) \leqslant c\}$ is compact.

Prove:

1. One has that $\lim_{\tau \to 0+} \alpha(\tau) = 0$ and the improper integral
$$\int_0^1 \frac{\mathrm{d}\tau}{\alpha(\tau)}$$
converges.
2. The metric d_g on M is complete if and only if the improper integral
$$\int_0^{+\infty} \frac{\mathrm{d}\tau}{\alpha(\tau)}$$
diverges.

Solution Using the unit geodesic vector field

$$U = \frac{\operatorname{grad} H}{|\operatorname{grad} H|}$$

we shall calculate the distance between the level sets $\{H = c_0\}$ and $\{H = c\}$ in M with respect to the distance function (metric) d_g induced by the metric tensor g.

Recall that by definition $|\operatorname{grad} H|(p) = \alpha(H(p)) > 0$ if $H(p) > 0$, that is, $\alpha(\tau) > 0$ for all $\tau > 0$. Consider some level set $\{H = c\}$ with positive $c > 0$. By compactness of this set, there exists a local one-parametric group φ_t of U defined on some neighbourhood of $\{H = c\}$ for all $t \in [0, t_0]$ with $t_0 > 0$ (the vector field U is smooth on the set $D = \{H > 0\} \subset M$). But

$$H\bigl(\varphi_{t_0}(p)\bigr) = H(p) + \int_0^{t_0} \alpha\bigl(H\bigl(\varphi_t(p)\bigr)\bigr)\,\mathrm{d}t > H(p) \qquad (\star)$$

because the function

$$UH = |\operatorname{grad} H| = \alpha \circ H$$

is a positive constant on the set $\{H = c\}$. Then, again by the compactness of $\{H = c\}$ one has

$$\min_{p \in \{H=c\}} H\bigl(\varphi_{t_0}(p)\bigr) > c.$$

In other words, for each $c > 0$ there exists $\delta(c) > 0$ such that any point $p \in \{H = c\}$ is connected by an integral curve of U with some point of the level set $\{H = d\}$, where $c \leqslant d \leqslant c + \delta(c)$. Therefore, for arbitrary $0 < c_0 \leqslant c < +\infty$ and any point $p_0 \in \{H = c_0\}$, there is an integral curve $\gamma(t)$, $t \in [0, t_0]$, of the vector field U with initial point $p_0 \in \{H = c_0\}$ such that $H(p) = c$, where $p = \gamma(t_0)$.

Let $\gamma(t)$, $t \in [0, t_0]$, be the integral curve of the vector field U with initial point $p_0 \in \{H = c_0\}$, $c_0 > 0$. There exists a smooth function $h\colon (0, +\infty) \to \mathbb{R}$ such that the function $h(H(\gamma(t)))$ is linear in t. It is easy to verify that

$$h'(s) = \frac{1}{\alpha(s)} \quad \text{and} \quad h(s) = \int_1^s \frac{\mathrm{d}\tau}{\alpha(\tau)}$$

because $\alpha(\tau) > 0$ for all $\tau > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} h\bigl(H\bigl(\gamma(t)\bigr)\bigr) = h'\bigl(H\bigl(\gamma(t)\bigr)\bigr)\,\mathrm{d}H\bigl(\gamma'(t)\bigr) = h'\bigl(H\bigl(\gamma(t)\bigr)\bigr) \cdot \alpha\bigl(H\bigl(\gamma(t)\bigr)\bigr) = 1.$$

Let $p = \gamma(t_0)$ and $H(p) = c$. As we showed above, $c > c_0$ (see ($\star$)). However, the curve $\gamma(t)$ is a geodesic. Therefore, the length of the curve $\gamma(t)$, $t \in [0, t_0]$, from p_0 to p is

$$t_0 = h\bigl(H(p)\bigr) - h\bigl(H(p_0)\bigr) = h(c) - h(c_0).$$

For any other curve $\lambda(t)$, $t \in [0, \tilde{t}_0]$, with $|\lambda'(t)| = 1$, starting from a point $\tilde{p}_0 \in \{H = c_0\}$, ended at a point $\tilde{p} \in \{H = c\}$ and belonging to the set $\{H > 0\}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} h\bigl(H\bigl(\lambda(t)\bigr)\bigr) &= h'\bigl(H\bigl(\lambda(t)\bigr)\bigr) \cdot \mathrm{d}H\bigl(\lambda'(t)\bigr) = h'\bigl(H\bigl(\lambda(t)\bigr)\bigr) \cdot g\bigl((\operatorname{grad} H)_{\lambda(t)}, \lambda'(t)\bigr) \\ &\leqslant h'\bigl(H\bigl(\lambda(t)\bigr)\bigr) \cdot |\operatorname{grad} H|\bigl(\lambda(t)\bigr) \cdot \bigl|\lambda'(t)\bigr| = \bigl|\lambda'(t)\bigr| = 1.\end{aligned}$$

Thus $h(H(\lambda(t))) - h(H(p_0)) \leqslant t$ and, consequently,

$$h\big(H\big(\lambda(t)\big)\big) \leqslant h(c_0) + t \leqslant h(c_0) + t_0 \leqslant h(c), \quad t \in [0, t_0].$$

Since h is an increasing function and $H(\lambda(t)) > 0$, we have

$$H\big(\lambda(t)\big) < c \quad \text{for all } 0 \leqslant t < t_0, \quad \text{and} \quad H\big(\lambda(t_0)\big) \leqslant c.$$

Therefore, $\tilde{t}_0 \geqslant t_0$ and the length of the curve $\lambda(t)$ from $\tilde{p}_0$ to $\tilde{p}$ is not smaller than the length of the geodesic curve $\gamma(t)$, $t \in [0, t_0]$. So the distance between the level sets $\{H = c_0\}$ and $\{H = c\}$, $0 < c_0 \leqslant c$, is $(h(c) - h(c_0))$. We have

$$\lim_{\tau \to 0+} \alpha(\tau) = 0$$

because all level sets $\{H = \tau\}$, $\tau \geqslant 0$, are compact and the functions H, $|\operatorname{grad} H|$ are continuous on the manifold M. Moreover, since the distance between the compact level sets $\{H = 0\}$ and $\{H = c\}$, $c > 0$, is finite and each smooth curve connecting these two sets intersects all sets $\{H = c_0\}$, where $0 \leqslant c_0 \leqslant c$, this distance is equal to the converging improper integral

$$\int_0^c \frac{\mathrm{d}t}{\alpha(t)} = h(c) - h(0).$$

2. Taking into account that all level surfaces $\{H = c\}$, $c \geqslant 0$, are compact sets, the metric d_g on M is complete if and only if the metric on the set $[0, +\infty)$ induced by the function h is complete, that is, if and only if

$$\int_0^{+\infty} \frac{\mathrm{d}t}{\alpha(t)} = +\infty.$$

Problem 6.62 Consider the standard linear action of the Lie group $\mathrm{SO}(n)$ on $\mathbb{R}^n$ with standard global coordinates $x = (x^1, \dots, x^n)$. Let g be an $\mathrm{SO}(n)$-invariant Riemannian metric on $\mathbb{R}^n$.

Prove:

(i) There exist unique smooth positive even functions $a, b\colon \mathbb{R} \to \mathbb{R}^+$, $a(0) = b(0)$, such that $g(x) = (g_{ij}(x))$, $i, j = 1, \dots, n$, where

$$\begin{aligned} g_{ij}(x) &= \frac{a(r) - b(r)}{r^2} x^i x^j + b(r)\delta_{ij}, \\ r = r(x) &= \sqrt{\left(x^1\right)^2 + \dots + \left(x^n\right)^2}. \end{aligned} \tag{$\star$}$$

(ii) For each pair a, b of smooth positive even functions on $\mathbb{R}$, the relations ($\star$) determine an $\mathrm{SO}(n)$-invariant metric on $\mathbb{R}^n$.

(iii) The metric d_g on $\mathbb{R}^n$ is complete if and only if the improper integral

$$\int_0^{+\infty} \sqrt{\frac{a(\sqrt{\tau})}{\tau}}\, \mathrm{d}\tau$$

diverges.

Solution

(i) Denote by P_x the matrix with entries

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x^i}\bigg|_x, \frac{\partial}{\partial x^j}\bigg|_x\right), \quad i, j = 1, \dots, n.$$

Since the action of the group $K = \mathrm{SO}(n)$ on $\mathbb{R}^n$ is linear and $A^t = A^{-1}$ for each $A \in \mathrm{SO}(n)$, the metric g is $\mathrm{SO}(n)$-invariant if and only if

$$P_x = A^{-1} \cdot P_{Ax} \cdot A, \quad x \in \mathbb{R}^n,\ A \in \mathrm{SO}(n).$$

In particular,

$$A \cdot P_x = P_x \cdot A \quad \text{for all elements in} \quad K_x = \{A \in K : Ax = x\} \subset K. \qquad (\star\star)$$

If x is a nonzero vector, then $\mathbb{R}^n = \langle x\rangle \oplus \langle x\rangle^\perp$, where $\langle x\rangle$ is a one-dimensional trivial K_x-module generated by the vector x and $\langle x\rangle^\perp$ is the irreducible K_x-module of all vectors orthogonal to x (as is well known, the group $K_x \cong \mathrm{SO}(n-1)$ acts transitively on the set of all vectors of constant length in $\langle x\rangle^\perp$). Since $\langle x\rangle$ and $\langle x\rangle^\perp$ are irreducible modules (over $\mathbb{R}$) of the group K_x and the matrix P_x is symmetric (always has an eigenvector belonging to $\langle x\rangle^\perp$), from condition ($\star\star$) above it follows that for each vector $y \in \mathbb{R}^n$ its two components corresponding to the decomposition $\mathbb{R}^n = \langle x\rangle \oplus \langle x\rangle^\perp$ are eigenvectors of P_x with eigenvalues $\alpha(x)$ and $\beta(x)$, respectively:

$$P_x y = \frac{\alpha(x) - \beta(x)}{r^2(x)}\left(\sum_{i=1}^n x^i y^i\right)x + \beta(x)y.$$

Moreover, $P_0 y = a_0 y$ for some $a_0 > 0$ because, by condition ($\star\star$), the symmetric matrix P_0 commutes with all elements of the isotropy group $K_0 = K = \mathrm{SO}(n)$ acting transitively on the set of all vectors of constant length in $\mathbb{R}^n$. Now taking into account that

$$P_{Ax}A = AP_x, \quad x \in \mathbb{R}^n,\ A \in K,$$

we obtain that $\alpha(Ax) = \alpha(x)$ and $\beta(Ax) = \beta(x)$ for all such x and A. Therefore, there exist uniquely defined functions

$$a, b\colon [0, +\infty) \to \mathbb{R}, \qquad \alpha(x) = a\big(r(x)\big), \qquad \beta(x) = b\big(r(x)\big).$$

Denote by a and b the even extensions of these functions to the whole line $\mathbb{R}$.

Let us prove that the functions a and b are smooth and $a(0) = b(0)$. Indeed, fix two non-zero vectors $x \perp y$ in $\mathbb{R}^n$ such that $r(x) = r(y) = 1$. Since the metric g is smooth, we obtain that the two vector-functions

$$\lambda \mapsto P_{\lambda x} x = a(\lambda) \cdot x, \qquad \lambda \mapsto P_{\lambda x} y = b(\lambda) \cdot y,$$

are smooth functions of the parameter $\lambda \in \mathbb{R}$ and, moreover, $P_0 x = a(0)x$ and $P_0 y = b(0)y$. Thus a and b are smooth even functions on $\mathbb{R}$ and $a(0) = b(0) = a_0$.

(ii) Now it is easy to see that relations ($\star$) define a smooth $\mathrm{SO}(n)$-invariant metric on $\mathbb{R}^n$ (with a smooth extension to the zero point). To prove the last part of the problem consider the smooth function $H(x) = r^2(x)$ on $\mathbb{R}^n$. It is clear that

$$\operatorname{grad} H(x) = \frac{2}{a(r)}\, x.$$

Then

$$|\operatorname{grad} H|(x) = \frac{2r(x)}{\sqrt{a(r(x))}} = 2\sqrt{\frac{H(x)}{a(\sqrt{H(x)})}},$$

that is, $|\operatorname{grad} H|(x) = \alpha(H(x))$ on the subset $\{H > 0\}$ with a smooth function $\alpha\colon (0, +\infty) \to \mathbb{R}$.

(iii) By Problem 6.61, the metric d_g on $\mathbb{R}^n$ is complete if and only if the improper integral

$$\int_0^{+\infty} \sqrt{\frac{a(\sqrt{\tau})}{\tau}}\, \mathrm{d}\tau$$

diverges.

Problem 6.63 Let (M, g) be a Riemannian manifold and let $\pi : T^*M \to M$ be the canonical projection of the cotangent bundle T^*M onto the manifold M. The metric g determines a natural isomorphism

$$\psi_g \colon TM \to T^*M.$$

Denote by H the real-valued function on T^*M such that

$$H(\omega) = \frac{1}{2}\, g\big(\psi_g^{-1}(\omega),\ \psi_g^{-1}(\omega)\big)\big(\pi(\omega)\big).$$

The Hamiltonian flow on T^*M generated by the Hamiltonian vector field X_H of H with respect to canonical symplectic form Ω on T^*M, where, recall,

$$i_{X_H} \Omega = -\mathrm{d}H,$$

is said to be the *geodesic flow of* g.

Prove:

(i) The projection $\gamma(t) = \pi(\sigma(t))$ of any integral curve $\sigma(t)$, $t \in (a, b) \subset \mathbb{R}$, of the vector field X_H is a geodesic of the Levi-Civita connection of the metric g.
(ii) For any geodesic $\gamma(t)$, $t \in (a, b) \subset \mathbb{R}$, on (M, g) the curve

$$\sigma(t) = \psi_g\big(\gamma'(t)\big) \subset T^*M$$

is an integral curve of the vector field X_H.
(iii) For any integral curve $\sigma(t)$ of X_H we have

$$\sigma(t) = \psi_g\big(\lambda'(t)\big),$$

where $\lambda(t) = \pi(\sigma(t))$ is a geodesic on (M, g); moreover, the Riemannian manifold (M, g) is geodesically complete if and only if the Hamiltonian vector field X_H is complete.

Solution

(i) It is sufficient to prove it locally. Let $q = (q^1, \dots, q^n)$ denote local coordinates on M. They induce local coordinates $(q, p) = (q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*M putting $\omega_x = \sum_i p_i(\omega_x)\, \mathrm{d}q^i|_x$ for $\omega_x \in T^*M$, $x \in M$, and local coordinates $(q, y) = (q^1, \dots, q^n, y^1, \dots, y^n)$ on TM letting

$$v_x = \sum_i y^i(v_x) \frac{\partial}{\partial q^i}\bigg|_x, \qquad v_x \in TM.$$

Then the isomorphisms ψ_g, ψ_g^{-1} and the function H have the following form in local coordinates:

$$\psi_g\bigg(\sum_i y^i \frac{\partial}{\partial q^i}\bigg) = \sum_{i,j} g_{ij}(q) y^i \,\mathrm{d}q^j,$$

$$\psi_g^{-1}\bigg(\sum_i p_i \,\mathrm{d}q^i\bigg) = \sum_{i,j} g^{ij}(q) p_j \frac{\partial}{\partial q^i},$$

$$H(q, p) = \frac{1}{2} \sum_{i,j} g^{ij}(q) p_i p_j,$$

$g(q) = \sum_{i,j} g_{ij}(q)\, \mathrm{d}q^i\, \mathrm{d}q^j$ being the metric tensor in the local coordinates and $\sum_j g_{ij}(q) g^{jk}(q) = \delta_i^k$ by definition. Since $\Omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$ (see Problem 5.59), the vector field X_H is described by the relation

$$X_H(q, p) = \sum_{j,k} g^{kj}(q) p_j \frac{\partial}{\partial q^k} - \frac{1}{2} \sum_{i,j,k} \frac{\partial g^{ij}}{\partial q^k}(q) p_i p_j \frac{\partial}{\partial p_k}.$$

Therefore, its integral curve $\sigma(t)$ is a solution of the following equations:

$$\frac{\mathrm{d}q^k}{\mathrm{d}t} = \sum_j g^{kj}(q)p_j, \qquad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{1}{2}\sum_{i,j}\frac{\partial g^{ij}}{\partial q^k}(q)p_i p_j. \tag{$\star$}$$

Putting $y^k(t) = (\mathrm{d}q^k/\mathrm{d}t)(t)$, we rewrite the first relation above as $y^k = \sum_j g^{kj}p_j$ and $\sum_k g_{jk}y^k = p_j$. Differentiating this (first) relation, we obtain that

$$\begin{aligned}\frac{\mathrm{d}y^k}{\mathrm{d}t} &= \sum_j\left(\sum_l \frac{\partial g^{kj}}{\partial q^l}\frac{\mathrm{d}q^l}{\mathrm{d}t}p_j + g^{kj}\frac{\mathrm{d}p_j}{\mathrm{d}t}\right)\\ &= \sum_{j,l}\left(\frac{\partial g^{kj}}{\partial q^l}y^l p_j - \frac{1}{2}\sum_i g^{kj}\frac{\partial g^{il}}{\partial q^j}p_i p_l\right).\end{aligned}$$

Let us show that the relations

$$\begin{aligned}\frac{\mathrm{d}y^k}{\mathrm{d}t} &= \sum_{i,j,l}\left(\frac{\partial g^{kj}}{\partial q^l}g^{li}p_i p_j - \frac{1}{2}g^{kj}\frac{\partial g^{il}}{\partial q^j}p_i p_l\right)\\ &= \sum_{i,j,l}\left(\frac{\partial g^{kj}}{\partial q^l}g^{li} - \frac{1}{2}g^{kl}\frac{\partial g^{ij}}{\partial q^l}\right)p_i p_j,\end{aligned} \tag{$\star\star$}$$

where $p_j = \sum_k g_{jk}y^k$, are equivalent to the equations of the geodesics:

$$\frac{\mathrm{d}y^k}{\mathrm{d}t} = -\sum_{l,s=1}^n \Gamma^k_{ls}y^l y^s = -\frac{1}{2}\sum_{j,l,s=1}^n g^{kj}\left(\frac{\partial g_{js}}{\partial q^l} + \frac{\partial g_{jl}}{\partial q^s} - \frac{\partial g_{ls}}{\partial q^j}\right)y^l y^s,$$

where Γ^k_{ls} are the Christoffel symbols of the Levi-Civita connection of the metric g. Indeed,

$$\begin{aligned}\frac{\mathrm{d}y^k}{\mathrm{d}t} &= \sum_{j,l=1}^n \frac{\partial g^{kj}}{\partial q^l}y^l p_j - \frac{1}{2}\sum_{i,j,l=1}^n g^{kj}\frac{\partial g^{il}}{\partial q^j}p_i p_l\\ &= \sum_{j,l,s=1}^n \frac{\partial g^{kj}}{\partial q^l}y^l g_{js}y^s - \frac{1}{2}\sum_{i,j,l,m,s=1}^n g^{kj}\frac{\partial g^{il}}{\partial q^j}g_{im}y^m g_{ls}y^s.\end{aligned}$$

Taking into account that $\sum_l g^{il}g_{ls} = \delta^i_s$, we obtain that

$$\sum_l\left(\frac{\partial g^{il}}{\partial q^j}g_{ls} + g_{il}\frac{\partial g_{ls}}{\partial q^j}\right) = 0$$

and, consequently,

$$\sum_{i,j,l,m,s=1}^{n} g^{kj}\left(\frac{\partial g^{il}}{\partial q^j} g_{ls}\right) g_{im} y^m y^s = -\sum_{i,j,l,m,s=1}^{n} g^{kj}\left(g_{im} g^{il}\right)\frac{\partial g_{ls}}{\partial q^j} y^m y^s$$
$$= -\sum_{j,l,s=1}^{n} g^{kj}\frac{\partial g_{ls}}{\partial q^j} y^l y^s.$$

Now, we can rewrite the previous expression for $\frac{\mathrm{d}y^k}{\mathrm{d}t}$ as

$$\begin{aligned}
\frac{\mathrm{d}y^k}{\mathrm{d}t} &= \sum_{j,l,s=1}^{n}\left(\frac{\partial g^{kj}}{\partial q^l} g_{js} + \frac{1}{2} g^{kj}\frac{\partial g_{ls}}{\partial q^j}\right) y^l y^s \\
&= \sum_{j,l,s=1}^{n}\left(\frac{1}{2} g^{kj}\frac{\partial g_{ls}}{\partial q^j} - g^{kj}\frac{\partial g_{js}}{\partial q^l}\right) y^l y^s \\
&= -\frac{1}{2}\sum_{j,l,s=1}^{n} g^{kj}\left(\frac{\partial g_{js}}{\partial q^l} + \frac{\partial g_{jl}}{\partial q^s} - \frac{\partial g_{ls}}{\partial q^j}\right) y^l y^s = -\sum_{l,s=1}^{n} \Gamma^k_{ls} y^l y^s
\end{aligned}$$

because $\sum_{l,s} \frac{\partial g_{js}}{\partial q^l} y^l y^s = \sum_{l,s} \frac{\partial g_{jl}}{\partial q^s} y^l y^s$. In other words, the curve $\gamma(t) = \pi(\sigma(t))$,

$$\gamma(t) = \left(q^1(t), \ldots, q^n(t)\right)$$

is a geodesic in the Riemannian manifold (M, g).

(ii) Consider a geodesic $\gamma(t) = (q^1(t), \ldots, q^n(t))$ and put $\gamma'(t) = (y^1(t), \ldots, y^n(t))$, where $y^k(t) = \mathrm{d}q^k(t)/\mathrm{d}t$. Then $\{y^k(t)\}$ satisfy the relations ($\star\star$), where $p_j = \sum_k g_{jk}(q) y^k$ (and $y^k = \sum_j g^{kj} p_j$). We remark here that, by the relations for the isomorphism

$$\psi_g \colon TM \to T^*M,$$

these functions $\{q^k(t)\}$ and $\{p_j(t)\}$ are the coordinates of the curve $\psi_g(\gamma'(t)) \subset T^*M$. Now using ($\star\star$) and differentiating the relations $p_m = \sum_k g_{mk} y^k$, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}p_m}{\mathrm{d}t} &= \sum_{i,j,k,l=1}^{n} g_{mk}\left(\frac{\partial g^{kj}}{\partial q^l} g^{li} - \frac{1}{2} g^{kl}\frac{\partial g^{ij}}{\partial q^l}\right) p_i p_j + \sum_{k,l=1}^{n} \frac{\partial g^{mk}}{\partial q^l} y^l y^k \\
&= \sum_{i,j,k,l=1}^{n}\left(g_{mk}\frac{\partial g^{kj}}{\partial q^l} g^{li} - \frac{1}{2} g_{mk} g^{kl}\frac{\partial g^{ij}}{\partial q^l} + \frac{\partial g^{mk}}{\partial q^l} g^{kj} g^{li}\right) p_i p_j.
\end{aligned}$$

But $\sum_k g_{mk}g^{kl} = \delta^l_m$ and, consequently, $\sum_k \frac{\partial}{\partial q^l}(g_{mk}g^{kj}) = 0$ and

$$\frac{\mathrm{d}p_m}{\mathrm{d}t} = -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial g^{ij}}{\partial q^m} p_i p_j.$$

So that

$$\sigma(t) = \big(q(t), p(t)\big)$$

satisfies the relations ($\star$), and therefore this curve is an integral curve of X_H ($\frac{\mathrm{d}q^k}{\mathrm{d}t}(t) = y^k(t) = \sum_j g^{kj}(q(t))p_j(t)$ by definition).

(iii) As it follows from the proofs of (i) and (ii), for any integral curve $\sigma(t)$ of X_H we have

$$\sigma(t) = \psi_g\big(\lambda'(t)\big), \qquad \lambda(t) = \pi\big(\sigma(t)\big).$$

Now to complete the proof of (iii) it is sufficient to note that for any point $x \in M$ the map $\gamma \mapsto \psi_g(\gamma')$ determines a one-to-one correspondence between the set of all geodesics on (M, g) through the point x and the set of all the integral curves of the vector field X_H through the points of the space $T^*_x M$. Therefore, every maximal geodesic $\gamma(t)$ on (M, g) is defined for every real value of the parameter t if and only if so is the maximal integral curve $\sigma(t) = \psi_g(\gamma'(t))$ of X_H.

6.5 The Exponential Map

Problem 6.64 Consider on $\mathbb{R}^n$ with the Euclidean metric the geodesic $\gamma(t)$ through p with unit initial velocity v_p, and let $(e_1, \dots, e_n)$ be an orthonormal frame along γ such that $e_1 = \gamma'(t)$. Compute the *Fermi coordinates* $(x^1, \dots, x^n)$ on $(\mathbb{R}^n, \gamma)$ relative to $(e_1, \dots, e_n)$ and p.

The relevant theory is developed, for instance, in Hicks [16].

Solution The geodesic $\gamma(t)$ through $p \in \mathbb{R}^n$ with the initial velocity vector $v_p \in T_p\mathbb{R}^n$ is the straight line $\gamma(t) = p + tv_p$. Thus

$$\begin{aligned} \exp_p \colon T_p\mathbb{R}^n &\to \mathbb{R}^n \\ v_p &\mapsto \sigma(1) = p + v_p, \end{aligned}$$

hence

$$x^1\left(\exp_{\gamma(t)}\left(\sum_{j=2}^{n} t^j e_j\bigg|_{\gamma(t)}\right)\right) = x^1\left(\gamma(t) + \sum_{j=2}^{n} t^j e_j\bigg|_{\gamma(t)}\right) = t,$$

$$x^i\left(\exp_{\gamma(t)}\left(\sum_{j=2}^{n} t^j e_j\bigg|_{\gamma(t)}\right)\right) = t^i, \quad 2 \leqslant i \leqslant n.$$

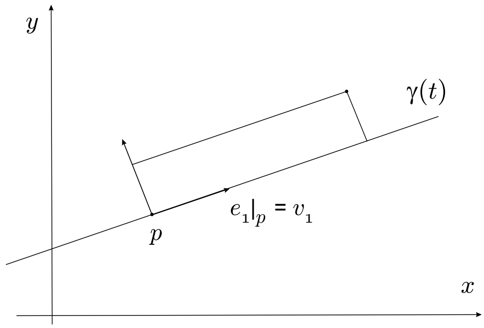


Fig. 6.2 A simple example of Fermi coordinates

Since $\exp_p$ is a global diffeomorphism, we have a new set of coordinates on $\mathbb{R}^n$. The first coordinate is the distance from the origin p along γ and the other coordinates are the orthogonal coordinates relative to $e_2, \ldots, e_n$ (see Fig. 6.2).

Problem 6.65 Let M be an n-dimensional complete Riemannian manifold and let $q \in M$. Identify $T_q M$ with $\mathbb{R}^n$ as a manifold by choosing an orthonormal basis at q. Then

$$\exp_q : T_q M \to M$$

is a C^∞ map of $\mathbb{R}^n$ onto M, mapping 0 to q.

(i) Suppose $M = S^n$, the unit sphere with its usual metric. Prove that

$$\operatorname{rank}(\exp_q)_{*X_q} < n \quad \text{if } |X_q| = k\pi, \; k = 1, 2, \ldots,$$

without using Jacobi fields.

(ii) Find $(\exp_N)_{*X}(e_1)$ and $(\exp_N)_{*X}(e_2)$ for two orthonormal vectors $e_1, e_2 \in T_X T_N S^2$, and $X = \lambda e_1$, $e_1 \in T_N S^2$. In particular, find the values of the two above vectors if $\lambda = 0$, $\pi/2$, or π.

Solution

(i) The geodesic through q with initial vector X_q is (see Problem 6.101) the great circle

$$\gamma(t) = \big(\cos |X_q| t\big) q + \big(\sin |X_q| t\big) \frac{X_q}{|X_q|},$$

hence (see Fig. 6.3)

$$\exp_q(X_q) = \gamma(1) = \big(\cos |X_q|\big) q + \big(\sin |X_q|\big) \frac{X_q}{|X_q|}.$$

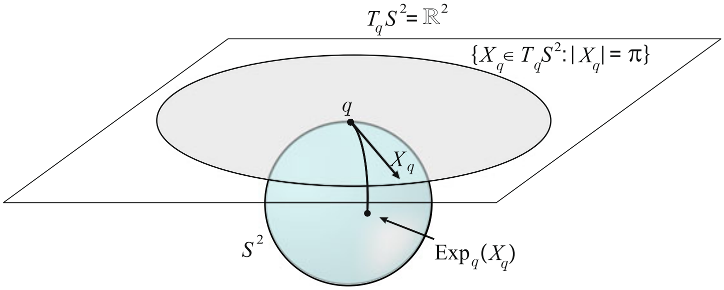


Fig. 6.3 The exponential map on S^2 at q

We can take, without loss of generality,

$$q = (0, \ldots, 0, 1) = N \in S^n \subset \mathbb{R}^{n+1}.$$

Thus, the map $\exp_q$ is given by

$$\begin{aligned} \exp_N : T_N S^n &\to S^n \\ X = (X_1, \ldots, X_n) &\mapsto \left(\frac{\sin|X|}{|X|} X_1, \ldots, \frac{\sin|X|}{|X|} X_n, -\cos|X| \right) \end{aligned}$$

(where we have simplified X_N to X) and has Jacobian matrix $(\exp_N)_*$ given by

$$\begin{pmatrix} \frac{\sin|X|}{|X|}(1 - \frac{X_1^2}{|X|^2}) + \frac{\cos|X|}{|X|^2} X_1^2 & (\cos|X| - \frac{\sin|X|}{|X|}) \frac{X_1 X_2}{|X|^2} & \cdots & (\cos|X| - \frac{\sin|X|}{|X|}) \frac{X_1 X_n}{|X|^2} \\ (\cos|X| - \frac{\sin|X|}{|X|}) \frac{X_1 X_2}{|X|^2} & \frac{\sin|X|}{|X|}(1 - \frac{X_2^2}{|X|^2}) + \frac{\cos|X|}{|X|^2} X_2^2 & \cdots & (\cos|X| - \frac{\sin|X|}{|X|}) \frac{X_2 X_n}{|X|^2} \\ \vdots & & & \vdots \\ (\cos|X| - \frac{\sin|X|}{|X|}) \frac{X_1 X_n}{|X|^2} & \cdots & & \frac{\sin|X|}{|X|}(1 - \frac{X_n^2}{|X|^2}) + \frac{\cos|X|}{|X|^2} X_n^2 \\ -\frac{\sin|X|}{|X|} X_1 & \cdots & & -\frac{\sin|X|}{|X|} X_n \end{pmatrix}$$

Suppose $|X| = k\pi$, $k = 1, 2, \ldots$, then

$$(\exp_N)_{*X} = {}^t\begin{pmatrix} \frac{(-1)^k}{k^2\pi^2}X_1^2 & \frac{(-1)^k}{k^2\pi^2}X_1X_2 & \ldots & \frac{(-1)^k}{k^2\pi^2}X_1X_n \\ \vdots & & & \vdots \\ \frac{(-1)^k}{k^2\pi^2}X_1X_n & \frac{(-1)^k}{k^2\pi^2}X_2X_n & \ldots & \frac{(-1)^k}{k^2\pi^2}X_n^2 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

Since

$$\det\left(\frac{(-1)^k}{k^2\pi^2}X_iX_j\right) = X_1^2X_2^2\cdots X_n^2\left(\frac{(-1)^k}{k^2\pi^2}\right)^n \det\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{pmatrix} = 0,$$

we obtain $\operatorname{rank}(\exp_N)_{*X} < n$.

(ii) We have $X = (X_1, X_2) = (\lambda, 0) = \lambda e_1 \in T_N S^2$, hence

$$(\exp_N)_{*\lambda e_1} = \begin{pmatrix} \cos\lambda & 0 \\ 0 & \frac{\sin\lambda}{\lambda} \\ -\sin\lambda & 0 \end{pmatrix}.$$

In particular,

$$(\exp_N)_{*0e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (\exp_N)_{*\frac{\pi}{2}e_1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{\pi} \\ -1 & 0 \end{pmatrix},$$

$$(\exp_N)_{*\pi e_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

hence

$$(\exp_N)_{*\lambda e_1}(e_1) = \cos\lambda e_1 - \sin\lambda\, e_3 = \begin{cases} e_1 & \text{if } \lambda = 0, \\ -e_3 & \text{if } \lambda = \frac{\pi}{2}, \\ -e_1 & \text{if } \lambda = \pi, \end{cases}$$

$$(\exp_N)_{*\lambda e_1}(e_2) = \frac{\sin\lambda}{\lambda}e_2 = \begin{cases} e_2 & \text{if } \lambda = 0, \\ \frac{2}{\pi}e_2 & \text{if } \lambda = \frac{\pi}{2}, \\ 0 & \text{if } \lambda = \pi, \end{cases}$$

where the vectors in parentheses $e_1, e_2 \in T_{\lambda e_1}(T_N S^2)$ (see Fig. 6.4).

Problem 6.66 Show that if a Riemannian manifold (M, g) is complete and contains a point which has no conjugate points, then M is covered by $\mathbb{R}^n$.

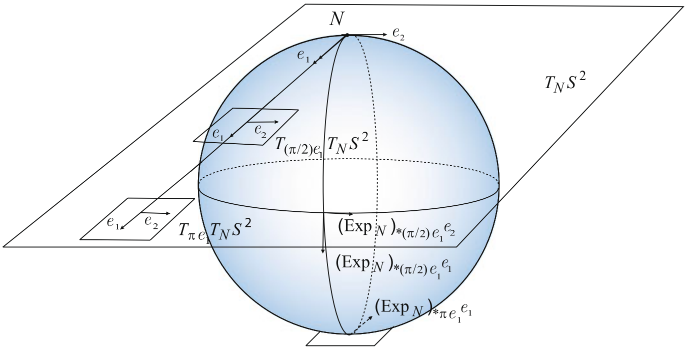


Fig. 6.4 The differential of the exponential map on S^2 at the north pole

Solution Let $p \in M$ be a point without conjugate points. As M is complete, the exponential map

$$\exp_p \colon T_p M \to M$$

is everywhere defined on the tangent space and it is surjective. Moreover, as is well known (see Definitions 6.6), $\exp_p X$ is conjugate to p if and only if $\exp_p$ is critical at X. Hence, by virtue of the hypothesis, $\exp_p$ has no critical point. Accordingly, $\exp_p$ is a surjective local diffeomorphism. Endow $T_p M$ with the metric $\exp_p^* g$ induced by $\exp_p$. Then it is clear that

$$\exp_p \colon \left(T_p M, \exp_p^* g\right) \to (M, g)$$

is a local isometry. Since it applies each ray $t \mapsto tv$ to the geodesic curve γ_v, one deduces that these rays are geodesics, so that the manifold $T_p M$ is complete at 0. The result thus follows from Theorem 6.20.

Problem 6.67 Compute the cut locus on:

(i) The sphere S^n with the round metric.
(ii) The real projective space $\mathbb{RP}^n$ with the constant curvature metric.
(iii) The square torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ with the flat metric.
(iv) The Klein bottle K with the flat metric.

The relevant theory is developed, for instance, in Kobayashi and Nomizu [19, vol. 2, VIII.7].

Solution

(i) All the geodesics are minimizing up to distance π (see Fig. 6.5(a)). For a point $p \in S^n$, we have that exp is a diffeomorphism on

$$U_p = B(p, \pi) \subset T_p S^n$$

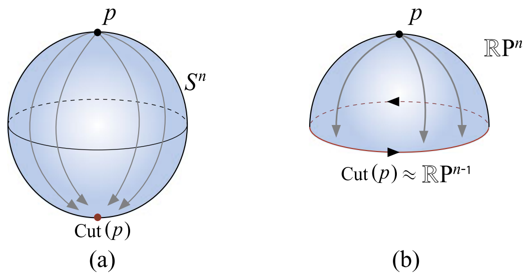


Fig. 6.5 (**a**) The cut locus on the sphere S^n. (**b**) The cut locus on the real projective space $\mathbb{R}P^n$

and that

$$\exp_p(U_p) = S^n \setminus \{-p\}.$$

Hence

$$\mathrm{Cut}(p) = \exp(\partial U_p) = \{-p\},$$

that is, the cut locus reduces to the antipodal point of p.

(ii) The round metric on S^n induces under the projection pr: $S^n \to \mathbb{R}P^{n-1}$ the metric of constant curvature on $\mathbb{R}P^n$. Let $p \in \mathbb{R}P^2$ correspond to the north and south poles in S^n, $p = \{N, S\}$. All the geodesics are minimizing up to distance $\pi/2$ (see Fig. 6.5(b)) so exp is a diffeomorphism on

$$U_p = B\left(p, \frac{\pi}{2}\right) \subset T_p\big(\mathbb{R}P^n\big).$$

Thus, denoting the equator of S^n simply by S^{n-1}, we have that

$$\exp_p(U_p) = \mathbb{R}P^n \Big\backslash \big\{\mathrm{pr}\big(S^{n-1}\big)\big\},$$

hence

$$\mathrm{Cut}(p) = \exp(\partial U_p) = \mathrm{pr}\big(S^{n-1}\big) \cong \mathbb{R}P^{n-1},$$

a naturally imbedded real projective space $\mathbb{R}P^{n-1}$.

(iii) Consider the usual identifications on the closed square $[0, 1] \times [0, 1] \in \mathbb{R}^2$ (see Fig. 6.6). The flat metric on $\mathbb{R}^2$ induces the flat metric on T^2. Let $p = (\frac{1}{2}, \frac{1}{2})$. It is immediate (see Fig. 6.6) that the cut locus of $p \in T^2$ consists of two closed curves which form a basis of the first integer homology group $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(iv) Consider the usual identifications on the closed square $[0, 1] \times [0, 1] \in \mathbb{R}^2$ (see Fig. 6.7). The flat metric on $\mathbb{R}^2$ induces the flat metric on K. Let $p = (\frac{1}{2}, \frac{1}{2})$. It is immediate (see Fig. 6.7) that the cut locus of $p \in K$ consists of two closed curves which form a basis of the first integer homology group $H_1(K, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

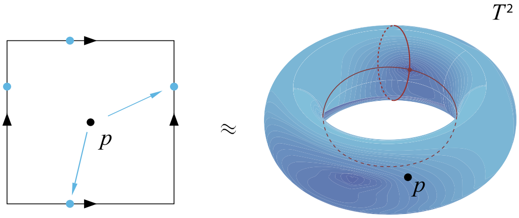


Fig. 6.6 The cut locus on the 2-torus

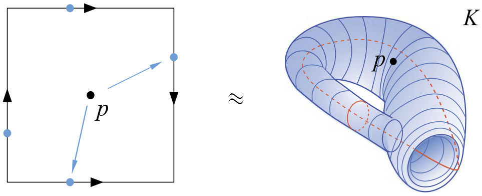


Fig. 6.7 The cut locus on the Klein bottle

6.6 Gauss' Lemma for Covariant Symmetric Tensors

The problems in this section have as common aim proving the Gauss Lemma for general covariant symmetric tensor fields of finite order. The explicit statement is given in Problem 6.72.

Problem 6.68 Let M be a differentiable manifold of dimension n, ∇ a linear connection on M, and $p \in M$. Let exp denote the restriction of the exponential map to T_pM. There exists a star-shaped open neighbourhood V of 0 in T_pM and an open neighbourhood U of p in M such that $\exp V = U$ and $\exp\colon V \to U$ is a diffeomorphism. A fixed basis $(u_1, \dots, u_n)$ of T_pM defines an isomorphism $u\colon \mathbb{R}^n \to T_pM$. Let $A = u^{-1}(V) \subset \mathbb{R}^n$. Then (see Definition 5.8) the map $\varphi = u^{-1} \circ \exp^{-1}$ is called a *system of normal coordinates at* p (related to the connection ∇), and the maps $x^i = r^i \circ \varphi$, r^i being the coordinate functions $r^i : \mathbb{R}^n \to \mathbb{R}$, are called the coordinate functions of φ; we thus have the diagram

$$\begin{array}{ccc} M \supset U & \xrightarrow{\exp^{-1}} & V \subset T_pM \\ x^i \downarrow & \searrow \varphi & \downarrow u^{-1} \\ \mathbb{R} & \xleftarrow{r^i} & A \subset \mathbb{R}^n . \end{array}$$

Consider now the functions $\Gamma^i_{jk}\colon A \to \mathbb{R}$ defined by

$$\Gamma^i_{jk} = \left(\mathrm{d}x^i \left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right) \right) \circ \varphi^{-1}, \quad i, j, k = 1, \dots, n.$$

Prove that if ∇ is torsionless, then

$$\Gamma^i_{jk}(0) = 0.$$

Hint Let $\zeta = (\zeta^1, \dots, \zeta^n) \in A$ and denote $v = u(\zeta)$. Consider the geodesic

$$\gamma_v \colon [0, 1] \to M$$

with $\gamma'(0) = v \in T_p M$.

The reader can find the relevant theory developed, for instance, in O'Neill [26, Chap. 5].

Solution For $t \in [0, 1]$ we have

$$(\varphi \circ \gamma_v)(t) = \big(u^{-1} \circ \exp^{-1}\big)\big(\exp(tv)\big) = u^{-1}\big(u(t\zeta)\big) = t\,\zeta.$$

Hence, the components of γ_v in the chart φ are

$$\gamma_v^i(t) = \varphi^i\big(\gamma_v(t)\big) = t\zeta^i,$$

so

$$\gamma_v'(t) = \sum_{i=1}^n \zeta^i \frac{\partial}{\partial x^i}\bigg|_{\gamma_v(t)}.$$

Notice that if $\zeta = e_i$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$, then $v = u_i$, so $\gamma_v' = \frac{\partial}{\partial x^i} \circ \gamma_v$. Thus, taking values at $t = 0$, we have $u_i = \frac{\partial}{\partial x^i}|_p$. The curve γ_v must satisfy the equations of the geodesics. Hence, we have

$$0 = \mathrm{d}x^i\big(\nabla_{\gamma_v'}\gamma_v'\big) = \mathrm{d}x^i\left(\nabla_{\gamma_v'}\left(\sum_{j=1}^n \zeta^j \frac{\partial}{\partial x^j} \circ \gamma_v\right)\right) = \mathrm{d}x^i\left(\sum_{j=1}^n \zeta^j \nabla_{\gamma_v'}\left(\frac{\partial}{\partial x^j} \circ \gamma_v\right)\right)$$

$$= \sum_{j,k=1}^n \zeta^j \zeta^k \big(\varGamma_{jk}^i \circ \varphi \circ \gamma_v\big).$$

So,

$$\sum_{j,k=1}^n \varGamma_{jk}^i(t\zeta)\zeta^j\zeta^k = 0.$$

Taking $t = 0$, we obtain that

$$\varGamma_{jk}^i(0) + \varGamma_{kj}^i(0) = 0, \quad i, j, k = 1, \dots, n,$$

hence $\varGamma_{jk}^i(0) = 0$ if ∇ is torsionless.

Problem 6.69 With the definitions and notations of Problem 6.68, and ∇ being a torsionless connection, consider now an r times ($r \geqslant 1$) covariant symmetric tensor $T_p \in \bigotimes^r T_p^* M$. We can write it as

$$T_p = \sum T_{i_1 \dots i_r} \left(\mathrm{d}x^{i_1}\right)_p \otimes \cdots \otimes \left(\mathrm{d}x^{i_r}\right)_p, \quad T_{i_1 \dots i_r} \in \mathbb{R}.$$

If $v = u(\zeta) \in T_p M$, then clearly

$$T_p(v, \dots, v) = \sum T_{i_1 \dots i_r} \zeta^{i_1} \cdots \zeta^{i_r}.$$

From this tensor we can define a $(0, r)$ tensor field T on U by

$$T = \sum T_{i_1 \dots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r}$$

which has constant components. Let $\rho \in \mathfrak{X}(U)$ be the vector field *vector radius* given by

$$\rho = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}.$$

We define the function $f \in C^\infty U$ by

$$f = T(\rho, \dots, \rho) = \sum T_{i_1 \dots i_r} x^{i_1} \cdots x^{i_r}.$$

Let $v = u(\zeta)$ and $a = T_p(v, \dots, v) \in \mathbb{R}$. Then

$$x^i(\exp v) = r^i\big(\varphi(\exp v)\big) = \zeta^i.$$

Hence

$$\begin{aligned} f\big(\exp(v)\big) &= \sum T_{i_1 \dots i_r} x^{i_1}\big(\exp(v)\big) \cdots x^{i_r}\big(\exp(v)\big) \\ &= \sum T_{i_1 \dots i_r} \zeta^{i_1} \cdots \zeta^{i_r} = T_p(v, \dots, v) = a. \end{aligned}$$

We now define open subsets B, B^+, $B^- \subset U$, letting

$$\begin{aligned} B^+ &= \big\{p_1 \in U : f(p_1) > 0\big\}, \\ B^- &= \big\{p_1 \in U : f(p_1) < 0\big\}, \qquad B = B^+ \cup B^-. \end{aligned}$$

It is clear that if $\zeta \in \varphi(B)$ then $t\zeta \in \varphi(B)$ for all $t \in (0, 1]$. We moreover define the *levels* of the function $v \in T_p M \mapsto T_p(v, \dots, v) \in \mathbb{R}$ and those of f as

$$H_a = \big\{v \in V : T_p(v, \dots, v) = a\big\}, \qquad \widetilde{H}_a = \big\{p_1 \in U : f(p_1) = a\big\},$$

so that obviously

$$\widetilde{H}_a = \exp(H_a).$$

For instance, if g is a Riemannian metric on M and $T_p = g_p$, then H_a would be the sphere of radius a in $T_p M$ and $\widetilde{H}_a$ the geodesic sphere of radius a centred at p.

We define on B the differentiable functions h and ξ^i, $i = 1, \dots, n$, and the vector field $\xi \in \mathfrak{X}(B)$ as follows:

$$h = |f|^{\frac{1}{r}}, \qquad \xi^i = \frac{x^i}{h}, \qquad \xi = \frac{1}{h}\rho = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}.$$

So,

$$T(\xi, \ldots, \xi) = \sum T_{i_1 \ldots i_r} \xi^{i_1} \cdots \xi^{i_r} = h^{-r} \sum T_{i_1 \ldots i_r} x^{i_1} \cdots x^{i_r} = \frac{f}{|f|}.$$

Hence,

$$T(\xi, \ldots, \xi) = \begin{cases} 1 & \text{if } \xi \in B^+, \\ -1 & \text{if } \xi \in B^-. \end{cases}$$

The function h is *semi-homogeneous of degree* 1, in the sense that

$$h\big(\exp(tv)\big) = th\big(\exp(v)\big), \quad t \geqslant 0,$$

and each function ξ^i is *semi-homogeneous of degree zero*, that is,

$$\xi^i\big(\exp(tv)\big) = \xi^i\big(\exp(v)\big).$$

Hence $\xi(\xi^i) = \rho(\xi^i) = 0$ and $\rho(h) = h$, that is,

$$\xi\big(\xi^i\big) = \mathrm{d}\xi^i(\xi) = 0, \qquad \xi(h) = \mathrm{d}h(\xi) = 1.$$

Then:

(i) Prove that

$$\nabla_\xi \xi = 0.$$

Now consider, for each $i = 1, \ldots, n$, the vector field $X_i \in \mathfrak{X}(U)$ obtained under radial parallel transport of u_i and let $\chi^i \in \Lambda^1 U$, $i = 1, \ldots, n$, be the differentiable 1-forms dual to the vector fields X_i. Then, the forms χ^i are also parallel along the curves

$$t \mapsto \varphi^{-1}(tv), \quad t \in [0,1], \; v \in V.$$

We thus have on B that

$$X_{i\,|p} = u_i = \left.\frac{\partial}{\partial x^i}\right|_p, \qquad \nabla_\xi X_i = \nabla_\rho X_i = 0,$$

$$\chi^i(X_j) = \delta^i_j, \qquad \big(\chi^i\big)_p = \big(\mathrm{d}x^i\big)_p, \qquad \nabla_\rho \chi^i = \nabla_\xi \chi^i = 0.$$

Denote

$$\varphi^i = \chi^i - \xi^i \,\mathrm{d}h \in A^1(B).$$

Then:

(ii) Prove that for each $i = 1, \ldots, n$, one has

$$\varphi^i(\xi) = 0, \qquad \chi^i(\xi) = \xi^i.$$

Hint (to (ii)) Compute the limit of

$$\chi^i(\xi) = \sum_{j=1}^{n} \xi^j \chi^i \frac{\partial}{\partial x^j},$$

when one goes to p along a radius.

Solution

(i) We have

$$\mathrm{d}x^i(\nabla_\xi \xi) = \xi\left(\xi^i\right) + \frac{1}{h^2} \sum_{j,k=1}^{n} \Gamma^i_{jk} x^j x^k = 0,$$

that is,

$$\nabla_\xi \xi = 0.$$

(ii) In fact, we have

$$\varphi^i(\xi) = \chi^i(\xi) - \xi^i \, \mathrm{d}h(\xi) = \chi^i(\xi) - \xi^i,$$

but

$$\xi\left(\chi^i(\xi) - \xi^i\right) = \left(\nabla_\xi \chi^i\right)(\xi) + \chi^i(\nabla_\xi \xi) - \xi\left(\xi^i\right) = 0,$$

so it follows that $\chi^i(\xi) - \xi^i$ is constant along the radiuses. Let us see the limit of

$$\chi^i(\xi) = \sum_{j=1}^{n} \xi^j \chi^i\left(\frac{\partial}{\partial x^j}\right)$$

when one goes to p along a radius. Since the functions ξ^j are constant along a radius because $\xi(\xi^j) = 0$, we get that this limit is equal to

$$\xi^j \left(\mathrm{d}x^i\right)_p \left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \xi^i.$$

So the limit of $\chi(\xi) - \xi^i$ when one goes to p along a radius is zero. Since that function is constant along radiuses, we conclude.

Problem 6.70 With the definitions and notations of Problems 6.68 and 6.69, suppose now that ∇ is torsionless and that T_p is the value at p of an r times covariant symmetric tensor field $\widetilde{T}$ such that

$$\nabla \widetilde{T} = 0.$$

We restrict ourselves to B^+ for the sake of simplicity, as the treatment for B^- is similar.

Since $\widetilde{T}$ is parallel, its value at a point $\exp(v)$ of U can be obtained by parallel transport from T_p along the curve $t \mapsto \exp(tv)$. As

$$T_p = \sum T_{i_1 \dots i_r} \big(\mathrm{d}x^{i_1}\big)_p \otimes \cdots \otimes \big(\mathrm{d}x^{i_r}\big) p = \sum T_{i_1 \dots i_r} \chi_p^{i_1} \otimes \cdots \otimes \chi_p^{i_r},$$

we have

$$\widetilde{T} = \sum T_{i_1 \dots i_r} \chi^{i_1} \otimes \cdots \otimes \chi^{i_r}.$$

Let β be a differential 1-form on B^+ such that

$$\beta(\xi) = 0, \qquad \iota_\xi \, \mathrm{d}\beta = 0,$$

where ι_ξ denotes the interior product by ξ.

Then prove that either $\beta \equiv 0$ or

$$\lim_{x \to p} \beta(x)$$

does not exist.

Solution Put $\beta = \sum_i \beta_i \, \mathrm{d}x^i$. Since ρ and ξ are proportional on B, we have $\beta(\rho) = \sum_i \beta_i x^i = 0$, from which

$$\sum_i \big(x^i \, \mathrm{d}\beta_i + \beta_i \, \mathrm{d}x^i\big) = 0.$$

We also have that $\iota_\rho \, \mathrm{d}\beta = 0$, so

$$\begin{aligned}
\iota_\rho \, \mathrm{d}\beta &= \iota_\rho \sum_{i=1}^n \big(\mathrm{d}\beta_i \wedge \mathrm{d}x^i\big) = \sum_{i=1}^n \big(\rho(\beta_i) \, \mathrm{d}x^i - x^i \, \mathrm{d}\beta_i\big) \\
&= \sum_{i,j=1}^n \frac{\partial \beta_i}{\partial x^j} x^j \, \mathrm{d}x^i + \sum_{i=1}^n \beta_i \, \mathrm{d}x^i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial \beta_i}{\partial x^j} x^j + \beta_i \right) \mathrm{d}x^i = 0.
\end{aligned}$$

Hence, for each $i = 1, \dots, n$, we have that

$$\sum_{j=1}^n \frac{\partial \beta_i}{\partial x^j} x^j = -\beta_i.$$

This implies that β_i is *semi-homogeneous of degree* -1, that is, for any $\zeta \in \varphi(B^+)$ and any $t \geqslant 0$ we have

$$\beta_i\big(\varphi^{-1}(t\zeta)\big) = \frac{1}{t} \, \beta_i\big(\varphi^{-1}(\zeta)\big),$$

so that either $\beta_i(\varphi^{-1}(\zeta)) = 0$ or $\lim_{t \to 0} \beta_i(\varphi^{-1}(t\zeta))$ does not exist.

Problem 6.71 With the definitions and notations of Problems 6.68, 6.69 and 6.70, consider now the differential 1-form $\kappa \in A^1(B^+)$ defined by

$$\kappa = \sum_i T\left(\xi, \dots, \xi, \frac{\partial}{\partial x^i}\right)\varphi^i.$$

Then prove:

(i)

$$\kappa \equiv 0.$$

(ii)

$$\nabla_\xi \, \mathrm{d}h = \nabla_\rho \, \mathrm{d}h = 0.$$

Solution

(i) Putting to be short

$$T\big(\xi^{(r)}, Y\big) = T\big(\xi, \overset{(r)}{\dots}, \xi, Y\big),$$

where Y stands for any vector field on B^+, on the one hand, we have

$$\sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\xi^i \, \mathrm{d}h = T\big(\xi^{(r)}, \xi\big)\, \mathrm{d}h = \mathrm{d}h,$$

so that

$$\kappa = \sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i - \mathrm{d}h.$$

Differentiating now the expression $h^r = \sum T_{i_1 \dots i_r} x^{i_1} \cdots x^{i_r}$ we have, because of the symmetry of T, that

$$\begin{aligned} r h^{r-1}\, \mathrm{d}h &= r \sum T_{i_1 \dots i_r} x^{i_1} \cdots x^{i_{r-1}} \, \mathrm{d}x^{i_r} = r h^{r-1} \sum T_{i_1 \dots i_r} \xi^{i_1} \cdots \xi^{i_{r-1}} \, \mathrm{d}x^{i_r} \\ &= r h^{r-1} \sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right) \mathrm{d}x^i. \end{aligned}$$

Hence

$$\sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\xi^i \, \mathrm{d}h = \sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right) \mathrm{d}x^i,$$

and consequently,

$$\kappa = \sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\big(\chi^i - \mathrm{d}x^i\big).$$

We deduce that

$$\kappa(X_j) = \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\left(\delta^i_j - \mathrm{d}x^i(X_j)\right)$$

goes to zero when we approximate to p, that is, κ goes to zero when we approach p.

On the other hand, we have

$$\kappa(\xi) = \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\theta^i(p) - \mathrm{d}h(p) = \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\xi^i - h = 0.$$

According to the result in Problem 6.70, it suffices to prove that $\iota_\xi\, \mathrm{d}\kappa = 0$. To this end, we shall prove that $\mathrm{d}\kappa(\xi, X_j) = 0$.

We have

$$\begin{aligned}
\mathrm{d}\kappa(\xi, X_j) &= \xi\left(\sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\theta^i(X_j)\right) - X_j\left(\sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\theta^i(p)\right) \\
&\quad - \kappa\left([\xi, X_j]\right) \\
&= \sum_{i=1}^{n} \xi\left(T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i(X_j) - T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\xi^i\, \mathrm{d}h(X_j)\right) \\
&\quad - \kappa\left([\xi, X_j]\right) \\
&= \xi\left(T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right) - X_j(h)\right) - \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i\left([\xi, X_j]\right) \\
&\quad + \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\xi^i\, \mathrm{d}h\left([\xi, X_j]\right) \\
&= \xi\left(T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right)\right) - \xi\left(X_j(h)\right) + \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i(\nabla_{X_j}\xi) \\
&\quad + \xi\left(X_j(h)\right) - X_j\left(\xi(h)\right) \\
&= \xi\left(T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right)\right) + \sum_{i=1}^{n} T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i(\nabla_{X_j}\xi)
\end{aligned}$$

(since $T(\xi, \overset{(r)}{\ldots}, \xi) = 1$, $\xi(h) = 1$, and $[\xi, X_j] = \nabla_\xi X_j - \nabla_{X_j}\xi = -\nabla_{X_j}\xi$).

Now,

$$\begin{aligned}
T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right) &= \sum T_{i_1 \ldots i_{r-1} j}\xi^{i_1}\cdots\xi^{i_{r-1}} \\
&= \sum T_{i_1 \ldots i_r}\chi^{i_1}(\xi)\cdots\chi^{i_{r-1}}(\xi)\chi^{i_r}(X_j) = \widetilde{T}\left(\xi^{(r)}, X_j\right),
\end{aligned}$$

so that

$$\xi\left(T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right)\right) = (\nabla_\xi \widetilde{T})\big(\xi^{(r)}, X_j\big) + (r-1)\widetilde{T}\big(\xi^{(r)}, \nabla_\xi \xi, X_j\big) \\ + \widetilde{T}\big(\xi^{(r)}, \nabla_\xi X_j\big) = 0.$$

Finally, we have

$$\begin{aligned}\sum_{i=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right)\chi^i(\nabla_{X_j}\xi) &= \sum_{i=1}^n \widetilde{T}\big(\xi^{(r)}, X_i\big)\chi^i(\nabla_{X_j}\xi) = \widetilde{T}\big(\xi^{(r)}, \nabla_{X_j}\xi\big) \\ &= \frac{1}{r}\nabla_{X_j}\big(\widetilde{T}(\xi, \dots, \xi)\big) = \frac{1}{r}X_j\big(\widetilde{T}(\xi, \dots, \xi)\big) \\ &= \frac{1}{r}X_j(1) = 0.\end{aligned}$$

So indeed $\kappa = 0$.

(ii) We have

$$\begin{aligned}\kappa(X_i) &= T\left(\xi^{(r)}, \frac{\partial}{\partial x^i}\right) - \sum_{j=1}^n T\left(\xi^{(r)}, \frac{\partial}{\partial x^j}\right)\xi^j \,\mathrm{d}h(X_i) \\ &= \widetilde{T}\big(\xi^{(r)}, X_i\big) - \mathrm{d}h(X_i) = 0,\end{aligned}$$

so that

$$\nabla_\xi\big(\widetilde{T}\big(\xi^{(r)}, X_i\big) - \mathrm{d}h(X_i)\big) = -(\nabla_\xi \,\mathrm{d}h)(X_i) = 0,$$

and one immediately concludes.

Problem 6.72 Here we follow the definitions and notations of Problems 6.68, 6.69, 6.70 and 6.71.

Prove that if $q \in \tilde{H}_a$, $a \neq 0$, and $X \in T_qM$ is tangent to $\tilde{H}_a$, then

$$\widetilde{T}_q(v, \dots, v, X) = 0,$$

where $v \in T_qM$ is tangent to the geodesic in U through p passing by q.

Remark Note that this result, when $\widetilde{T}$ is a Riemannian metric on M and ∇ the Levi-Civita connection, is the classical *Gauss Lemma*.

The reader can find the relevant theory developed, for instance, in O'Neill [26, Chap. 5].

Solution We have that $q \in \widetilde{H}_a$ if and only if $f(p) = a$. Thus $X(f) = 0$, and from this, $\mathrm{d}h(X) = 0$. Hence

$$
\begin{aligned}
0 = \kappa_q(X) &= T_q\left(\xi_q^{(r)}, \frac{\partial}{\partial x^i}\right)\big(\chi^i(X) - \xi^i\,\mathrm{d}h(X)\big) \\
&= \widetilde{T}_q\big(\xi_q^{(r)}, X_{i|p}\big)\chi^i(X) = \widetilde{T}_q\big(\xi_p^{(r)}, X\big).
\end{aligned}
$$

As ξ_q is tangent at q to the geodesic in B with origin p through q, we conclude.

Problem 6.73 With the definitions and notations of Problems 6.68, 6.69, 6.70, 6.71 and 6.72, prove that, as a consequence of the result in Problem 6.72, one has

(i) If $r = 1$,

$$
\begin{cases} \widetilde{T} = \mathrm{d}h & \text{on } B^+, \\ \widetilde{T} = -\mathrm{d}h & \text{on } B^-. \end{cases}
$$

(ii) If $r = 2$,

$$
\widetilde{T} = \begin{cases} \mathrm{d}h \otimes \mathrm{d}h + \sum_{i,j} T_{ij}\theta^i \otimes \theta^j & \text{on } B^+, \\ -\mathrm{d}h \otimes \mathrm{d}h + \sum_{i,j} T_{ij}\theta^i \otimes \theta^j & \text{on } B^-. \end{cases}
$$

(iii) If $\widetilde{T}$ is a pseudo-Riemannian metric, and the basis u determining a normal coordinate system is orthonormal, that is,

$$
\widetilde{T}(u_i, u_i) = \varepsilon_i \in \{-1, 1\}, \qquad \widetilde{T}(u_i, u_j) = 0, \quad i \neq j,
$$

then we have on B^+ that

$$
\widetilde{T} = \mathrm{d}h \otimes \mathrm{d}h + \sum_{i=1}^{n} \varepsilon_i \theta^i \otimes \theta^i,
$$

and on B^- that

$$
\widetilde{T} = -\mathrm{d}h \otimes \mathrm{d}h + \sum_{i=1}^{n} \varepsilon_i \theta^i \otimes \theta^i.
$$

Solution

(i) We have

$$
\widetilde{T} = \sum_i T_i \chi^i = \sum_i \big(T_i\theta^i + T_i\xi^i\,\mathrm{d}h\big) = T(\xi)\,\mathrm{d}h + \kappa = T(\xi)\,\mathrm{d}h.
$$

Notice that in the case $r = 1$ we could have taken $h = f$ instead of $h = |f|$, so we had obtained $\widetilde{T} = \mathrm{d}h$. This result is consistent with the fact that, $\widetilde{T}$ being parallel with respect to a torsionless connection ∇, we have

$$
(\mathrm{d}\widetilde{T})(X, Y) = (\nabla_X \widetilde{T})(Y) - (\nabla_Y \widetilde{T})(X) = 0,
$$

that is, $\mathrm{d}\widetilde{T} = 0$.

(ii) One similarly has

$$
\begin{aligned}
\widetilde{T} &= \sum_{i,j=1}^{n} T_{ij}\, \chi^i \otimes \chi^j = \sum_{i,j=1}^{n} T_{ij} \left(\theta^i + \xi^i \, \mathrm{d}h\right) \otimes \left(\theta^j + \xi^j \, \mathrm{d}h\right) \\
&= T(\xi, \xi)\, \mathrm{d}h \otimes \mathrm{d}h + \sum_{i,j=1}^{n} \left(T_{ij}\theta^i \otimes \theta^j + T_{ij}\xi^i \, \mathrm{d}h \otimes \theta^j + T_{ij}\theta^i \otimes \xi^j \, \mathrm{d}h\right) \\
&= T(\xi, \xi)\, \mathrm{d}h \otimes \mathrm{d}h + \sum_{i,j=1}^{n} T_{ij}\theta^i \otimes \theta^j + \mathrm{d}h \otimes \sum_{j=1}^{n} T\left(\xi, \frac{\partial}{\partial x^j}\right)\theta^j \\
&\quad + \sum_{j=1}^{n} T\left(\xi, \frac{\partial}{\partial x^j}\right)\theta^j \otimes \mathrm{d}h \\
&= T(\xi, \xi)\, \mathrm{d}h \otimes \mathrm{d}h + \sum_{i,j=1}^{n} T_{ij}\theta^i \otimes \theta^j,
\end{aligned}
$$

since

$$
\sum_{j=1}^{n} T\left(\xi, \frac{\partial}{\partial x^j}\right)\theta^j = \kappa = 0.
$$

(iii) Immediate.

6.7 Curvature and Ricci Tensors

Problem 6.74 Find the Riemann–Christoffel curvature tensor of the Riemannian manifold (U, g), where U denotes the unit open disk of the plane $\mathbb{R}^2$ and

$$
g = \frac{1}{1 - x^2 - y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).
$$

Solution We have

$$
g^{-1} \equiv \begin{pmatrix} 1 - x^2 - y^2 & 0 \\ 0 & 1 - x^2 - y^2 \end{pmatrix}.
$$

So, taking $x = x^1$, $y = x^2$, the Christoffel symbols are

$$
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \frac{x}{1 - x^2 - y^2},
$$

$$
-\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \frac{y}{1 - x^2 - y^2}.
$$

Therefore,

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = g\left(R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) = -\frac{2}{\left(1 - x^2 - y^2\right)^2}.$$

Problem 6.75 Consider on $\mathbb{R}^3$ the metric

$$g = \mathrm{e}^{2z}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right).$$

Compute $R(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z})$, where R denotes the Riemann–Christoffel curvature tensor.

Solution

$$g^{-1} \equiv \begin{pmatrix} \mathrm{e}^{-2z} & 0 & 0 \\ 0 & \mathrm{e}^{-2z} & 0 \\ 0 & 0 & \mathrm{e}^{-2z} \end{pmatrix}.$$

So, taking $x = x^1$, $y = x^2$, $z = x^3$, the only non-vanishing Christoffel symbols are

$$\Gamma^1_{13} = \Gamma^2_{23} = -\Gamma^3_{11} = -\Gamma^3_{22} = \Gamma^3_{33} = 1.$$

Therefore,

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = 0.$$

Problem 6.76 Let (M, g) be a Riemannian n-manifold. Consider an orthonormal basis $\{e_1, \dots, e_{n-1}, X\}$ of $T_pM,\ p \in M$. Let P_i be the plane section generated by e_i and X; $K(P_i)$ the sectional curvature of P_i; and $\mathbf{r}$ the Ricci tensor. Prove that

$$\mathbf{r}(X, X) = \sum_{i=1}^{n-1} K(P_i).$$

Solution For $R(X, e_i)e_i$ and $R(X, e_i, X, e_i)$ as in Definition 6.5, we have

$$K(P_i) = \frac{R(X, e_i, X, e_i)}{g(X, X)g(e_i, e_i) - g(X, e_i)^2} = \frac{g(R(X, e_i)e_i, X)}{g(X, X)g(e_i, e_i) - g(X, e_i)^2}$$
$$= g\left(R(X, e_i)e_i, X\right).$$

On the other hand, with respect to the given orthonormal basis we have

$$\mathbf{r}(X, X) = \sum_{i=1}^{n-1} g\left(R(e_i, X)X, e_i\right).$$

Therefore, $\mathbf{r}(X, X) = \sum_{i=1}^{n-1} K(P_i)$.

Problem 6.77 Prove the following consequence of the second Bianchi identity on a Riemannian manifold (M, g):

$$\mathrm{d}\mathbf{s} = 2\,\mathrm{div}\,\mathbf{r},$$

where $\mathbf{r}$ and $\mathbf{s}$ denote the Ricci tensor and the scalar curvature of the Levi-Civita connection.

The relevant theory is developed, for instance, in O'Neill [26, Chap. 3].

Solution Let us fix a point $p \in M$ and consider the normal coordinates with origin p, associated to an orthonormal basis $\{\varepsilon_i\}$ of T_pM. We can get a local orthonormal moving frame (e_i) by parallel transport of $\{\varepsilon_i\}$ along radial geodesics, so $e_i|_p = \varepsilon_i$, and $\nabla e_i = 0$, along a radial geodesic; in particular, $(\nabla e_i)_p = 0$, where ∇ stands for the Levi-Civita connection.

Further, recall that $(R(X, Y)Z)(p)$, the curvature tensor field at any point p, depends only on the values of the vector fields X, Y, Z at p, so that if either X_p, or Y_p, or Z_p is zero, then $(R(X, Y)Z)(p) = 0$.

On the other hand, since ∇ is torsionless, the second Bianchi identity can be written as

$$g\big((\nabla_X R)(Y, Z)W, U\big) + g\big((\nabla_Y R)(Z, X)W, U\big) + g\big((\nabla_Z R)(X, Y)W, U\big) = 0,$$

$X, Y, Z, W, U \in \mathfrak{X}(M)$. Interchanging X and Y in the third summand and then contracting all the summands with respect to X and U, we have

$$\sum_i \big\{g\big((\nabla_{e_i} R)(Y, Z)W, e_i\big) + g\big((\nabla_Y R)(Z, e_i)W, e_i\big) - g\big((\nabla_Z R)(Y, e_i)W, e_i\big)\big\} = 0. \tag{$\star$}$$

For the second summand in ($\star$), we have

$$\begin{aligned}
&\sum_i g\big((\nabla_Y R)(Z, e_i)W, e_i\big)(p) \\
&\quad = \sum_i \big\{g\big(\nabla_Y\big(R(Z, e_i)W\big), e_i\big) - g\big(R(\nabla_Y Z, e_i)W, e_i\big) \\
&\qquad - g\big(R(Z, \nabla_Y e_i)W, e_i\big) - g\big(R(Z, e_i)\nabla_Y W, e_i\big)\big\}(p) \\
&\quad = \Big\{\sum_i Y\big(g\big(R(Z, e_i)W, e_i\big)\big) - \mathbf{r}(\nabla_Y Z, W) - \mathbf{r}(Z, \nabla_Y W)\Big\}(p) \\
&\quad = \big\{Y\big(\mathbf{r}(Z, W)\big) - \mathbf{r}(\nabla_Y Z, W) - \mathbf{r}(Z, \nabla_Y W)\big\}(p) = (\nabla_Y \mathbf{r})(Z, W)(p).
\end{aligned}$$

Similarly, for the third summand in ($\star$), we have

$$-\sum_i g\big((\nabla_Z R)(Y, e_i)W, e_i\big)(p) = -(\nabla_Z \mathbf{r})(Y, W)(p).$$

So, at the point p we can write ($\star$) as

$$\Big\{\sum_i g\big((\nabla_{e_i} R)(Y,Z)W,e_i\big) + (\nabla_Y \mathbf{r})(Z,W) - (\nabla_Z \mathbf{r})(Y,W)\Big\}(p) = 0. \qquad (\star\star)$$

Contracting ($\star\star$) with respect to Y and W, we obtain

$$\sum_{i,j}\big\{g\big((\nabla_{e_i} R)(e_j,Z)e_j,e_i\big) + (\nabla_{e_j}\mathbf{r})(Z,e_j) - (\nabla_Z\mathbf{r})(e_j,e_j)\big\}(p) = 0,$$

or equivalently,

$$\begin{aligned}
0 &= \Big\{\sum_{i,j}\{g\big(\nabla_{e_i}\big(R(e_j,Z)e_j\big),e_i\big) - g\big(R(\nabla_{e_i}e_j,Z)e_j,e_i\big) \\
&\quad - g\big(R(e_j,\nabla_{e_i}Z)e_j,e_i\big) - g\big(R(e_j,Z)\nabla_{e_i}e_j,e_i\big)\big\} + (\operatorname{div}\mathbf{r})Z \\
&\quad - \sum_j\{Z\big(\mathbf{r}(e_j,e_j)\big) - \mathbf{r}(\nabla_Z e_j,e_j) - \mathbf{r}(e_j,\nabla_Z e_j)\}\Big\}(p) \\
&= \Big\{\sum_{j,j}\{e_i\big(\mathbf{r}(Z,e_i)\big) - g\big(R(e_j,Z)e_j,\nabla_{e_i}e_i\big) - \mathbf{r}(\nabla_{e_i}Z,e_i)\} + (\operatorname{div}\mathbf{r})Z - Z\mathbf{s}\Big\}(p) \\
&= \Big\{\sum_i(\nabla_{e_i}\mathbf{r})(e_i,Z) + (\operatorname{div}\mathbf{r})Z - Z\mathbf{s}\Big\}(p) = \big\{2(\operatorname{div}\mathbf{r})Z - Z\,\mathbf{s}\big\}(p),
\end{aligned}$$

for every $p \in M$, that is, $((2\operatorname{div}\mathbf{r} - \mathrm{d}\mathbf{s})Z)(p) = 0$ for all $Z \in \mathfrak{X}(M)$ and all $p \in M$.

6.8 Characteristic Classes

Problem 6.78 Consider the complex projective space $\mathbb{C}\mathrm{P}^1$ equipped with the Hermitian metric

$$g = h(z)(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z),$$

where

$$h(z) = \frac{1}{(1+|z|^2)^2}. \qquad (\star)$$

If $w = 1/z$ is the coordinate at infinity, then the metric is given by

$$g = h(w)(\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w).$$

Prove that the Chern class of the tangent bundle $T\mathbb{C}\mathrm{P}^1$ is nonzero.

Solution Since the Chern classes of a complex vector bundle do not depend on the particular connection chosen to define them, we choose here the canonical Hermitian connection, which, for a given h defined by

$$h(z_0) = h\left(\frac{\partial}{\partial z}\bigg|_{z_0}, \frac{\partial}{\partial z}\bigg|_{z_0}\right),$$

is the connection with connection form and curvature form relatives to the holomorphic moving frame $\partial/\partial z$, given by

$$\widetilde{\omega} = h^{-1}\partial h = h^{-1}\frac{\partial h}{\partial z}\,\mathrm{d}z, \qquad \widetilde{\Omega} = \bar{\partial}\widetilde{\omega} = \frac{\partial \widetilde{\omega}}{\partial \bar{z}}\,\mathrm{d}\bar{z},$$

respectively. Then we have, for the metric h in ($\star$), the Chern form

$$\begin{aligned} c_1\big(T\mathbb{C}\mathrm{P}^1, \widetilde{\omega}\big) &= \frac{\mathrm{i}}{2\pi}\,\widetilde{\Omega} = \frac{\mathrm{i}}{2\pi}\,\bar{\partial}\frac{\partial h(z)}{h(z)} = \frac{\mathrm{i}}{2\pi}\,\bar{\partial}\left(\big(1+|z|^2\big)^2\partial\frac{1}{(1+|z|^2)^2}\right) \\ &= \frac{\mathrm{i}}{\pi(1+|z|^2)^2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = \frac{2}{\pi(1+|z|^2)^2}\,\mathrm{d}x\wedge\mathrm{d}y. \end{aligned}$$

By taking polar coordinates, it is easily seen that

$$\int_{\mathbb{C}\mathrm{P}^1} c_1\big(T\mathbb{C}\mathrm{P}^1, \widetilde{\omega}\big) = 2.$$

By Stokes' Theorem, the Chern form $c_1(T\mathbb{C}\mathrm{P}^1, \widetilde{\omega})$ cannot be exact. Thus the Chern class is $c_1(T\mathbb{C}\mathrm{P}^1) = 2\alpha \neq 0$, where α denotes the standard generator of the cohomology group $H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z}) \cong \mathbb{Z}$; and $T\mathbb{C}\mathrm{P}^1$ is thus a non-trivial complex line bundle.

Problem 6.79 Prove that the Pontrjagin forms of a space M of constant curvature K vanish.

The relevant theory is developed, for instance, in Spivak [32].

Solution The curvature forms Ω^i_j of the Levi-Civita connection on the bundle of orthonormal frames are given in terms of the components θ^k of the canonical form on the bundle of orthonormal frames (which is the restriction of the canonical form on the bundle of linear frames) by

$$\Omega^i_j = K\theta^i \wedge \theta^j.$$

Hence, by the formula on p. 576, the rth Pontrjagin form, denoted here by p_r, is given, for $r = 1, \dots, \dim M/4$, by

$$p^*(p_r) = \frac{1}{(2\pi)^{2r}(2r)!}\sum \delta^{j_1\dots j_{2r}}_{i_1\dots i_{2r}}\,\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_{2r}}_{j_{2r}}$$

$$= \frac{K^{2r}}{(2\pi)^{2r}(2r)!} \sum_{\substack{i_1,\dots,i_{2r}\\ j_1,\dots,j_{2r}}} \delta^{j_1\dots j_{2r}}_{i_1\dots i_{2r}} \theta^{i_1} \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{i_{2r}} \wedge \theta^{j_{2r}},$$

where p denotes the projection map of the bundle of orthonormal frames.

The δ's vanish unless $j_1, \dots, j_{2r}$ is a permutation of $i_1, \dots, i_{2r}$, but then the wedge product of θ's has repeated factors, so $p^*(p_r)$ vanishes. As p^* is injective, p_r also vanishes.

Problem 6.80 Let M be a 4-dimensional compact oriented C^∞ manifold. Let Ω^i_j, $i, j = 1, \dots, 4$, be the curvature forms of a linear connection on M, and let $\widetilde{\Omega}^i_j$ be given by $\Omega^j_i = \sigma^* \widetilde{\Omega}^i_j$, the curvature forms relative to any fixed orthonormal moving frame σ on M. Prove that the signature $\tau(M)$ can be expressed by

$$\tau(M) = -\frac{1}{24\pi^2} \int_M \sum_{i<j} \widetilde{\Omega}^i_j \wedge \widetilde{\Omega}^j_i.$$

Hint Use Hirzebruch's formula in Theorem 6.16.

Solution To apply the Hirzebruch Theorem, we need to compute a representative form of the first Pontrjagin class of M, that is, of the first Pontrjagin class of the tangent bundle TM. The principal GL$(4, \mathbb{R})$-bundle corresponding to TM is the frame bundle (FM, p, M), where the given connection is defined. Now, by Weil's Theorem, the characteristic class does not depend on the chosen connection. Thus, as we can always reduce the structure group to the orthogonal group O(4) (here, even to SO(4), since M is oriented) or, equivalently, take a Riemannian metric on M, the matrix of curvature 2-forms of the connection is skew-symmetric, as it takes values in the Lie algebra $\mathfrak{so}(4)$. We shall compute the Pontrjagin form $p_1(M)$ in terms of the curvature forms of the metric connection in two related ways, which is perhaps instructive. The form $p_1(M)$ is given by

$$\begin{aligned} p^*\big(p_1(M)\big) &= \text{term of } \det\left(I - \frac{1}{2\pi}\Omega\right) \text{ quadratic in the } \Omega\text{'s} \\ &= \frac{1}{4\pi^2}\big(-\Omega^1_2 \wedge \Omega^2_1 - \Omega^1_3 \wedge \Omega^3_1 - \Omega^1_4 \wedge \Omega^4_1 - \Omega^2_3 \wedge \Omega^3_2 \\ &\quad - \Omega^2_4 \wedge \Omega^4_2 - \Omega^3_4 \wedge \Omega^4_3\big) \\ &= \frac{1}{4\pi^2}\left(-\frac{1}{2}\operatorname{tr}(\Omega \wedge \Omega)\right). \end{aligned}$$

We can also directly use the formula on p. 576, as follows. Let (i_1, i_2) be an ordered subset of $\{1, 2, 3, 4\}$, (j_1, j_2) a permutation of (i_1, i_2), and $\delta^{j_1 j_2}_{i_1 i_2}$ the sign of the permutation. Then

$$\begin{aligned}
p^*\big(p_1(M)\big) &= \frac{1}{(2\pi)^2 2!} \sum \delta^{j_1 j_2}_{i_1 i_2}\, \Omega^{i_1}_{j_1} \wedge \Omega^{i_2}_{j_2} \\
&= \frac{1}{8\pi^2}\big(-\Omega^1_2 \wedge \Omega^2_1 - \Omega^1_3 \wedge \Omega^3_1 - \Omega^1_4 \wedge \Omega^4_1 - \Omega^2_1 \wedge \Omega^1_2 \\
&\quad - \Omega^2_3 \wedge \Omega^3_2 - \Omega^2_4 \wedge \Omega^4_2 - \Omega^3_1 \wedge \Omega^1_3 - \Omega^3_2 \wedge \Omega^2_3 \\
&\quad - \Omega^3_4 \wedge \Omega^4_3 - \Omega^4_1 \wedge \Omega^1_4 - \Omega^4_2 \wedge \Omega^2_4 - \Omega^4_3 \wedge \Omega^3_4\big) \\
&= -\frac{1}{8\pi^2}\operatorname{tr}(\Omega \wedge \Omega).
\end{aligned}$$

Furthermore, since for any given invariant polynomial in the curvature, the corresponding differential form on the base space (see, for instance, [24, p. 295], [29, vol. IV, L. 22]) does not depend on the chosen orthonormal moving frame, by applying the Hirzebruch formula, we can write:

$$\tau(M) = \frac{1}{3}\int_M p_1(M) = -\frac{1}{3}\int_M \frac{1}{8\pi^2}\sum_{i<j} \widetilde{\Omega}^i_j \wedge \widetilde{\Omega}^j_i,$$

as stated.

Problem 6.81 Let M be a differentiable manifold and let $F \subset TM$ be an involutive sub-bundle of TM. A linear connection ∇ on the quotient bundle TM/F is said to be a *basic connection* if the following property holds:

$$\nabla_X\big(Y(\operatorname{mod} F)\big) = [X, Y](\operatorname{mod} F), \quad X \in \Gamma(F),\ Y \in \mathfrak{X}(M).$$

Prove:

(i) Basic connections exist.
(ii) For any basic connection ∇, one has

$$R^\nabla(X, Y) = 0, \quad X, Y \in \Gamma(F).$$

Suppose that the normal vector bundle

$$E = TM/F,$$

has rank q (that is, the dimension of the fiber equals q).

Let $\operatorname{Pont}^k(E)$ be the vector subspace of the cohomology group $H^k(M, \mathbb{R})$ consisting of the homogeneous polynomials of degree k in the Pontrjagin classes $p_i(E) \in H^{4i}(M, \mathbb{R})$.

(iii) Prove the *Bott Theorem on Pontrjagin classes of foliated manifolds*, namely

$$\operatorname{Pont}^k(E) = 0, \quad k > 2q.$$

Remark The basic connections are also called *adapted linear connections*. The relevant theory is developed, for instance, in Bott [3], Heitsch [15], and Moscovici [25].

Solution

(i) If g is a Riemannian metric on M, then $TM = F \oplus F^{\perp}$ and every $X \in \mathfrak{X}(M)$ decomposes uniquely as $X = X' + X''$, with $X' \in \Gamma(F)$, $X'' \in \Gamma(F^{\perp})$. We define

$$\nabla_X\big(Y(\operatorname{mod} F)\big) = \big[X', Y\big](\operatorname{mod} F), \quad X, Y \in \mathfrak{X}(M),$$

and the definition makes sense, as for all $Z \in \Gamma(F)$ one has

$$\begin{aligned}\nabla_X\big(Y + Z(\operatorname{mod} F)\big) &= \big[X', Y + Z\big](\operatorname{mod} F) = \big(\big[X', Y\big] + \big[X', Z\big]\big)(\operatorname{mod} F)\\ &= \big[X', Y\big](\operatorname{mod} F) = \nabla_X\big(Y(\operatorname{mod} F)\big)\end{aligned}$$

because $[X', Z]$ belongs to $\Gamma(F)$, the sub-bundle F being involutive. Next we check the properties of a connection for ∇.

If $f \in C^\infty(M)$, then $(fX)' = fX'$ and one has

$$\begin{aligned}\nabla_{fX}\big(Y(\operatorname{mod} F)\big) &= \big[fX', Y\big](\operatorname{mod} F) = \big(f\big[X', Y\big] - (Yf)X'\big)(\operatorname{mod} F)\\ &= f\big[X', Y\big](\operatorname{mod} F) = f\,\nabla_X\big(Y(\operatorname{mod} F)\big),\end{aligned}$$

as $(Yf)X' \in \Gamma(F)$. Finally,

$$\begin{aligned}\nabla_X\big(fY(\operatorname{mod} F)\big) &= \big[X', fY\big](\operatorname{mod} F) = \big(\big(X'f\big)Y + f\big[X', Y\big]\big)(\operatorname{mod} F)\\ &= \big(X'f\big)\big(Y(\operatorname{mod} F)\big) + f\nabla_X\big(Y(\operatorname{mod} F)\big).\end{aligned}$$

(ii) If $X, Y \in \Gamma(F)$, then from the definition of ∇ one obtains

$$\begin{aligned}R^\nabla(X, Y)\big(Z(\operatorname{mod} F)\big) &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\big(Z(\operatorname{mod} F)\big)\\ &= \big(\big[X, [Y, Z]\big] - \big[Y, [X, Z]\big] - \big[[X, Y], Z\big]\big)(\operatorname{mod} F)\\ &= \big(\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]\big)(\operatorname{mod} F)\\ &= 0.\end{aligned}$$

(iii) Since by the Weil Theorem the characteristic classes do not depend on the chosen connection, we compute the Pontrjagin forms in terms of the curvature forms of a basic connection ∇. Let $\{e_1, \dots, e_n\}$ be a local frame field and let $\{\theta^1, \dots, \theta^n\}$ be its dual basis of differential 1-forms. The curvature 2-form Ω of ∇ is locally given by

$$\Omega = \sum_{i<j} \Omega^i_j \theta^i \wedge \theta^j,$$

and the kth Pontrjagin form $p_k(\Omega)$ is furnished by the term of

$$\det\left(I - \frac{1}{2\pi}\Omega\right)$$

of order $2k$ in the Ω's, so it is given (up to a coefficient) by

$$p_k(\Omega) = \operatorname{tr}\bigl(\Omega \wedge \overset{(2k)}{\cdots} \wedge \Omega\bigr) = \sum \operatorname{tr}\bigl(\Omega^{i_1}_{j_1}, \dots, \Omega^{i_{2k}}_{j_{2k}}\bigr)\, \theta^{i_1} \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{i_{2k}} \wedge \theta^{j_{2k}}.$$

If $k > 2q$, at least one pair belongs to F. From (ii) above one gets

$$\theta^{i_1} \wedge \cdots \wedge \theta^{i_{2k}} = 0.$$

Problem 6.82 Let g be the bi-invariant metric on SO(3). Calculate the *Chern–Simons invariant* $J(\mathrm{SO}(3), g)$.

Remark For the related definitions and results, see Definition 6.17. The relevant theory is developed, for instance, in Chern and Simons [8].

Solution Let X_1, X_2, X_3 be the standard basis of TS^3, that is,

$$X_1 = \frac{1}{2}\left(-x^1 \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}\right),$$

$$X_2 = \frac{1}{2}\left(-x^2 \frac{\partial}{\partial x^0} - x^3 \frac{\partial}{\partial x^1} + x^0 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3}\right),$$

$$X_3 = \frac{1}{2}\left(-x^3 \frac{\partial}{\partial x^0} + x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + x^0 \frac{\partial}{\partial x^3}\right).$$

From Problem 4.90, it follows that X_1, X_2, X_3 are left-invariant vector fields, and as the Lie groups S^3 and SO(3) have the same Lie algebra, X_1, X_2, X_3 can be considered as left-invariant vector fields on SO(3).

We have

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2.$$

Let $\alpha^i_j = -\alpha^j_i$, $i, j = 1, 2, 3$, be the Maurer–Cartan forms on SO(3), determined by

$$\alpha^3_2(X_i) = \delta_{1i}, \qquad \alpha^3_1(X_i) = \delta_{2i}, \qquad \alpha^2_1(X_i) = \delta_{3i},$$

so that the structure equations of the Lie group, are

$$\mathrm{d}\alpha^i_j = \sum_{k=1}^{3} \alpha^i_k \wedge \alpha^k_j, \quad i, j = 1, 2, 3. \tag{$\star$}$$

The bi-invariant metric on SO(3) is given by

$$g = \alpha^1_2 \otimes \alpha^1_2 + \alpha^1_3 \otimes \alpha^1_3 + \alpha^2_3 \otimes \alpha^2_3.$$

By writing these equations, one has chosen a basis of the Lie algebra of SO(3) and hence, by right translations, a frame field on the manifold SO(3). It is convenient to choose the notation so that the equations remain invariant under a cyclic permutation of 1, 2, 3. Setting $\theta^i = \alpha^k_j$, $i, j, k =$ cyclic permutation of 1, 2, 3, the invariant metric becomes

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3.$$

The connection and curvature forms $\omega^i_j = -\omega^j_i$, $\Omega^i_j = -\Omega^j_i$, are determined by the Cartan structure equations

$$\mathrm{d}\theta^i = -\sum_j \omega^i_j \wedge \theta^j, \qquad \mathrm{d}\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j.$$

Comparing these equations with the structure equations ($\star$) of the Lie group, one finds

$$\omega^i_j = \frac{1}{2}\theta^k, \qquad \Omega^i_j = -\frac{1}{4}\theta^i \wedge \theta^j.$$

Hence

$$\begin{aligned} \frac{1}{2} T P_1(\Omega) &= \frac{1}{8\pi^2} \sum_{1 \leqslant i < j \leqslant 3} \omega^i_j \wedge \Omega^i_j - \frac{1}{8\pi^2} \omega^1_2 \wedge \omega^2_3 \wedge \omega^3_1 \\ &= -\frac{1}{16\pi^2} \theta^1 \wedge \theta^2 \wedge \theta^3. \end{aligned} \tag{$\star\star$}$$

Let us compute the total volume of SO(3) with its bi-invariant metric. We have

$$\mathrm{vol}_g\big(\mathrm{SO}(3)\big) = \mathrm{vol}_g\big(\mathbb{R}\mathrm{P}^3\big) = \frac{1}{2}\mathrm{vol}_g\big(S^3\big).$$

Hence we only need to calculate $\mathrm{vol}_g(S^3)$. Write

$$\theta^1 \wedge \theta^2 \wedge \theta^3 = \rho\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

By calculation we obtain

$$\big(\theta^1 \wedge \theta^2 \wedge \theta^3\big)(X_1, X_2, X_3) = \det\big(\theta^i(X_j)\big) = -1$$

and

$$\big(\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\big)(X_1, X_2, X_3) = \frac{1}{8} \det \begin{pmatrix} -x^1 & -x^2 & -x^3 \\ x^0 & -x^3 & x^2 \\ x^3 & x^0 & -x^1 \end{pmatrix} = -\frac{x^3}{8}.$$

Hence $\rho = \frac{8}{x^3}$. By considering the parametrisation of S^3 (see Remark 1.4) given by

$$x^0 = \sin u, \qquad x^1 = \cos u \sin v, \qquad x^2 = \cos u \cos v \sin w,$$
$$x^3 = \cos u \cos v \cos w,$$

with $u, v \in (-\pi/2, \pi/2)$, $w \in (-\pi, \pi)$, we compute

$$\int_{S^3} \theta^1 \wedge \theta^2 \wedge \theta^3 = 8\left(\int_{-\pi/2}^{\pi/2} \cos^2 u \,\mathrm{d}u\right)\left(\int_{-\pi/2}^{\pi/2} \cos v \,\mathrm{d}v\right)\left(\int_{-\pi}^{\pi} \mathrm{d}w\right) = 16\pi^2.$$

Therefore,

$$\mathrm{vol}_g\big(\mathrm{SO}(3)\big) = 8\pi^2,$$

and from ($\star\star$) we conclude

$$J\big(\mathrm{SO}(3), g\big) = \frac{1}{2}.$$

Remark The reader can check that the metric g is really bi-invariant by proving that the forms θ^i are also right-invariant. This readily follows from formula ($\star\star$) in Problem 4.71.

6.9 Isometries

Problem 6.83 Let (M, g) be the Poincaré upper half-plane. We define an action of $\mathrm{SL}(2, \mathbb{R})$ on M as follows: Identifying $\mathbb{R}^2$ with $\mathbb{C}$ as usual, let $z = x + \mathrm{i}y$. Given $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, we define

$$w = sz = \frac{az + b}{cz + d}.$$

(i) Prove that $\mathrm{SL}(2, \mathbb{R})$ is a group of isometries of (M, g) (called the *group of fractional linear transformations* of the Poincaré upper half-plane).
(ii) Prove that under these isometries the half-circles with centre at the x-axis are transformed either in the same type of half-circles or in vertical lines.

Hint One can obtain an Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$, writing each $s \in \mathrm{SL}(2, \mathbb{R})$ as the product of a matrix of $\mathrm{SO}(2)$ by a diagonal matrix with determinant equal to 1 by an upper triangular matrix with the elements of the diagonal equal to 1.

Solution

(i) Given $z = x + \mathrm{i}y \in M$ we have $y = \mathrm{Im}\, z > 0$. We also have $\mathrm{Im}\, w > 0$. In fact,

$$\operatorname{Im} w = \operatorname{Im}\frac{az+b}{cz+d} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0,$$

and thus $w \in M$.

Moreover, $s_2(s_1 z) = (s_2 s_1)z$. In fact, putting

$$s_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad s_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

we have

$$s_2(s_1 z) = \frac{(a'a + b'c)z + a'b + b'd}{(c'a + d'c)z + c'b + d'd} = (s_2 s_1)z.$$

The metric g can be written on $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ as

$$g = \frac{1}{2}\frac{\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z}{(\operatorname{Im} z)^2}.$$

Moreover, it is easy to compute that

$$\frac{\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w}{(\operatorname{Im} w)^2} = \frac{\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z}{(\operatorname{Im} z)^2}.$$

As the expression of g is preserved in the new coordinates, the action of s is an isometry. Thus $\mathrm{SL}(2,\mathbb{R})$ acts on M as a group of isometries.

(ii) From the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ in the hint, we can write each element $s \in \mathrm{SL}(2,\mathbb{R})$ uniquely as a product

$$s = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \lambda \neq 0.$$

Considering the previous property $s_2(s_1 z) = (s_2 s_1)z$, in order to study the action of $\mathrm{SL}(2,\mathbb{R})$ on a half-circle of M with centre at the x-axis, it suffices to see the consecutive action of the elements of the previous decomposition. The action

$$z \mapsto \frac{az+b}{cz+d}$$

by an element of the type $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is $z \mapsto z + a$. That is, the translation of the half-circle by the vector $a + 0\,\mathrm{i}$, $a \in \mathbb{R}$ (see Fig. 6.8(a)).

The action by an element of the type $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$, $\lambda \in \mathbb{R} \setminus \{0\}$, is $z \mapsto \lambda^2 z$. That is, a homothety of ratio $\lambda^2 \in \mathbb{R}^+$ (see Fig. 6.8(b)).

From these results, it follows that to study the whole action it suffices to consider the unit half-circle C with centre at $(0,0)$.

We can parametrise that half-circle as

$$(x, y) = (\cos\beta, \sin\beta), \quad \beta \in (0, \pi).$$

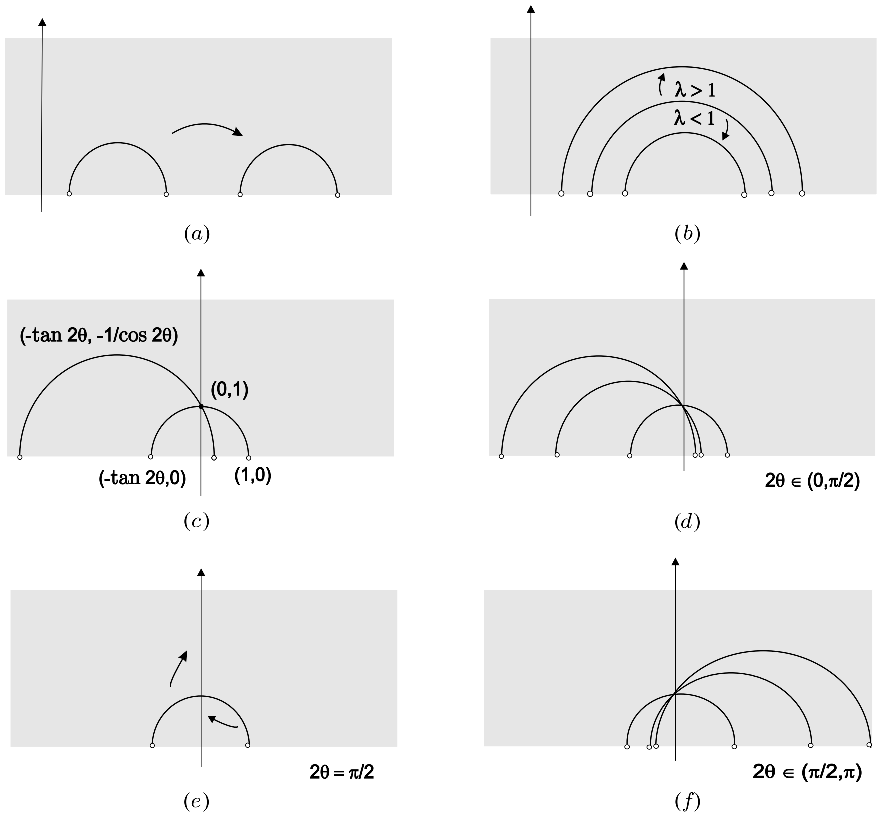


Fig. 6.8 Variations of the image of the half-circle C

The action of an element $s = \left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right) \in \mathrm{SO}(2)$ is

$$z \mapsto \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta} \equiv \left(\frac{x\cos 2\theta}{1 + x\sin 2\theta}, \frac{y}{1 + x\sin 2\theta}\right).$$

The limits when $y \to 0$, that is when $x \to \pm 1$, are

$$\left(-\frac{\cos 2\theta}{1 - \sin 2\theta}, 0\right), \qquad \left(\frac{\cos 2\theta}{1 + \sin 2\theta}, 0\right),$$

respectively. So the centre of the half-circle image is at the point of the x-axis with abscissa

$$\frac{1}{2}\frac{\cos 2\theta}{1 - \sin^2 2\theta}(1 - \sin 2\theta - 1 - \sin 2\theta) = -\tan 2\theta.$$

Moreover, from

$$\left(\frac{x\cos 2\theta}{1+x\sin 2\theta}+\tan 2\theta\right)^2+\frac{y^2}{(1+x\sin 2\theta)^2}=r^2,$$

one has $r=1/\cos 2\theta$. The image of C is thus the half-circle of centre $-\tan 2\theta$ and radius $1/\cos 2\theta$, if $\cos 2\theta \neq 0$. The image of $(0,1)$ is $(0,1)$ (see Fig. 6.8(c)).

Let us now see how the image of C changes as a function of $\theta \in [0, 2\pi]$.

If $\theta = 0$, we have the identity $C \to C$.

For the interval $2\theta \in (0, \pi/2)$ we obtain as images a family of half-circles of the previous type, and $(0,1)$ is preserved (see Fig. 6.8(d)).

For $2\theta = \frac{\pi}{2}$ the image is $\{(0, \sqrt{\frac{1-x}{1+x}})\}$, so the part at the first quadrant goes to the vertical segment from $(0,0)$ to $(0,1)$, and the part at the second quadrant into the vertical half-line $\{(0,y) : y \in (1,\infty)\}$. Thus C is transformed in the half-line $\{(0,y) : y \in (0,\infty)\}$ (see Fig. 6.8(e)).

For the interval $2\theta \in (\pi/2, \pi)$ we have the images obtained by reflection in the y-axis of the ones corresponding to the interval $(0, \pi/2)$ because $\cos 2\theta$ changes its sign, but $\sin 2\theta$ does not (see Fig. 6.8(f)).

For $2\theta = \pi$ the image is $(-x, y)$, that is $g(C) = C$ by reflection on the y-axis.

For the interval $2\theta \in (\pi, 3\pi/2)$, the values of $\cos 2\theta$ and $\sin 2\theta$ change their sign with respect to their values when $2\theta \in (0, \pi/2)$. Therefore, changing the sign of x, the values of $x\sin 2\theta$ and $x\cos 2\theta$ are preserved and if the value of y does not change, we obtain that the image sets are the reflections with regard to the y-axis of the ones corresponding to the interval $(0, \pi/2)$ (see Fig. 6.9(g)).

For $2\theta = 3\pi/2$ one has

$$(x,y) \mapsto \left(0, \sqrt{\frac{1+x}{1-x}}\right),$$

that is, again the half-line $\{(0,y) : y > 0\}$, but obtained from C in a different way, as we can see in Fig. 6.9(h).

For the interval $2\theta \in (3\pi/2, 2\pi)$, we have that $\sin 2\theta$ changes its sign with respect to $2\theta \in (0, \pi/2)$ and $\cos 2\theta$ preserves it. Changing x by $-x$, we have the symmetric situation with respect to the y-axis (see Fig. 6.9(i)).

For $2\theta = 2\pi$ we have $(x,y) \to (x,y)$; again the identity.

Summarising, from a half-circle of radius r with centre at the x-axis we can obtain all the half-circles with centre at the x-axis and any radius, and all the vertical lines (see Fig. 6.9(j)).

Problem 6.84

(i) Prove that the isometry group $I(S^n)$ of S^n with the round metric, is $\mathrm{O}(n+1)$.

(ii) Prove that the isometry group of the hyperbolic space H^n, equipped with the canonical metric of negative constant curvature, is the *proper Lorentz group*

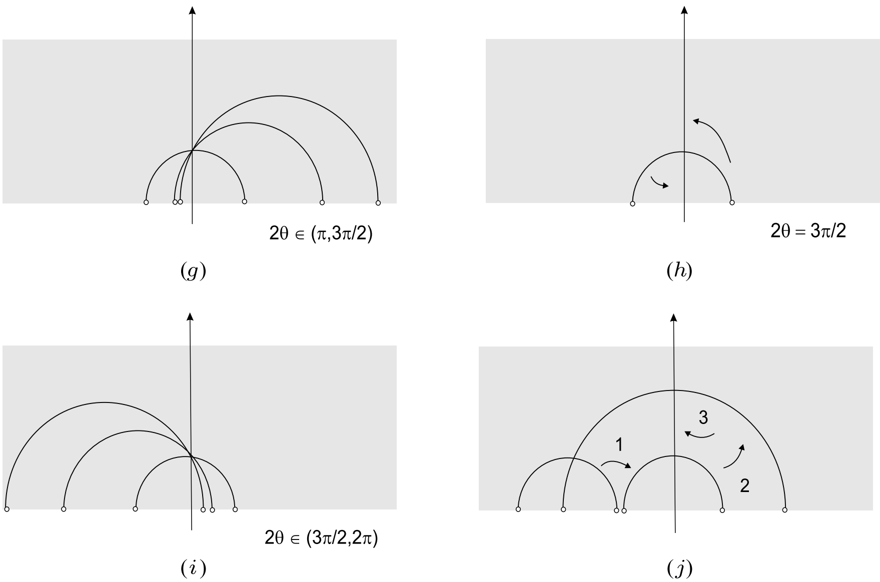


Fig. 6.9 Variations of the image of the half-circle C

$O_+(1, n)$, which is the group of all linear transformations of $\mathbb{R}^{n+1}$ which leave invariant the Lorentz product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$, defined by

$$\big\langle (x^0, x^1, \dots, x^n), (y^0, y^1, \dots, y^n) \big\rangle = -x^0 y^0 + \sum_{i=1}^{n} x^i y^i.$$

Hint Apply Theorem 6.22.

Solution

(i) Let $\langle \cdot, \cdot \rangle$ denote the Euclidean metric on $\mathbb{R}^{n+1}$. The round metric on S^n is defined by letting the embedding of S^n into $\mathbb{R}^{n+1}$ be isometric, i.e.

$$\big\langle (p, u), (p, v) \big\rangle_{S^n} = \big\langle (p, u), (p, v) \big\rangle = \langle u, v \rangle \qquad (\star)$$

for $p \in S^n$ and $(p, u), (p, v) \in TS^n$. The group $O(n+1)$ acts on S^n by isometries. In fact,

$$\begin{aligned} \big\langle a_*(p, u), a_*(p, v) \big\rangle_{S^n} &= \big\langle (ap, au), (ap, av) \big\rangle_{S^n} && \big(a \in O(n+1)\big) \\ &= \langle au, av \rangle && (\text{by } (\star)) \\ &= \langle u, v \rangle \\ &= \big\langle (p, u), (p, v) \big\rangle_{S^n} && (\text{by } (\star)). \end{aligned}$$

Now let $a \in I(S^n)$. Fix $p = e_0 \in S^n$ and the orthonormal basis $\{(p, e_j)\}$ for T_pS^n. Let $(q, \tilde{e}_j) = a_*(p, e_j) \in T_qS^n$ for $q = a(p)$. Let $b \in \mathrm{O}(n+1)$ take p to q and e_j to $\tilde{e}_j$, $j \geqslant 1$. Since $\mathrm{O}(n+1)$ acts on S^n by isometries we have $b \in I(S^n)$. As moreover $b_{*p} = a_{*p}$, applying Theorem 6.22, we obtain $a = b \in \mathrm{O}(n+1)$.

(ii) The proof is similar to that of (i) since the hyperbolic space H^n can be viewed as the component

$$\left\{\left(x^0, x^1, \dots, x^n\right) \in \mathbb{R}^{n+1} : -\left(x^0\right)^2 + \sum_{i=1}^{n}\left(x^i\right)^2 = -1,\ x^0 > 0\right\}$$

of the hyperboloid $\langle\cdot\,,\cdot\rangle^{-1}(-1)$.

Problem 6.85 Let $f\colon (M, g) \to (M', g')$ be an isometry of Riemannian manifolds. Prove that f preserves the Riemann–Christoffel curvature tensor field and the sectional curvature.

Solution An isometry is a diffeomorphism that preserves the metric, that is,

$$g'(f \cdot X, f \cdot Y) = \left(f^* g'\right)(X, Y) \circ f^{-1} = g(X, Y) \circ f^{-1}, \tag{$\star$}$$

then it is an affine map for the Levi-Civita connection, i.e. $f \cdot \nabla_X Y = \nabla'_{f \cdot X} f \cdot Y$, where ∇ and ∇' denote the Levi-Civita connections of g and g', respectively. Hence, we conclude as in Problem 5.43 that for the respective Riemann–Christoffel curvature tensors one has that

$$R'(f \cdot X, f \cdot Y, f \cdot Z, f \cdot W) = R(X, Y, Z, W) \circ f^{-1}. \tag{$\star\star$}$$

Put $p' = f(p)$. If $\{X_p, Y_p\}$ is a basis of the 2-plane P of T_pM, then $\{(f_*X)_{p'}, (f_*Y)_{p'}\}$ is a basis of the 2-plane $P = f_*P$ of $T_{p'}M'$. Then, according to Definition 6.5, from ($\star$) and ($\star\star$) above we obtain for the sectional curvature

$$\begin{aligned} K'(P) &= \frac{R'((f_*X)_{p'}, (f_*Y)_{p'}, (f_*X)_{p'}, (f_*Y)_{p'})}{g'((f_*X)_{p'}, (f_*X)_{p'})g'((f_*Y)_{p'}, (f_*Y)_{p'}) - g'((f_*X)_{p'}, (f_*Y)_{p'})^2} \\ &= \frac{R(X_p, Y_p, X_p, Y_p)}{g(X_p, X_p)g(Y_p, Y_p) - g(X_p, Y_p)^2} = K(P). \end{aligned}$$

Problem 6.86 Let (M, g) be a Riemannian manifold.

(i) Prove that if f is an isometry of (M, g) and ∇ denotes the Levi-Civita connection, then we have

$$f^* \nabla_{e_j} \beta = \nabla_{f^{-1} \cdot e_j} f^* \beta, \quad \beta \in \Lambda^1 M,$$

where (e_j) stands for a local orthonormal frame.

(ii) Prove that the *codifferential* δ, defined by

$$\delta\beta = -\operatorname{div}\beta = -\sum_k i_{e_k}\nabla_{e_k}\beta, \quad \beta \in \Lambda^* M, \tag{$\star$}$$

$\{e_k\}$ being an orthonormal basis, commutes with isometries.

The relevant theory is developed, for instance, in Poor [28, Chap. 4].

Solution

(i) As f is an isometry, f preserves the Levi-Civita connection, that is, $f \cdot \nabla_X Y = \nabla_{f \cdot X} f \cdot Y$, so that $\nabla_{e_j}(f \cdot X) = f \cdot \nabla_{f^{-1} \cdot e_j} X$. Moreover, we recall that we have $(f^*\omega)(X) \circ f^{-1} = \omega(f \cdot X)$ for every $\omega \in \Lambda^1 M$, as it is readily checked. Letting $\omega = \nabla_{e_j}\beta$, we obtain that

$$\begin{aligned}
\big(f^*\nabla_{e_j}\beta\big)(X) \circ f^{-1} &= (\nabla_{e_j}\beta)(f \cdot X) = \nabla_{e_j}\big(\beta(f \cdot X)\big) - \beta\big(\nabla_{e_j}(f \cdot X)\big) \\
&= e_j\big(\big(f^*\beta\big)(X) \circ f^{-1}\big) - \beta(f \cdot \nabla_{f^{-1} \cdot e_j} X) \\
&= \big\{\big(f^{-1} \cdot e_j\big)\big(\big(f^*\beta\big)(X)\big) - \big(f^*\beta\big)(\nabla_{f^{-1} \cdot e_j} X)\big\} \circ f^{-1} \\
&= \big\{\nabla_{f^{-1} \cdot e_j}\big(\big(f^*\beta\big)(X)\big) - \big(f^*\beta\big)(\nabla_{f^{-1} \cdot e_j} X)\big\} \circ f^{-1} \\
&= \big\{\big(\nabla_{f^{-1} \cdot e_j} f^*\beta\big)(X)\big\} \circ f^{-1}.
\end{aligned}$$

(ii)

$$\begin{aligned}
f^*\delta\beta &= -f^* \operatorname{div}\beta && \text{(by definition of div, and locally)} \\
&= -\sum_j f^*(i_{e_j}\nabla_{e_j}\beta) && \text{(by } (\star)) \\
&= -\sum_j i_{f^{-1} \cdot e_j} f^*\nabla_{e_j}\beta \\
&= -\sum_j i_{f^{-1} \cdot e_j}\nabla_{f^{-1} \cdot e_j} f^*\beta && \text{(by part (i) of this problem)} \\
&= -\operatorname{div} f^*\beta = \delta f^*\beta.
\end{aligned}$$

6.10 Left-Invariant Metrics on Lie Groups

Problem 6.87 Let G be a Lie group equipped with a left-invariant Riemannian metric g. Prove that the scalar curvature is constant.

Solution Let e denote the identity element of G. Since the left translations by elements $a \in G$ are isometries, they preserve (see Problem 6.85) the Riemann–Christoffel curvature tensor

$$R(L_{a*}X_1, L_{a*}X_2, L_{a*}X_3, L_{a*}X_4) = R(X_1, X_2, X_3, X_4), \quad X_i \in T_e(G).$$

As $(L_{a*})_e$ sends an orthonormal basis $\{e_i\}$ at e to an orthonormal basis, the scalar curvature

$$\mathbf{s} = \sum_{i,j} R(e_i, e_j, e_i, e_j)$$

satisfies $\mathbf{s}(a) = \mathbf{s}(e)$ for all $a \in G$, that is, $\mathbf{s}$ is a constant function.

Problem 6.88 Consider the Heisenberg group H (see Problem 4.42) equipped with the left-invariant metric

$$g = \mathrm{d}x^2 + (\mathrm{d}y - x\,\mathrm{d}z)^2 + \mathrm{d}z^2.$$

1. Find the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Ricci tensor and the scalar curvature of (H, g):
   (a) In terms of the coordinate frame

   $$\left(Y_1 = \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial y},\ Y_3 = \frac{\partial}{\partial z}\right),$$

   by using direct calculation.
   (b) With respect to the orthonormal moving frame

   $$\sigma = \left(X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$$

   dual to the orthonormal moving coframe

   $$\left(\tilde{\theta}^1 = \mathrm{d}x,\ \tilde{\theta}^2 = \mathrm{d}y - x\,\mathrm{d}z,\ \tilde{\theta}^3 = \mathrm{d}z\right), \qquad (\star)$$

   by using the Koszul formula.
   (c) With respect to the moving frame in (b), by using Cartan's structure equations.
2. Is (H, g) a space of constant curvature?
3. Is (H, g) a space of constant scalar curvature?

Hint (to 1(c)) Since one considers an orthonormal moving frame and the Levi-Civita connection is torsionless, one should consider Cartan's structure equations for orthonormal moving frames (in Riemannian signature), that we can write (see p. 597) as

$$\mathrm{d}\tilde{\theta}^i = -\sum_j \tilde{\omega}^i_j \wedge \tilde{\theta}^j, \qquad \tilde{\omega}^i_j + \tilde{\omega}^j_i = 0,$$

$$\widetilde{\Omega}^i_j = \mathrm{d}\widetilde{\omega}^i_j + \sum_k \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j.$$

Solution 1(a) The matrices of g and its inverse g^{-1} are given in terms of the coordinate frame by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & -x & 1+x^2 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & x \\ 0 & x & 1 \end{pmatrix}.$$

From this we have that the non-zero Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

are easily computed to be

$$\Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{23} = -\Gamma^1_{32} = -\Gamma^3_{13} = -\Gamma^3_{31} = -\frac{1}{2}x,$$

$$\Gamma^3_{12} = \Gamma^3_{21} = -\Gamma^1_{23} = -\Gamma^1_{32} = -\frac{1}{2}, \qquad \Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{2}\big(x^2-1\big), \qquad \Gamma^1_{33} = -x.$$

The non-vanishing components of the curvature tensor

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \sum_r \big(\Gamma^r_{lj}\,\Gamma^i_{kr} - \Gamma^r_{kj}\,\Gamma^i_{lr}\big)$$

are then given by

$$R^1_{212} = -R^1_{221} = -R^2_{112} = R^2_{121} = -R^3_{223} = R^3_{232} = \frac{1}{4},$$

$$R^1_{213} = -R^1_{231} = R^1_{312} = -R^1_{321} = R^1_{223} = R^1_{232} = R^2_{223} = -R^2_{232}$$

$$= -R^3_{323} = -R^3_{332} = -\frac{1}{4}x,$$

$$R^1_{313} = -R^1_{331} = \frac{1}{4}\big(x^2-3\big), \qquad R^2_{113} = -R^2_{131} = x,$$

$$R^2_{323} = -R^2_{332} = \frac{1}{4}\big(x^2+1\big), \qquad R^3_{113} = -R^3_{131} = \frac{3}{4},$$

so the nonzero components of the Riemann–Christoffel curvature tensor

$$R_{ijkl} = \sum_h g_{ih} R^h_{jkl}$$

are given by

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = R_{2323} = -R_{2332} = -R_{3223} = R_{3232} = \frac{1}{4},$$

$$R_{1213} = -R_{1231} = R_{1312} = -R_{1321} = R_{1223} = R_{1232} = -R_{2113} = R_{2131} = -\frac{1}{4}x,$$

$$R_{1313} = -R_{1331} = -R_{3113} = R_{3131} = \frac{1}{4}\big(x^2 - 3\big).$$

From the components of either the curvature tensor or the Riemann–Christoffel curvature tensor we get that the non-null components of the Ricci tensor

$$\mathbf{r}_{ij} = \sum_k R^k_{ikj} = \sum_{k,l} g^{kl} R_{ikjl}$$

are

$$\mathbf{r}_{11} = -\mathbf{r}_{22} = -\frac{1}{2}, \qquad \mathbf{r}_{23} = \mathbf{r}_{32} = -\frac{1}{2}x, \qquad \mathbf{r}_{33} = \frac{1}{2}\big(x^2 - 1\big),$$

and thus the scalar curvature

$$\mathbf{s} = \sum_{i,j,k,l} g^{ij} g^{kl} R_{ikjl} = \sum_{i,j} g^{ij} \mathbf{r}_{ij}$$

is

$$\mathbf{s} = -\frac{1}{2}.$$

1(b). This time, the matrices of g and its inverse g^{-1} are obviously given in terms of the frame by

$$g = \operatorname{diag}(1, 1, 1), \qquad g^{-1} = \operatorname{diag}(1, 1, 1).$$

The Levi-Civita connection of g is given by the Koszul formula

$$2g(\nabla_X Y, Z) = g\big([X, Y], Z\big) - g\big([Y, Z], X\big) + g\big([Z, X], Y\big), \quad X, Y, Z \in \mathfrak{g}.$$

Now, since (X_1, X_2, X_3) is a basis of $\mathfrak{h}$, to determine ∇ we only have to know $\nabla_{X_i} X_j$. The nonzero brackets $[X_i, X_j]$, for $i, j = 1, 2, 3$, are

$$[X_1, X_3] = -[X_3, X_1] = X_2.$$

Hence,

$g(\nabla_{X_1} X_1, X_i) = 0$, $i = 1, 2, 3$, so $\nabla_{X_1} X_1 = 0$. Similarly, $\nabla_{X_2} X_2 = \nabla_{X_3} X_3 = 0$;
$g(\nabla_{X_1} X_2, X_1) = 0$, $g(\nabla_{X_1} X_2, X_2) = 0$ and $2g(\nabla_{X_1} X_2, X_3) = -1$, thus $\nabla_{X_1} X_2 = -\frac{1}{2} X_3$. So, as ∇ is torsionless, it follows that $\nabla_{X_2} X_1 = -\frac{1}{2} X_3$;
$g(\nabla_{X_1} X_3, X_1) = 0$, $2g(\nabla_{X_1} X_3, X_2) = 1$ and $g(\nabla_{X_1} X_3, X_3) = 0$, therefore $\nabla_{X_1} X_3 = \frac{1}{2} X_2$, and $\nabla_{X_3} X_1 = -\frac{1}{2} X_2$;

$2g(\nabla_{X_2}X_3, X_1) = 1$, $g(\nabla_{X_2}X_3, X_2) = 0$ and $g(\nabla_{X_2}X_3, X_3) = 0$, so $\nabla_{X_2}X_3 = \frac{1}{2}X_1$ and $\nabla_{X_3}X_2 = \frac{1}{2}X_1$.

That is, we have that

$$\nabla_{X_1}X_1 = 0, \qquad \nabla_{X_1}X_2 = \nabla_{X_2}X_1 = -\frac{1}{2}X_3, \qquad \nabla_{X_1}X_3 = -\nabla_{X_3}X_1 = \frac{1}{2}X_2,$$
$$\nabla_{X_2}X_2 = 0, \qquad \nabla_{X_2}X_3 = \nabla_{X_3}X_2 = \frac{1}{2}X_1, \qquad \nabla_{X_3}X_3 = 0.$$

The Riemann–Christoffel curvature tensor has thus components computed as, for instance,

$$\begin{aligned} R_{1212} &= g\big(R(X_1, X_2)X_2, X_1\big) = g(\nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\nabla_{X_1}X_2 - \nabla_{[X_1,X_2]}X_2, X_1) \\ &= \frac{1}{2}g(\nabla_{X_2}X_3, X_1) = \frac{1}{4}g(X_1, X_1) = \frac{1}{4}, \end{aligned}$$

giving that the nonzero components are

$$R_{1212} = R_{2323} = \frac{1}{4}, \qquad R_{1313} = -\frac{3}{4}.$$

Hence, the non-null components of the Ricci tensor $\mathbf{r}_{ij} = \sum_k R_{ikjk}$ are

$$\mathbf{r}_{11} = \mathbf{r}_{33} = -\frac{1}{2}, \qquad \mathbf{r}_{22} = \frac{1}{2},$$

and the scalar curvature $\mathbf{s} = \sum_i \mathbf{r}_{ii}$ is

$$\mathbf{s} = -\frac{1}{2}.$$

1(c). The Levi-Civita connection forms $\widetilde{\omega}^i_j$ relative to σ satisfy Cartan's first structure equation

$$\mathrm{d}\tilde{\theta}^i = -\sum_j \widetilde{\omega}^i_j \wedge \tilde{\theta}^j. \tag{$\star\star$}$$

From ($\star$) we have

$$\mathrm{d}\tilde{\theta}^1 = \mathrm{d}\tilde{\theta}^3 = 0, \qquad \mathrm{d}\tilde{\theta}^2 = -\tilde{\theta}^1 \wedge \tilde{\theta}^2.$$

Thus ($\star\star$) reduces to

$$\begin{aligned} 0 &= -\widetilde{\omega}^1_2 \wedge (\mathrm{d}y - x\,\mathrm{d}z) - \widetilde{\omega}^1_3 \wedge \mathrm{d}z, \\ -\mathrm{d}x \wedge \mathrm{d}z &= -\widetilde{\omega}^2_1 \wedge \mathrm{d}x - \widetilde{\omega}^2_3 \wedge \mathrm{d}z, \\ 0 &= -\widetilde{\omega}^3_1 \wedge \mathrm{d}x - \widetilde{\omega}^3_2 \wedge (\mathrm{d}y - x\,\mathrm{d}z). \end{aligned}$$

The second equation is satisfied taking $\widetilde{\omega}_1^2 = -\frac{1}{2}\,\mathrm{d}z$, $\widetilde{\omega}_3^2 = \frac{1}{2}\,\mathrm{d}x$, and we have from the other equations that

$$0 = -\frac{1}{2}\,\mathrm{d}z \wedge (\mathrm{d}y - x\,\mathrm{d}z) - \widetilde{\omega}_3^1 \wedge \mathrm{d}z, \qquad 0 = -\widetilde{\omega}_1^3 \wedge \mathrm{d}x + \frac{1}{2}\,\mathrm{d}x \wedge (\mathrm{d}y - x\,\mathrm{d}z),$$

which are satisfied i $\widetilde{\omega}_3^1 = \frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z)$. Since the forms $\tilde{\theta}^i$ determine uniquely a set of connection forms $\widetilde{\omega}_j^i$, we have that

$$\widetilde{\omega}_2^1 = \frac{1}{2}\,\tilde{\theta}^3, \qquad \widetilde{\omega}_3^1 = \frac{1}{2}\,\tilde{\theta}^2, \qquad \widetilde{\omega}_3^2 = \frac{1}{2}\,\tilde{\theta}^1.$$

From Cartan's structure equation $\widetilde{\Omega}_j^i = \mathrm{d}\widetilde{\omega}_j^i + \sum_k \widetilde{\omega}_k^i \wedge \widetilde{\omega}_j^k$, we thus get the curvature forms relative to σ:

$$\widetilde{\Omega}_2^1 = \frac{1}{4}\,\theta^1 \wedge \theta^2, \qquad \widetilde{\Omega}_3^1 = -\frac{3}{4}\,\theta^1 \wedge \theta^3, \qquad \widetilde{\Omega}_3^2 = \frac{1}{4}\,\theta^2 \wedge \theta^3.$$

Hence, from

$$\begin{aligned}
\widetilde{\Omega}_2^1 &= R_{212}^1\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R_{213}^1\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R_{223}^1\tilde{\theta}^2 \wedge \tilde{\theta}^3,\\
\widetilde{\Omega}_3^1 &= R_{312}^1\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R_{313}^1\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R_{323}^1\tilde{\theta}^2 \wedge \tilde{\theta}^3,\\
\widetilde{\Omega}_3^2 &= R_{312}^2\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R_{313}^2\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R_{323}^2\tilde{\theta}^2 \wedge \tilde{\theta}^3,
\end{aligned}$$

we deduce that the non-vanishing components of the Riemann–Christoffel curvature tensor are

$$R_{1212} = R_{2323} = \frac{1}{4}, \qquad R_{1313} = -\frac{3}{4},$$

as in part 1(b), so concluding that the scalar curvature is given by

$$\mathbf{s} = -\frac{1}{2}.$$

2. From the previous results it follows that (H, g) is not a space of constant curvature.

3. Yes. (Note that this is an instance of the result in Problem 6.87).

Problem 6.89 Let

$$G = H \times \mathbb{R}$$

be the direct product of the three-dimensional Heisenberg group with $\mathbb{R}$. Write any element of G as

$$(x, y, z, t), \quad (x, y, z) \in H,\ t \in \mathbb{R}.$$

The group law is given by

$$(x, y, z, t) \cdot \left(x', y', z', t'\right) = \left(x + x', y + y' + xz', z + z', t + t'\right).$$

Let $\Gamma \subset G$ be the integral lattice

$$\Gamma = \{(x, y, z, t) : x, y, z, t \in \mathbb{Z}\}.$$

(i) Prove that the subgroup Γ of G is not normal.

The *Kodaira–Thurston manifold* is defined as the compact quotient

$$M = \Gamma \backslash G.$$

Prove:

(ii) M is symplectic.
(iii) M is a symplectic 2-torus bundle over a 2-torus.
(iv) M is not a Kähler manifold.

Hint (to (ii)) Consider the 2-form

$$\omega = \mathrm{d}x \wedge (\mathrm{d}y - x\,\mathrm{d}z) + \mathrm{d}z \wedge \mathrm{d}t$$

on G.

Hint (to (iii)) Take the base space of the bundle given by x, y and the fiber by z, t.

Hint (to (iv)) The odd cohomology groups $H^{2r+1}(M, \mathbb{Z})$ of a Kähler manifold M vanish. Compute the first Betti number of M, recalling that, $\pi_1(M)$ denoting the first homotopy group of M, the integer homology group of degree one is given by

$$H_1(M, \mathbb{Z}) = \pi_1(M) / [\pi_1(M), \pi_1(M)].$$

Remark This was the first known example of a symplectic non-Kähler manifold.

Solution

(i) In fact, given $(a, b, c, d) \in \Gamma$, one has (see, for instance, Problem 4.42(ii))

$$\begin{aligned}
&(x, y, z, t)(a, b, c, d)(-x, xz - y, -z, -t) \\
&\quad = (x + a, y + b + xc, z + c, t + d)(-x, xz - y, -z, -t) \\
&\quad = (a, b + xc - az, c, d) \notin \Gamma \quad \text{for some } x, z \in \mathbb{R}.
\end{aligned}$$

(ii) We know (see, e.g. Problem 6.88) that

$$\alpha_1 = \mathrm{d}x, \qquad \alpha_2 = \mathrm{d}y - x\,\mathrm{d}z, \qquad \alpha_3 = \mathrm{d}z, \qquad \alpha_4 = \mathrm{d}t$$

is a left G-invariant orthonormal coframe on G. Then it is immediate that the differential 2-forms

$$\omega_1 = \mathrm{d}x \wedge (\mathrm{d}y - x\,\mathrm{d}z) + \mathrm{d}z \wedge \mathrm{d}t, \qquad \omega_2 = \mathrm{d}z \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}t$$

are symplectic left G-invariant forms on G (ω_2 is invariant because $\mathrm{d}z \wedge (\mathrm{d}y - x\,\mathrm{d}z) = \mathrm{d}z \wedge \mathrm{d}y$). In particular, these symplectic forms are (left) Γ-invariant. Hence the quotient space $\Gamma\backslash G$ also is a symplectic manifold with the symplectic structure ω_1' (resp., ω_2') induced by the form ω_1 (resp., ω_2).

(iii) We now prove that (M, ω_1') is a symplectic 2-torus bundle over a 2-torus. In fact, take the fiber of the bundle given by x, y and the base space by z, t. The bundle with projection map

$$\pi_1 \colon (x,\ y,\ z,\ t) \mapsto (z,\ t)$$

is trivial (a product) along the t-direction, but non-trivial (only a local product) along the z-direction.

The left G-invariant symplectic form ω_1' is compatible with the bundle structure. This means that ω_1' is the sum of two forms. It is clear that the restriction of the form $\mathrm{d}x \wedge (\mathrm{d}y - x\,\mathrm{d}z)$ to every fiber is non-degenerate, and the form $\mathrm{d}z \wedge \mathrm{d}t$ is the pull-back under π_1^* of the symplectic form on the base space.

To prove that (M, ω_2') is a symplectic 2-torus bundle over a 2-torus note that the action of the group Γ is generated by the four maps

$$\begin{aligned}
\tau_1 &\colon (x, y, z, t) \mapsto (1, 0, 0, 0) \cdot (x, y, z, t) = (x + 1, y + z, z, t),\\
\tau_2 &\colon (x, y, z, t) \mapsto (0, 1, 0, 0) \cdot (x, y, z, t) = (x, y + 1, z, t),\\
\tau_3 &\colon (x, y, z, t) \mapsto (0, 0, 1, 0) \cdot (x, y, z, t) = (x, y, z + 1, t),\\
\tau_4 &\colon (x, y, z, t) \mapsto (0, 0, 0, 1) \cdot (x, y, z, t) = (x, y, z, t + 1).
\end{aligned}$$

In this case, consider the bundle with projection map

$$\pi_2 \colon (x, y, z, t) \mapsto (x, t),$$

taking the fiber of the bundle given by y, z and the base space by x, t.

Moreover, we can write the manifold M as a torus bundle over a torus, putting

$$M = \mathbb{R}^2 \times_{\mathbb{Z}^2} T^2,$$

where the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ is the usual one, and that on T^2 is

$$\begin{aligned}
\mathbb{Z}^2 \times T^2 &\longrightarrow T^2\\
\big((m, n), (y, z)\big) &\longmapsto (y + m\,z, z).
\end{aligned}$$

Therefore,

$$M = \big(\mathbb{R} \times T^1 \times T^2\big)/\sim,$$

where

$$(x, t, y, z) \sim (x + m, t, y + mz, z), \quad m \in \mathbb{Z}.$$

(iv) M cannot admit any Kähler structure because its first Betti number is

$$b_1(M) = 3 \neq 0.$$

In fact, the subgroup of Γ generated by the transformations τ_2, τ_3, τ_4 is commutative, $[\tau_1, \tau_2] = \tau_1\tau_2\tau_1^{-1}\tau_2^{-1} = 1$ and the only non-trivial element $[\tau_i, \tau_j]$ is

$$[\tau_1, \tau_3] = \tau_1\tau_3\tau_1^{-1}\tau_3^{-1} = \tau_2,$$

hence,

$$H_1(M, \mathbb{Z}) = \pi_1(M) \big/ \big[\pi_1(M), \pi_1(M)\big] = \Gamma/[\Gamma, \Gamma] = \Gamma/\langle\tau_2\rangle \cong \mathbb{Z}^3.$$

Problem 6.90

(i) Let G be a compact Lie group equipped with a bi-invariant metric g and let $\mathfrak{g}$ be its Lie algebra. If X and Y are left-invariant vector fields on G and ∇ is the Levi-Civita connection of g, prove that

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(ii) Compute $R(X, Y)Z$, $X, Y, Z \in \mathfrak{g}$.
(iii) Show that the sectional curvature of g is non-negative.

Hint (to (i) and (iii)) If a metric is bi-invariant, each ad_X, $X \in \mathfrak{g}$, is skew-symmetric with respect to g (see [28, p. 114]).

Solution

(i) By the result quoted in the hint, the Koszul formula reduces to

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(g\big([X, Y], Z\big) + g(\mathrm{ad}_Z Y, X) + g(\mathrm{ad}_Z X, Y)\Big) = \frac{1}{2}g\big([X, Y], Z\big).$$

That is, we have $\nabla_X Y = \frac{1}{2}[X, Y]$.
(ii) From (i) one has for the curvature,

$$\begin{aligned} R(X, Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= \frac{1}{4}\big[X, [Y, Z]\big] - \frac{1}{4}\big[Y, [X, Z]\big] - \frac{1}{2}\big[[X, Y], Z\big] \\ &= -\frac{1}{4}\Big(\mathop{\mathfrak{S}}_{XYZ}\big[[X, Y], Z\big]\Big) - \frac{1}{4}\big[[X, Y], Z\big] \\ &= -\frac{1}{4}\big[[X, Y], Z\big] \quad \text{(by Jacobi identity).} \end{aligned}$$

(iii) Let $X, Y \in \mathfrak{g}$ be orthonormal. Then, again by the result in the hint, we have for the sectional curvature at e, hence at all points:

$$K(X,Y) = g\big(R(X,Y)Y, X\big) = -\frac{1}{4} g\big(\big[[X,Y],Y\big],X\big) = \frac{1}{4} g\big(\mathrm{ad}_Y[X,Y],X\big)$$
$$= -\frac{1}{4} g\big([X,Y],\mathrm{ad}_Y X\big) = \frac{1}{4} g\big([X,Y],[X,Y]\big) = \frac{1}{4}\big|[X,Y]\big|^2 \geqslant 0.$$

Problem 6.91 Consider the Lie group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} : x, y \in \mathbb{R},\ y > 0 \right\}.$$

(i) Prove that its Lie algebra is $\mathfrak{g} = \langle y\,\partial/\partial x,\ y\,\partial/\partial y \rangle$.
(ii) Write the left-invariant metric g on G built with the dual basis to that in (i).
(iii) Determine the Levi-Civita connection ∇ of g.
(iv) Is (G, g) a space of constant curvature?
(v) Prove (without using (iv)) that (G, g) is an Einstein manifold.

Solution

(i) G is a Lie group with the product of matrices and with only one chart,

$$(G,\varphi), \qquad G \xrightarrow{\varphi} U, \qquad \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \mapsto (x, y),$$

where U denotes the open subset $\{(x,y) \in \mathbb{R}^2 : y > 0\}$ of $\mathbb{R}^2$, hence $\dim G = 2$. Thus $\dim \mathfrak{g} = 2$, and so to prove that $\mathfrak{g}$ is generated by $X_1 = y\partial/\partial x$, $X_2 = y\partial/\partial y$, we shall only have to prove that X_1, X_2 are linearly independent and left-invariant. They are linearly independent, as $y > 0$. To prove that they are left-invariant vector fields, we have to prove that for all $A \in G$, one has

$$(L_A)_{*B}(X_i|_B) = X_i|_{AB}, \qquad B \in G,\ i = 1, 2. \tag{$\star$}$$

Let $A = \left(\begin{smallmatrix} 1 & 0 \\ a & b \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 0 \\ x_0 & y_0 \end{smallmatrix}\right)$. As $AB = \left(\begin{smallmatrix} 1 & 0 \\ a+bx_0 & by_0 \end{smallmatrix}\right)$, the right-hand side of ($\star$) is

$$X_1|_{AB} = by_0 \frac{\partial}{\partial x}\bigg|_{AB}, \qquad X_2|_{AB} = by_0 \frac{\partial}{\partial y}\bigg|_{AB}.$$

To determine the left-hand side of ($\star$), we compute the Jacobian matrix of L_A using the diagram

$$\begin{tikzcd} G \arrow[r, "L_A"] & G \arrow[d, "\varphi"] \\ U \arrow[u, "\varphi^{-1}"] \arrow[r, "\varphi \circ L_A \circ \varphi^{-1}"] & U \end{tikzcd}$$

with

$$\begin{tikzcd}
\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \arrow[r, mapsto, "L_A"] & \begin{pmatrix} 1 & 0 \\ a+bx & by \end{pmatrix} \arrow[d, mapsto, "\varphi"] \\
(x, y) \arrow[u, mapsto, "\varphi^{-1}"] \arrow[r, mapsto, "\varphi \circ L_A \circ \varphi^{-1}"] & (a+bx, by).
\end{tikzcd}$$

It follows that the Jacobian matrix of $\varphi \circ L_A \circ \varphi^{-1}$ is $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, hence

$$(L_A)_{*B} X_1|_B \equiv \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} y_0 \\ 0 \end{pmatrix} \equiv by_0 \left.\frac{\partial}{\partial x}\right|_{AB} = X_1|_{AB},$$
$$(L_A)_{*B} X_2|_B \equiv \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ y_0 \end{pmatrix} \equiv by_0 \left.\frac{\partial}{\partial y}\right|_{AB} = X_2|_{AB}.$$

(ii) The dual basis $\{\beta_1, \beta_2\}$ to $\{X_1, X_2\}$ is $\{\beta_1 = \mathrm{d}x/y,\ \beta_2 = \mathrm{d}y/y\}$. Therefore, the left-invariant metric on G we are looking for is $g = (1/y^2)(\mathrm{d}x^2 + \mathrm{d}y^2)$.

(iii) From the formula for the Levi-Civita connection of a left-invariant metric g on a Lie group and from

$$[X_1, X_1] = [X_2, X_2] = 0, \qquad [X_1, X_2] = -[X_2, X_1] = -X_1,$$

we have:

$g(\nabla_{X_1} X_1, X_1) = 0$, $2g(\nabla_{X_1} X_1, X_2) = 2$; thus $\nabla_{X_1} X_1 = X_2$;
$2g(\nabla_{X_1} X_2, X_1) = -2$, $g(\nabla_{X_1} X_2, X_2) = 0$; so $\nabla_{X_1} X_2 = -X_1$. As ∇ is torsionless, one has $\nabla_{X_2} X_1 = 0$;
$g(\nabla_{X_2} X_2, X_1) = g(\nabla_{X_2} X_2, X_2) = 0$; that is, $\nabla_{X_2} X_2 = 0$.

(iv)

$$\begin{aligned} R(X_1, X_2, X_1, X_2) &= g(\nabla_{X_1} \nabla_{X_2} X_2 - \nabla_{X_2} \nabla_{X_1} X_2 - \nabla_{[X_1, X_2]} X_2, X_1) \\ &= -g(X_1, X_1) = -1. \end{aligned}$$

Thus G is a space of constant curvature -1.

(v) Let $X, Y \in \mathfrak{X}(G)$, $X = f_1 X_1 + f_2 X_2$, $Y = h_1 X_1 + h_2 X_2$. Thus,

$$\begin{aligned} \mathbf{r}(X, Y) &= R(X_1, Y, X_1, X) + R(X_2, Y, X_2, X) \\ &= R(X_1, h_2 X_2, X_1, f_2 X_2) + R(X_2, h_1 X_1, X_2, f_1 X_1) \\ &= (f_1 h_1 + f_2 h_2) R(X_1, X_2, X_1, X_2) = -(f_1 h_1 + f_2 h_2) = -g(X, Y). \end{aligned}$$

Therefore, G is an Einstein manifold.

Remark In Problem 6.107 below, it is proved that every Riemannian manifold (M, g) of dimension n and constant curvature K is an Einstein manifold, with Ricci tensor $\mathbf{r}(X, Y) = K(n-1)g(X, Y)$. Here we have a verification of this formula in this example.

Problem 6.92 Let G be a Lie group and let Γ be a discrete co-compact subgroup of G which acts on the left on G. Denote by $\Gamma\backslash G$ the quotient space of right cosets

$$\Gamma\backslash G = \{\Gamma g : g \in G\}.$$

(i) Compute the de Rham cohomology of the compact quotient $\Gamma\backslash H$ of the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \ x, y, z \in \mathbb{R} \right\}$$

by its discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in \mathbb{Z} \right\}.$$

Consider now (see Definition 6.26) the *generalised Heisenberg group* $H(2, 1)$, that is, the group of real matrices of the form

$$H(2,1) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & y \\ 0 & 1 & 0 & z_1 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ x_1, x_2, y, z_2, z_2 \in \mathbb{R} \right\},$$

which is a five-dimensional connected, simply connected nilpotent Lie group, and its discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & b \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ a_1, a_2, b, c_1, c_2 \in \mathbb{Z} \right\}.$$

(ii) Compute the cohomology of the compact quotient

$$\Gamma\backslash H(2, 1).$$

Hint Apply Stokes' Theorem 3.3 and the definition of the de Rham cohomology groups (see, for instance, Definitions 3.7).

Remark The given quotient manifolds are two examples of a *nilmanifold*, that is, the quotient of a nilpotent Lie group by a closed subgroup.

Solution

(i) We know (see, for instance, Problem 6.88) that

$$\{\mathrm{d}x,\ \mathrm{d}y - x\,\mathrm{d}z,\ \mathrm{d}z\}$$

is a basis of left-invariant differential 1-forms on H. In particular, they are preserved by Γ, and so they descend to 1-forms α, β, γ on the nilmanifold $\Gamma\backslash H$, that is, if π denotes the canonical projection map

$$\pi : H \to \Gamma\backslash H,$$

then

$$\pi^*\alpha = \mathrm{d}x, \qquad \pi^*\beta = \mathrm{d}y - x\,\mathrm{d}z, \qquad \pi^*\gamma = \mathrm{d}z. \tag{$\star$}$$

From ($\star$) we have

$$\mathrm{d}\alpha = \mathrm{d}\gamma = 0, \qquad \mathrm{d}\beta = -\alpha \wedge \gamma. \tag{$\star\star$}$$

In fact, we have, for instance, $\pi^*(\mathrm{d}\alpha) = \mathrm{d}(\pi^*\alpha) = 0$. Moreover,

$$\begin{aligned} &\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}(\alpha \wedge \gamma) = \mathrm{d}(\beta \wedge \gamma) = 0, \\ &\mathrm{d}(\alpha \wedge \beta \wedge \gamma) = 0. \end{aligned} \tag{$\star\star\star$}$$

From ($\star\star$) and ($\star\star\star$) we have that $\alpha \wedge \gamma$ is closed but also exact. Hence the cohomology classes $[\alpha]$ and $[\gamma]$ generate the first cohomology group $H^1_{dR}(\Gamma\backslash H, \mathbb{R})$. Actually, they are a basis of that group. In fact, because of the definition of Γ, the manifold $\Gamma\backslash H$ contains the 2-torus product of

$$C_1 = \{y \bmod \mathbb{Z} = 0,\ z \bmod \mathbb{Z} = 0\}$$

and

$$C_2 = \{x \bmod \mathbb{Z} = 0,\ y \bmod \mathbb{Z} = 0\},$$

so that the variable is $x \bmod \mathbb{Z}$ on C_1 and $z \bmod \mathbb{Z}$ on C_2.

Suppose that

$$r[\alpha] + s[\beta] = 0, \tag{$\diamond$}$$

for certain $r, s \in \mathbb{R}$. Since the submanifolds C_1 and C_2 of $\Gamma\backslash H$ are closed manifolds, i.e. $\partial C_1 = \emptyset$ and $\partial C_2 = \emptyset$, then from ($\diamond$) and Stokes' Theorem 3.3 we have that

$$0 = \int_{C_1} (r\alpha + s\gamma) = \int_{x=0}^{1} \big(r\pi^*\alpha + s\pi^*\gamma\big) = \int_{x=0}^{1} (r\,\mathrm{d}x + s\,\mathrm{d}z) = r,$$

and similarly for C_2 and s. Thus $r = s = 0$.

Analogously, $[\alpha \wedge \beta]$ and $[\beta \wedge \gamma]$ generate $H^2_{dR}(\varGamma\backslash H, \mathbb{R})$. Suppose now that

$$r[\alpha \wedge \beta] + s[\beta \wedge \gamma] = 0, \tag{$\diamond\diamond$}$$

for certain r, s. As the submanifold $C_1 \times C_2$ of $\varGamma\backslash H$ is a closed manifold, then from $(\diamond\diamond)$ and again by Stokes' Theorem 3.3 one has

$$\begin{aligned} 0 &= \int_{C_1\times C_2} (r\alpha \wedge \beta + s\beta \wedge \gamma) = \int_{x=0}^{1}\int_{z=0}^{1} \big(r\pi^*(\alpha\wedge\beta) + s\pi^*(\beta\wedge\gamma)\big) \\ &= \int_{x=0}^{1}\int_{z=0}^{1} \big(r(\mathrm{d}x \wedge \mathrm{d}y - x\,\mathrm{d}x\wedge \mathrm{d}z) + s\,\mathrm{d}y \wedge \mathrm{d}z\big) = -\frac{1}{2}r. \end{aligned}$$

Consider now the surface S in $\varGamma\backslash H$ defined by

$$S = \{x \bmod \mathbb{Z} = 0\}$$

(whose coordinates are $y \bmod \mathbb{Z}$ and $z \bmod \mathbb{Z}$), which is a closed submanifold of $\varGamma\backslash H$. Then, since

$$\pi^*(\beta \wedge \gamma) = \mathrm{d}y \wedge \mathrm{d}z,$$

we get

$$\int_S \beta \wedge \gamma = \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = 1.$$

On account of $(\diamond\diamond)$ and Stokes' Theorem 3.3, this also implies that $s = 0$.

Let us prove that the 3-form $\alpha \wedge \beta \wedge \gamma$ is not exact. The manifold $\varGamma\backslash H$ is oriented because the form $\alpha \wedge \beta \wedge \gamma$ is nowhere vanishing on this manifold. This manifold is closed because the manifold $\mathbb{R}^3 \equiv H$ is closed and the natural projection $\pi\colon H \to \varGamma\backslash H$ is a local homeomorphism. On account of Stokes' Theorem 3.3, it is sufficient to show that

$$\int_{\varGamma\backslash H} (\alpha \wedge \beta \wedge \gamma) \neq 0.$$

To this end consider the open subset

$$U = \big\{(x, y, z) \in H : x, y, z \in (0, 1)\big\}$$

in $H \equiv \mathbb{R}^3$ and its closure $\bar{U} = [0, 1] \times [0, 1] \times [0, 1] \subset H$. It is easy to verify that the map $\pi|_U\colon U \to \pi(U) \subset \varGamma\backslash H$ is a diffeomorphism. Moreover, $\pi(\bar{U}) = \varGamma\backslash H$ and the difference $(\varGamma\backslash H) \setminus \{\pi(U)\}$ is a set of zero measure, as it is the union of a finite number of k-dimensional compact submanifolds with $k = 0, 1, 2$. Therefore, since our 3-form on $\varGamma\backslash H$ is smooth, we have

$$\int_{\varGamma\backslash H} (\alpha \wedge \beta \wedge \gamma) = \int_{\pi(U)} (\alpha \wedge \beta \wedge \gamma),$$

and since

$$\pi^*(\alpha \wedge \beta \wedge \gamma) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

we get

$$\int_{\pi(U)} (\alpha \wedge \beta \wedge \gamma) = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = 1.$$

Hence, the de Rham cohomology groups of $\Gamma\backslash H$ are

$$H^0_{dR}(\Gamma\backslash H, \mathbb{R}) = \langle [1] \rangle, \qquad H^1_{dR}(\Gamma\backslash H, \mathbb{R}) = \langle [\alpha], [\gamma] \rangle,$$
$$H^2_{dR}(\Gamma\backslash H, \mathbb{R}) = \langle [\alpha \wedge \beta], [\beta \wedge \gamma] \rangle, \qquad H^3_{dR}(\Gamma\backslash H, \mathbb{R}) = \langle [\alpha \wedge \beta \wedge \gamma] \rangle.$$

(ii) The set (x^1, x^2, y, z^1, z^2) is a global system of coordinates on $H(2,1)$. Moreover, the differential 1-forms

$$\mathrm{d}x^1, \quad \mathrm{d}x^2, \quad \mathrm{d}y - x^1\,\mathrm{d}z^1 - x^2\,\mathrm{d}z^2, \quad \mathrm{d}z^1, \quad \mathrm{d}z^2,$$

are a basis of left-invariant 1-forms on $H(2,1)$. In fact, denoting (x^1, x^2, y, z^1, z^2) simply by (x, y, z), we have for a fixed (x_0, y_0, z_0) that

$$\begin{aligned} \begin{pmatrix} 1 & x' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & x_0 & y_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x + x_0 & y + x_0 z + y_0 \\ 0 & 1 & z + z_0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

that is,

$$x' = x + x_0, \qquad y' = y + x_0 z + y_0, \qquad z' = z + z_0,$$

so

$$\mathrm{d}x' = \mathrm{d}x, \qquad \mathrm{d}y' - x'\,\mathrm{d}z' = \mathrm{d}y - x\,\mathrm{d}z, \qquad \mathrm{d}z' = \mathrm{d}z.$$

Hence, these forms are preserved by Γ, so they descend to 1-forms

$$\alpha_1, \quad \alpha_2, \quad \beta, \quad \gamma_1, \quad \gamma_2,$$

respectively, on the nilmanifold $\Gamma\backslash H(2,1)$. That is, if π denotes the canonical projection map

$$\pi\colon H(2,1) \to \Gamma\backslash H(2,1),$$

we have that

$$\pi^*\alpha_i = \mathrm{d}x^i, \qquad \pi^*\beta = \mathrm{d}y - x^1\,\mathrm{d}z^1 - x^2\,\mathrm{d}z^2, \qquad \pi^*\gamma_i = \mathrm{d}z^i, \quad i = 1, 2. \tag{†}$$

As in (i) above, from (†) we have

$$\mathrm{d}\alpha_i = \mathrm{d}\gamma_i = 0, \qquad \mathrm{d}\beta = -\alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2, \quad i = 1, 2.$$

Moreover,

$$\begin{aligned}
\mathrm{d}(\alpha_i \wedge \alpha_j) &= \mathrm{d}(\alpha_i \wedge \gamma_j) = \mathrm{d}(\gamma_i \wedge \gamma_j) = 0,\\
\mathrm{d}(\alpha_1 \wedge \beta) &= -\alpha_1 \wedge \alpha_2 \wedge \gamma_2, \qquad \mathrm{d}(\alpha_2 \wedge \beta) = \alpha_1 \wedge \alpha_2 \wedge \gamma_1,\\
\mathrm{d}(\gamma_1 \wedge \beta) &= \alpha_2 \wedge \gamma_1 \wedge \gamma_2, \qquad \mathrm{d}(\gamma_2 \wedge \beta) = -\alpha_1 \wedge \gamma_1 \wedge \gamma_2,\\
\mathrm{d}(\alpha_i \wedge \alpha_j \wedge \gamma_k) &= \mathrm{d}(\alpha_i \wedge \gamma_j \wedge \gamma_k) = 0, \quad i, j, k = 1, 2,\\
\mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \beta) &= \mathrm{d}(\alpha_1 \wedge \gamma_2 \wedge \beta) = \mathrm{d}(\alpha_2 \wedge \gamma_1 \wedge \beta)\\
&= \mathrm{d}(\gamma_1 \wedge \gamma_2 \wedge \beta) = 0,\\
\mathrm{d}(\alpha_1 \wedge \gamma_1 \wedge \beta) &= \mathrm{d}(\alpha_2 \wedge \gamma_2 \wedge \beta) = \alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \gamma_2,\\
\mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \gamma_2) &= \mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \gamma_2 \wedge \beta) = \mathrm{d}(\alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta)\\
&= \mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \beta)\\
&= \mathrm{d}(\alpha_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta) = 0,
\end{aligned}$$

$$\mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta) = 0 \quad \text{(as the form is of maximal rank).}$$

Proceeding analogously to part (i) above, one gets that the cohomology classes of the given differential forms are a basis of the de Rham cohomology groups of $\Gamma\backslash H(2,1)$, that is,

$$\begin{aligned}
H^0\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [1] \big\rangle,\\
H^1\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [\alpha_1], [\alpha_2], [\gamma_1], [\gamma_2] \big\rangle,\\
H^2\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2], [\alpha_1 \wedge \gamma_2],\\
&\qquad [\alpha_2 \wedge \gamma_1], [\gamma_1 \wedge \gamma_2] \big\rangle,\\
H^3\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [\alpha_1 \wedge \alpha_2 \wedge \beta], \big[(\alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2) \wedge \beta\big],\\
&\qquad [\alpha_1 \wedge \gamma_2 \wedge \beta], [\alpha_2 \wedge \gamma_1 \wedge \beta], [\gamma_1 \wedge \gamma_2 \wedge \beta] \big\rangle,\\
H^4\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [\alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \beta], [\alpha_1 \wedge \alpha_2 \wedge \gamma_2 \wedge \beta],\\
&\qquad [\alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta], [\alpha_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta] \big\rangle,\\
H^5\big(\Gamma\backslash H(2,1), \mathbb{R}\big) &= \big\langle [\alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \beta] \big\rangle.
\end{aligned}$$

Notice that this is consistent with (the corresponding particular cases of) the formula

$$\dim H^r\big(\Gamma\backslash H(2,1), \mathbb{R}\big) = \binom{4}{r} - \binom{4}{r-2}, \quad r = 0, 1, 2,$$

given in [31] and then, by Poincaré duality, with the other present cases.

6.11 Homogeneous Riemannian Manifolds and Riemannian Symmetric Spaces

Problem 6.93 Let (M, g) be a Riemannian manifold. Prove that if a Lie group G of isometries of M acts transitively on M, then the metric g is complete.

Solution Consider an arbitrary point $p \in M$. There is $\varepsilon > 0$ such that for each direction at this point, i.e. for each $v \in T_pM$, $g(v, v) = 1$, there exists a geodesic segment of length ε in this direction with initial point p. Since G acts transitively on M, this value ε is independent of $p \in M$. This implies that each geodesic can be continued indefinitely in any direction.

Indeed, for any $T > 0$, the chain of geodesic ε-segments $\gamma_n(t)$, $t \in [0, \varepsilon]$, such that

$$\gamma_{n-1}(\varepsilon) = \gamma_n(0), \qquad \gamma'_{n-1}(\varepsilon) = \gamma'_n(0), \quad n \in \{1, 2, \ldots, k\},\ k = \left[\frac{T}{\varepsilon}\right],$$

defines a unique geodesic $\gamma \colon [0, T] \to M$, with

$$\gamma(\varepsilon n + t) = \gamma_n(t), \quad t \in [0, \varepsilon],\ n = 0, 1, \ldots, k,$$

such that $\gamma(0) = \gamma_0(0)$ and $\gamma'(0) = \gamma'_0(0)$.

Problem 6.94 Let $(M = G/H, g)$ be a homogeneous Riemannian space where the metric g is G-invariant (see Definition 6.10). Prove that the scalar curvature is constant.

Solution Let e be the identity element of G and let o denote the base point $o = eH$ of G/H. Since the translations by elements $a \in G$ are isometries, they preserve (see Problem 6.85) the Riemann–Christoffel curvature tensor,

$$R(a_*X_1, a_*X_2, a_*X_3, a_*X_4) = R(X_1, X_2, X_3, X_4), \qquad X_i \in T_oM, \quad i = 1, \ldots, 4.$$

Since a_{*o} sends an orthonormal basis $\{e_i\}$ at o to an orthonormal basis, the scalar curvature

$$\mathbf{s} = \sum_{i,j} R(e_i, e_j, e_i, e_j)$$

satisfies $\mathbf{s}(a(o)) = \mathbf{s}(o)$. As the action of G is transitive, $\mathbf{s}$ is a constant function.

Problem 6.95 Let G be a connected closed subgroup of the Lie group $E(n)$ of all the motions (i.e. isometries of the Euclidean metric) of $\mathbb{R}^n$, acting transitively on $\mathbb{R}^n$.

Must G contain the full group of translations?

Solution G need not contain the full group of translations, as the following counterexample shows. Let $n = 3$, and let Φ_t be the rotation around the z-axis through an angle t. Let X_t, Y_t, Z_t be the translations by $(t, 0, 0)$, $(0, t, 0)$ and $(0, 0, t)$, respectively. Let

$$\Psi_t = Z_t \circ \Phi_t,$$

so Ψ_t is a screw motion around the z-axis. Then, the group generated by the Ψ_t, X_t, and Y_t, as t varies over $\mathbb{R}$, acts simply transitively on $\mathbb{R}^3$ but does not contain the translation in the z-direction.

Problem 6.96 Consider the action of the orthogonal group $\mathrm{O}(n)$ on the Riemannian manifold $(\mathbb{R}^n, g)$, where g denotes the Euclidean metric.

(i) Is $(\mathbb{R}^n, g)$ a homogeneous Riemannian manifold with respect to that action?
(ii) Describe the possible isotropy groups H_p.

Solution

(i) No, because the action is not transitive. In fact, the origin is a fixed point (take the origin 0 as one of the points p, q of $\mathbb{R}^n$ such that there might exist $\sigma \in \mathrm{O}(n)$ with $\sigma(p) = q$).
(ii) $H_0 = \mathrm{O}(n)$ and H_p are mutually conjugate subgroups isomorphic to $\mathrm{O}(n-1)$ for every $p \neq 0$.

Problem 6.97 Define a product on

$$E(n) = \left\{(a, A) : a \in \mathbb{R}^n,\ A \in \mathrm{O}(n)\right\}$$

by

$$(a, A) \cdot (b, B) = (a + Ab, AB).$$

Prove:

1. $(E(n), \cdot)$ is a Lie group (in fact, this is a semi-direct product of the Abelian group $(\mathbb{R}^n, +)$ and $\mathrm{O}(n)$, and it is called the *group of Euclidean motions*, or simply the *Euclidean group of* $\mathbb{R}^n$).
2. The subgroup of translations $T(n) = \{(a, I) : a \in \mathbb{R}^n\}$ is a normal subgroup of $E(n)$.

Let $E(n)$ act on $\mathbb{R}^n$ by setting $(a, A) \cdot x = a + Ax$. Then:

3. Prove that the map $x \mapsto (a, A) \cdot x$ is an isometry of the Euclidean metric.
4. Compute the isotropy of a point $x \in \mathbb{R}^n$. Are all these groups isomorphic? And conjugated in $E(n)$?
5. Prove that $E(n)/\mathrm{O}(n) \cong \mathbb{R}^n$.

Let $p\colon E(n) \to \mathbb{R}^n$ be the map $p(a, A) = a$. Prove:

6. The map p is the projection map of a principal $\mathrm{O}(n)$-bundle with respect to the action of $\mathrm{O}(n)$ on $E(n)$ given by $(a, A) \cdot B = (a, AB)$.
7. The bundle above can be identified to the bundle of orthonormal frames over $\mathbb{R}^n$ with respect to the Euclidean metric.

Solution

1(a) Associativity:

$$\begin{aligned}
\big((a, A) \cdot (b, B)\big) \cdot (c, C) &= (a + Ab, AB) \cdot (c, C) \\
&= \big(a + Ab + (AB)c, (AB)C\big), \\
(a, A) \cdot \big((b, B) \cdot (c, C)\big) &= (a, A) \cdot (b + Bc, BC) = \big(a + A(b + Bc), A(BC)\big) \\
&= \big(a + Ab + (AB)c, (AB)C\big).
\end{aligned}$$

1(b) Identity element: $(a, A) \cdot (0, I) = (0, I) \cdot (a, A) = (a, A)$.
1(c) Inverse element: $(a, A)^{-1} = (-A^{-1} a, A^{-1})$.

We endow $E(n)$ with the differentiable structure corresponding to $E(n) \cong \mathbb{R}^n \times \mathrm{O}(n)$. As $\mathrm{O}(n)$ is a Lie group, it follows from part 1(c) and the very definition of the product law in $E(n)$ that $E(n)$ is also a Lie group.

2. We have

$$\begin{aligned}
(a, A) \cdot (b, I) \cdot (a, A)^{-1} &= (a + Ab, A) \cdot \big(-A^{-1} a, A^{-1}\big) \\
&= \big(a + Ab + A\big(-A^{-1} a\big), I\big) \\
&= (Ab, I) \in T(n).
\end{aligned}$$

3. Trivial.
4. The isotropy group $E(n)_x$ of a point $x \in \mathbb{R}^n$ is defined by

$$E(n)_x = \big\{(a, A) \in E(n) : (a, A) \cdot x = x\big\}.$$

So, $(a, A) \in E(n)_x$ if and only if $a + Ax = x$, or equivalently, $(I - A)x = a$. In particular,

$$E(n)_0 = \mathrm{O}(n) = \big\{(0, A) : A \in \mathrm{O}(n)\big\}.$$

For every $A \in \mathrm{O}(n)$, we have

$$(x, I) \cdot (0, A) \cdot (x, I)^{-1} = \big((I - A)x, A\big) \in E(n)_x.$$

Hence the map $\psi \colon E(n)_0 \to E(n)_x$ is an isomorphism and all the isotropy groups are conjugated in $E(n)$, thus isomorphic.
5. We have a diffeomorphism $x \mapsto (x, I) \bmod \mathrm{O}(n)$.
6. For every $B \in \mathrm{O}(n)$ we have $p((a, A) \cdot B) = p(a, AB) = a$. Hence $(a, A) \cdot \mathrm{O}(n) = p^{-1}(a)$. Moreover, as $(a, A) \cdot B = (a, A)$ implies $B = I$, the $\mathrm{O}(n)$-action is free. Thus, $p \colon E(n) \to \mathbb{R}^n$ is a principal $\mathrm{O}(n)$-bundle.

7. Let

$$\pi : \mathscr{O}\big(\mathbb{R}^n\big) \to \mathbb{R}^n$$

be the orthonormal frame bundle over $\mathbb{R}^n$ for the metric $g = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^n \otimes \mathrm{d}x^n$. Define a map $\varphi\colon E(n) \to \mathscr{O}(\mathbb{R}^n)$ by setting

$$\varphi(a, A) = \left(\frac{\partial}{\partial x^1}\bigg|_a, \ldots, \frac{\partial}{\partial x^n}\bigg|_a \right) \cdot A.$$

It is immediate that $\pi \circ \varphi = p$. Moreover, we have

$$\begin{aligned}\varphi\big((a, A) \cdot B\big) = \varphi(a, AB) &= \left(\frac{\partial}{\partial x^1}\bigg|_a, \ldots, \frac{\partial}{\partial x^n}\bigg|_a \right) \cdot (AB)\\ &= \left(\left(\frac{\partial}{\partial x^1}\bigg|_a, \ldots, \frac{\partial}{\partial x^n}\bigg|_a \right) \cdot A \right) \cdot B = \varphi(a, A) \cdot B.\end{aligned}$$

Finally, $\varphi(a, A) = \varphi(b, B)$ means

$$\left(\frac{\partial}{\partial x^1}\bigg|_a, \ldots, \frac{\partial}{\partial x^n}\bigg|_a \right) \cdot A = \left(\frac{\partial}{\partial x^1}\bigg|_b, \ldots, \frac{\partial}{\partial x^n}\bigg|_b \right) \cdot B.$$

This implies $a = b$ and $A = B$, thus proving that φ is a principal bundle isomorphism.

Problem 6.98 Let H be a Lie group acting freely on a Riemannian manifold (M, g). Suppose that each diffeomorphism $h \in H$ is an isometry, i.e. preserves the metric g. Suppose that on the quotient space $\widetilde{M} = M/H$ there exists a structure of manifold such that the canonical projection $\pi : M \to \widetilde{M}$ is a submersion. Then there is an induced metric $\tilde{g}$ on $\widetilde{M}$ defined by

$$\tilde{g}(\pi_{*x} v_x, \pi_{*x} u_x) = g(v_x, u_x),$$

where $v_x, u_x \in T_x M$ are vectors orthogonal to the subspace $\ker \pi_{*x} \subset T_x M$.

Prove:

(i) For each geodesic $\gamma(t)$ on (M, g) such that $g(\gamma'(0), \ker \pi_{*\gamma(0)}) = 0$ (it is orthogonal to the sub-bundle $\ker \pi_* \subset TM$ at the initial point) its direction $\gamma'(t)$ is orthogonal to the space $\ker \pi_{*\gamma(t)} \subset T_{\gamma(t)} M$ for all t.

(ii) The metric $\tilde{g}$ is well-defined and smooth. If each geodesic on (M, g) with a direction orthogonal to $\ker \pi_* \subset TM$ at the initial point can be continued indefinitely, the metric $\tilde{g}$ is complete.

Hint Use the fact that the image of the curve $(\gamma(t), \gamma'(t)) \subset TM$ under the natural isomorphism $TM \to T^*M$ induced by the metric is an integral curve of the geodesic flow on T^*M (see Problem 6.63).

Solution

(i) It is sufficient to prove it locally. Let $q = (q^1, \dots, q^n)$ denote local coordinates on some open subset $U \subset M$. Since the natural projection $\pi : M \to \widetilde{M} = M/H$ is a submersion we can suppose that $\tilde{q} = (\tilde{q}^1, \dots, \tilde{q}^m)$, $m \leqslant n$, are local coordinates on the open subset $\widetilde{U} = \pi(U) \subset \widetilde{M}$ such that the restriction $\pi|_U$ is given in these coordinates by $\pi(q^1, q^2, \dots, q^n) = (q^1, q^2, \dots, q^m)$. These local coordinates on M induce local coordinates $(q, p) = (q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*M putting $\omega_x = \sum_i p_i(\omega_x)\, \mathrm{d}q^i|_x$ for $\omega_x \in T^*M$, $x \in M$, and local coordinates $(q, y) = (q^1, \dots, q^n, y^1, \dots, y^n)$ on TM putting

$$v_x = \sum_{i=1}^n y^i(v_x) \frac{\partial}{\partial q^i}\bigg|_x, \qquad v_x \in TM.$$

Similarly, the local coordinates $\tilde{q}$ on $\widetilde{M}$ induce local coordinates $(\tilde{q}, \tilde{p}) = (\tilde{q}^1, \dots, \tilde{q}^m, \tilde{p}_1, \dots, \tilde{p}_m)$ on $T^*\widetilde{M}$ and local coordinates $(\tilde{q}, \tilde{y}) = (\tilde{q}^1, \dots, \tilde{q}^m, \tilde{y}^1, \dots, \tilde{y}^m)$ on $T\widetilde{M}$.

The metric g on M induces the natural isomorphism $\psi_g : TM \to T^*M$. The maps ψ_g, ψ_g^{-1} have the following form in local coordinates,

$$\psi_g\left(\sum_i y^i \frac{\partial}{\partial q^i}\right) = \sum_{i,j} g_{ij}(q) y^i\, \mathrm{d}q^j,$$

$$\psi_g^{-1}\left(\sum_i p_i\, \mathrm{d}q^i\right) = \sum_{i,j} g^{ij}(q) p_j \frac{\partial}{\partial q^i},$$

$g(q) = \sum_{i,j} g_{ij}(q)\, \mathrm{d}q^i\, \mathrm{d}q^j$ being the metric tensor in the local coordinates and $\sum_j g_{ij}(q) g^{jk}(q) = \delta_i^k$ by definition.

Note also that the kernel of the map $\pi|_U$ is generated by the $n - m$ vector fields $\partial/\partial q^k$, $k = m+1, \dots, n$, and that it is tangent to the orbits

$$H \cdot x = \{hx : h \in H\} \subset M, \qquad x \in U,$$

of the group H. Since the metric g is H-invariant, the local functions $g_{ij}(q)$, $i, j = 1, \dots, n$, are independent of the variables $q^{m+1}, \dots, q^n$ (g is invariant with respect to the one-parameter groups

$$\left(q^1, \dots, q^k, \dots, q^n\right) \mapsto \left(q^1, \dots, q^k + t, \dots, q^n\right), \quad m+1 \leqslant k \leqslant n,$$

generated by the vector fields $\partial/\partial q^k$, $m+1 \leqslant k \leqslant n$).

Recall now that by Problem 6.63, the curve $\gamma(t) = q(t) = (q^1(t), \dots, q^n(t))$ is a geodesic on (M, g) if and only if the curve $(p(t), q(t))$, for some smooth map $t \mapsto p(t)$, is an integral curve of the vector field X_H described by the relation

$$X_H(q,p)=\sum_{j,k} g^{kj}(q)p_j\frac{\partial}{\partial q^k}-\sum_{i,j,k}\frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}(q)p_ip_j\frac{\partial}{\partial p_k} \qquad (\star)$$

(see Solution of Problem 6.63). Then $p_j(t)=y^i(t)g_{ij}(q(t))$, where $y^i(t)=\frac{\mathrm{d}q^i(t)}{\mathrm{d}t}$. This vector field X_H is the Hamiltonian vector field of the function

$$H(q,p)=\frac{1}{2}\sum_{i,j} g^{ij}(q)p_ip_j$$

with respect to the canonical symplectic form on T^*M.

Let $\gamma(t)=(q^1(t),\dots,q^n(t))$ be a geodesic in (M,g) and let $\psi_g(\gamma(t),\gamma'(t)) = (p(t),q(t))$ be the corresponding curve on T^*M. Put $y^i(t)=\frac{\mathrm{d}q^i(t)}{\mathrm{d}t}$. If

$$g\big(\sigma'(0),\ker\pi_{*\sigma(0)}\big)=0$$

then $y^i(0)g_{ik}(\sigma(0))=0$ for all $k=m+1,\dots,n$, because $\ker\pi|_U$ is generated by $\partial/\partial q^k$, $k=m+1,\dots,n$. Taking into account that $p_j(t)=\sum_i y^i(t)g_{ij}(q(t))$, $j=1,\dots,n$, to prove (i) it is sufficient to show that $p_k(t)=0$ for all t if $p_k(0)=0$ with $k=m+1,\dots,n$. But as we remarked above the curve $(p(t),q(t))$ is an integral curve of the vector field X_H. So we only need to prove that X_H is tangent to the submanifold

$$X_0=\big\{(p,q)\in T^*U : p_{m+1}=0,\dots,p_n=0\big\}$$

of $T^*U\subset T^*M$. This fact follows immediately from the expression ($\star$) for X_H and the independence of the functions $g_{ij}(q)$ on the variables $q^{m+1},\dots,q^n$.

(ii) The metric $\tilde g$ is well-defined because the metric g is H-invariant and π is a submersion. Indeed, the restriction map $\pi_{*x}\colon V_x\to T_{\pi(x)}\widetilde M$ of the orthogonal complement V_x to $\ker\pi_{*x}$ in T_xM is an isomorphism and for $v_x\in T_xM$ and $v_{hx}\in T_{hx}M$, $h\in H$, we have $\pi_{*x}v_x=\pi_{*hx}v_{hx}$ if and only if $v_{hx}=h_{*x}v_x$ (as $\pi\circ h=\pi$ and $h\colon M\to M$ is a diffeomorphism). Therefore,

$$\begin{aligned}\tilde g(\pi_{*x}v_x,\pi_{*x}u_x)&=g(v_x,u_x)=g(h_{*x}v_x,h_{*x}u_x)=g(v_{hx},u_{hx})\\&=\tilde g(\pi_{*hx}v_{hx},\pi_{*hx}u_{hx}).\end{aligned}$$

To prove the second assertion of (ii) it is sufficient to show that for each geodesic $\gamma(t)$ on (M,g) with directions $\gamma'(t)$ orthogonal to $\ker\pi_*\subset TM$ its image $\pi(\gamma(t))$ is a geodesic of the metric $\tilde g$. It is enough to prove this fact locally.

Let $\tilde g(\tilde q)=\sum_{a,b}\tilde g_{ab}(\tilde q)\,\mathrm{d}q^a\,\mathrm{d}q^b$, $a,b=1,\dots,m$, be the metric tensor $\tilde g$ in local coordinates, with $\sum_b\tilde g_{ab}(\tilde q)\tilde g^{bc}(\tilde q)=\delta_a^c$ by definition. Let $\tilde\gamma(t)=(\tilde q^1(t),\dots,\tilde q^m(t))$ be some smooth curve on $\widetilde M$. The curve $\tilde\gamma(t)$ is a geodesic of $\tilde{g}$ if and only if there exists a map $t \mapsto \tilde{p}(t)$ such that the curve $(\tilde{p}(t), \tilde{q}(t))$ is an integral curve of the vector field

$$X_{\tilde{H}} = \sum_{a,b} \tilde{g}^{ab}(\tilde{q})\tilde{p}_b \frac{\partial}{\partial \tilde{q}^a} - \frac{1}{2}\sum_{a,b,c} \frac{\partial \tilde{g}^{ab}}{\partial \tilde{q}^c}(\tilde{q})\tilde{p}_a\tilde{p}_b \frac{\partial}{\partial \tilde{p}_c},$$

i.e. if and only if

$$\frac{\mathrm{d}\tilde{q}^a}{\mathrm{d}t} = \sum_b \tilde{g}^{ab}(\tilde{q})\tilde{p}_b, \qquad \frac{\mathrm{d}\tilde{p}^c}{\mathrm{d}t} = -\frac{1}{2}\sum_{a,b} \frac{\partial \tilde{g}^{ab}}{\partial \tilde{q}^c}(\tilde{q})\tilde{p}_a\tilde{p}_b$$

(see Problem 6.63). In this case $\tilde{p}_b(t) = \sum_a \tilde{y}^a(t)\tilde{g}_{ab}(\tilde{q}(t))$, where $\tilde{y}^a(t) = \frac{\mathrm{d}\tilde{q}^a(t)}{\mathrm{d}t}$.

But $\pi(q^1, \dots, q^n) = (q^1, \dots, q^m)$ and any geodesic of (M, g) lying in the submanifold $X_0 \subset T^*U$ satisfies the following relations (as it is an integral curve of $X_H|_{X_0}$):

$$\frac{\mathrm{d}q^a}{\mathrm{d}t} = \sum_b g^{ab}(q)p_b, \qquad \frac{\mathrm{d}q^j}{\mathrm{d}t} = \sum_b g^{jb}(q)p_b, \quad j = m+1, \dots, n,$$

$$\frac{\mathrm{d}p^c}{\mathrm{d}t} = -\frac{1}{2}\sum_{a,b} \frac{\partial g^{ab}}{\partial q^c}(q)p_a p_b,$$

where $1 \leqslant a, b, c \leqslant m$. Therefore,

$$\big(q^1(t), \dots, q^n(t), p_1(t), \dots, p_m(t)\big), \qquad \big(\tilde{q}^1(t), \dots, \tilde{q}^m(t), \tilde{p}_1(t), \dots, \tilde{p}_m(t)\big),$$

where $\tilde{q}^a(t) = q^a(t)$ and $\tilde{p}^a(t) = p^a(t)$ are simultaneously the solutions of the corresponding relations if $\tilde{g}^{ab}(q^1, \dots, q^m) = g^{ab}(q^1, \dots, q^m, q^{m+1}, \dots, q^n)$ (the matrix $g_{ij}(q)$ is independent of $q^{m+1}, \dots, q^n$). Let us prove this fact.

Fix some point $x \in U \subset M$ with coordinates $(q^1, \dots, q^m, \dots, q^n)$ and its image $\pi(x)$ with coordinates $(q^1, \dots, q^m)$. Put $e_i = \partial/\partial q^i$, $i = 1, \dots, n$. Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be the matrix $q_{ij}(x) = g(e_i, e_j)$, where A and D are respectively $m \times m$ and $(n-m) \times (n-m)$ symmetric matrices and ${}^tB = C$. For each tangent vector e_a, $a = 1, \dots, m$, there exists a unique vector $v_a = \sum_{\beta=m+1}^n v_{a\beta}e_\beta$ such that $g(e_a + v_a, e_\beta) = 0$, where $\beta = m+1, \dots, n$. Then by definition $\tilde{g}_{ab}(\pi(x)) = g(e_a + v_a, e_b + v_b)$. Put $\tilde{A} = (\tilde{g}_{ab})$. It is easy to verify that the $m \times (n-m)$-matrix $V = (v_{a\beta})$ equals $-BD^{-1}$ and

$$\tilde{A} = A + VC + B\,{}^tV + VD\,{}^tV = A - BD^{-1}C.$$

In particular, the metric $\tilde{g}$ is smooth because the matrix-functions $A(q)$, $B(q)$, $C(q)$ and $D(q)$ are smooth.

Let $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ be the matrix $q^{ij}(x)$, where A' and D' are respectively $m \times m$ and $(n-m) \times (n-m)$ matrices. We need to prove that $(\tilde{A})^{-1} = A'$. Indeed, since $\sum_j q_{ij}(x) q^{jk}(x) = \delta_i^k$, we have

$$AA' + BC' = \mathrm{Id}_m, \qquad CA' + DC' = 0, \qquad AB' + BD' = 0,$$
$$CB' + DD' = \mathrm{Id}_{n-m}.$$

Therefore,

$$\tilde{A}A' = \big(A - BD^{-1}C\big)A' = AA' - BD^{-1}\big(CA'\big) = AA' - BD^{-1}\big(-DC'\big) = \mathrm{Id}_m.$$

Problem 6.99 Identify as usual the sphere S^3 and the Lie group $\mathrm{SU}(2)$ by the map

$$\begin{aligned} S^3 \subset \mathbb{C}^2 &\longrightarrow \quad \mathrm{SU}(2) \\ (z, w) &\longmapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}. \end{aligned}$$

Consider the basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$ given by

$$X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Then

$$[X_1, X_2] = 2X_3, \qquad [X_2, X_3] = 2X_1, \qquad [X_3, X_1] = 2X_2.$$

The one-parameter family

$$\{g_\varepsilon : \varepsilon > 0\}$$

of left-invariant Riemannian metrics on $S^3 \equiv \mathrm{SU}(2)$ given at the identity, with respect to the basis of left-invariant vector fields X_1, X_2, X_3, by

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

are called the *Berger metrics* on S^3; if $\varepsilon = 1$ we have the canonical (bi-invariant) metric. The *Berger spheres* are the simply connected complete Riemannian manifolds

$$S^3_\varepsilon = \big(S^3, g_\varepsilon\big), \quad \varepsilon > 0.$$

Then:

(i) Compute the curvature tensor field, the Ricci tensor and the scalar curvature of Berger 3-spheres.
(ii) Prove that for $\varepsilon \neq 1$, a Berger 3-sphere is not a Riemannian symmetric space.

The relevant theory is developed, for instance, in Petersen [27].

Solution

(i) The Koszul formula for the Levi-Civita connection of g_ε,

$$2g_\varepsilon(\nabla_X Y, Z) = g_\varepsilon\big([X, Y], Z\big) - g_\varepsilon\big([Y, Z], X\big) + g_\varepsilon\big([Z, X], Y\big),$$

for all $X, Y, Z \in \mathfrak{su}(2)$, gives us that the non-zero covariant derivatives between generators are given (omitting the ε in ∇^ε for the sake of simplicity) by

$$\begin{aligned} &\nabla_{X_1} X_2 = (2-\varepsilon)X_3, \qquad \nabla_{X_1} X_3 = (\varepsilon-2)X_2, \qquad \nabla_{X_2} X_1 = -\varepsilon X_3,\\ &\nabla_{X_2} X_3 = X_1, \qquad \nabla_{X_3} X_1 = \varepsilon X_2, \qquad \nabla_{X_3} X_2 = -X_1. \end{aligned}$$

The nonzero components of the curvature tensor field R, that is,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}\big(S^3_\varepsilon\big),$$

are thus given by

$$\begin{aligned} &R(X_1, X_2)X_1 = -\varepsilon^2 X_2, \qquad R(X_1, X_2)X_2 = \varepsilon X_1,\\ &R(X_1, X_3)X_1 = -\varepsilon^2 X_3, \qquad R(X_1, X_3)X_3 = \varepsilon X_1,\\ &R(X_2, X_3)X_2 = (3\varepsilon-4)X_3, \qquad R(X_2, X_3)X_3 = (4-3\varepsilon)X_2. \end{aligned}$$

We can obtain the nonzero components of the Riemann–Christoffel tensor

$$R(X, Y, Z, W) = g_\varepsilon\big(R(Z, W)Y, X\big),$$

given in terms of the orthonormal basis

$$\{X, X_2, X_3\} = \left\{\frac{1}{\sqrt{\varepsilon}} X_1, X_2, X_3\right\}$$

by

$$R(X, X_2, X, X_2) = R(X, X_3, X, X_3) = \varepsilon, \qquad R(X_2, X_3, X_2, X_3) = 4 - 3\varepsilon.$$

The non-zero components of the Ricci tensor are thus given by

$$\mathbf{r}(X, X) = 2\varepsilon, \qquad \mathbf{r}(X_2, X_2) = \mathbf{r}(X_3, X_3) = 4 - 2\varepsilon,$$

and the scalar curvature by

$$\mathbf{s} = \mathbf{r}(X, X) + \mathbf{r}(X_2, X_2) + \mathbf{r}(X_3, X_3) = 2(4 - \varepsilon).$$

(ii) It suffices to find some vectors fields X, Y, Z, W, U satisfying

$$(\nabla_X R)(Y, Z, W, U) \neq 0.$$

Now, for instance, we have

$$\begin{aligned}
&(\nabla_{X_2} R)(X_1, X_2, X_2, X_3) \\
&\quad = X_2\big(R(X_1, X_2, X_2, X_3)\big) \\
&\qquad - R(\nabla_{X_2} X_1, X_2, X_2, X_3) - R(X_1, \nabla_{X_2} X_2, X_2, X_3) \\
&\qquad - R(X_1, X_2, \nabla_{X_2} X_2, X_3) - R(X_1, X_2, X_2, \nabla_{X_2} X_3) \\
&\quad = -R(\nabla_{X_2} X_1, X_2, X_2, X_3) - R(X_1, X_2, X_2, \nabla_{X_2} X_3) \\
&\quad = -\varepsilon R(X_2, X_3, X_2, X_3) + R(X_1, X_2, X_1, X_2) \\
&\quad = -\varepsilon(4 - 3\varepsilon) + \varepsilon = 3\varepsilon(\varepsilon - 1),
\end{aligned}$$

which only vanishes for $\varepsilon = 1$, that is, for the canonical bi-invariant metric. Hence a Berger sphere different from the canonical one, is not a Riemannian symmetric space.

Problem 6.100 Consider the Riemannian manifold (M, g) given by

$$M = \mathbb{R}^3, \qquad g = A\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) + \big(\mathrm{d}z + B(x\,\mathrm{d}y - y\,\mathrm{d}x)\big)^2, \quad A > 0,\ B > 0.$$

Prove:

(i) The Lie algebra of the isometry group $I(M, g)$ of (M, g) is the vector space

$$\mathfrak{i}(M, g) = \left\{(ay + b)\frac{\partial}{\partial x} + (c - ax)\frac{\partial}{\partial y} + \big(B(cx - by) + d\big)\frac{\partial}{\partial z} : a, b, c, d \in \mathbb{R}\right\}.$$

(ii) The manifold (M, g) is not a space of constant curvature.
(iii) The manifold (M, g) is not a Riemannian symmetric space.

Hint Let $I(M, g)$ denote the isometry group of (M, g). Since $n = 3$, according to Theorem 6.33, we have $\dim I(M, g) \neq 5$. Moreover, for dimension $n = 3$, if (M, g) is not a space of constant curvature, then $\dim I(M, g) \leq 4$.

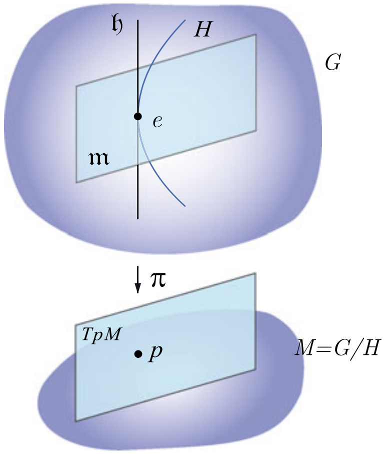
The reader can find the relevant theory developed in É. Cartan [6, Chap. XII].

Solution

(i) As a direct calculation shows, the vector fields

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial x} - By\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} + Bx\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial z},$$

Fig. 6.10 For a reductive homogeneous space one identifies $\mathfrak{m} \equiv T_p M$



are $\mathbb{R}$-linearly independent Killing vector fields. Since $\dim \mathfrak{i}(M, g) \leqslant 4$ it follows that

$$\dim \mathfrak{i}(M, g) = 4.$$

(ii) Let $o = (0,0,0)$. In order to apply the results in [19, Chap. X, §3], let us check that this homogeneous space is actually a *naturally reductive space* (see Fig. 6.10 for the general case of a reductive homogeneous space), that is, that the Lie algebra $\mathfrak{g} = \mathfrak{i}(M, g)$ decomposes into the direct sum of $\mathfrak{h} = \langle K \rangle$ and $\mathfrak{m} = \langle X, Y, Z \rangle$, with $\mathfrak{m}$ being $\mathrm{Ad}(H)$-invariant, satisfying

$$\big\langle U, [W, V]_{\mathfrak{m}} \big\rangle + \big\langle [W, U]_{\mathfrak{m}}, V \big\rangle = 0, \quad U, V, W \in \mathfrak{m},$$

where $\langle \cdot\, , \cdot \rangle$ is the positive definite, $\mathrm{Ad}(H)$-invariant, symmetric bilinear form on $\mathfrak{m}$, given by $\langle U, V \rangle = g(U, V)_o$.

Let us denote $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_3 = \partial/\partial z$ and take the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{h} = \langle K \rangle, \ \mathfrak{m} = \langle X, Y, Z \rangle,$$

with

$$K = y\partial_1 - x\,\partial_2, \qquad X = \partial_1 - By\,\partial_3, \qquad Y = \partial_2 + Bx\,\partial_3, \qquad Z = \partial_3 + \lambda K, \tag{$\star$}$$

$\lambda \in \mathbb{R}$ being a parameter to be determined so that the decomposition be naturally reductive. Calculating the brackets, we get

$$\begin{aligned} &[K, X] = Y, \qquad [K, Y] = -X, \qquad [K, Z] = 0, \\ &[X, Y] = 2B(Z - \lambda K), \qquad [X, Z] = -\lambda Y, \qquad [Y, Z] = \lambda X. \end{aligned} \tag{$\star\star$}$$

From (⋆⋆) we deduce that $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. Using these formulas, we have, for $U, V \in \mathfrak{m}$, that

$$[U,V] = 2B\big(U^1V^2 - U^2V^1\big)(Z-\lambda K) - \lambda\big(U^1V^3 - U^3V^1\big)Y + \lambda\big(U^2V^3 - U^3V^2\big)X,$$

so that

$$[U,V]_{\mathfrak{m}} = 2B\big(U^1V^2 - U^2V^1\big)Z - \lambda\big(U^1V^3 - U^3V^1\big)Y + \lambda\big(U^2V^3 - U^3V^2\big)X.$$

If $U, V, W \in \mathfrak{m}$ then

$$\big\langle U, [W,V]_{\mathfrak{m}}\big\rangle = 2BU^3\big(W^1V^2 - W^2V^1\big) - \lambda AU^2\big(W^1V^3 - W^3V^1\big) + \lambda AU^1\big(W^2V^3 - W^3V^2\big), \qquad (\star\star\star)$$

hence

$$\big\langle [W,U]_{\mathfrak{m}}, V\big\rangle = 2BV^3\big(W^1U^2 - W^2U^1\big) - \lambda AV^2\big(W^1U^3 - W^3U^1\big) + \lambda AV^1\big(W^2U^3 - W^3U^2\big). \qquad (\dagger)$$

Summing up equalities (⋆⋆⋆) and (†), one has

$$\big\langle U, [W,V]_{\mathfrak{m}}\big\rangle + \big\langle [W,U]_{\mathfrak{m}}, V\big\rangle = (2B-\lambda A)\big(U^3V^2W^1 - U^3V^1W^2 + U^2V^3W^1 - U^1V^3W^2\big),$$

so if

$$\lambda = \frac{2B}{A}$$

the previous decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is naturally reductive.

Let R denote the Riemann–Christoffel curvature tensor of (M, g). According to [19, Chap. X, Proposition 3.4], the sectional curvature is given by

$$R(U,V,U,V)_o = \frac{1}{4}\big\langle [U,V]_{\mathfrak{m}}, [U,V]_{\mathfrak{m}}\big\rangle - \big\langle \big[[U,V]_{\mathfrak{h}}, V\big], U\big\rangle \qquad (\dagger\dagger)$$

for $U, V \in \mathfrak{m}$. From the polarization formula

$$R(U,W,V,W) = \frac{1}{2}\big\{R(U+V,W,U+V,W) - R(U,W,U,W) - R(V,W,V,W)\big\},$$

it is clear that we can know the tensor R only calculating the three next possibilities, obtained by using (††),

$$R(X,Y,X,Y)_o = -3B^2, \qquad R(X,Z,X,Z)_o = \frac{B^2}{A},$$

$$R(Y, Z, Y, Z)_o = \frac{B^2}{A}.$$

Since the sectional curvatures do not coincide, the manifold does not have constant curvature.

(iii) By using the aforementioned results in [19], we now compute the covariant derivative of the previous curvature tensor. We should calculate

$$(\nabla R)(U_1, \dots, U_5)_o = (\nabla_{U_1} R)(U_2, \dots, U_5)_o,$$

where $U_1 = X, Y, Z$ and $(U_2, \dots, U_5)$ denotes any of the previous quadruples (X, Y, X, Y), (X, Z, X, Z) and (Y, Z, Y, Z), so that we have 9 values to consider. Let us calculate, for instance,

$$\begin{aligned}(\nabla_X R)(X, Y, Z, X)_o = X\big(R(X, Y, Z, X)\big)_o &- R(X, \nabla_X Y, Z, X)_o \\ &- R(X, Y, \nabla_X Z, X)_o.\end{aligned}$$

Then, denoting again by R the curvature tensor of type $(1, 3)$, we have on the one hand that

$$\begin{aligned}X\big(R(X, Y, Z, X)\big) = X\big(R(Z, X, X, Y)\big) &= X\big(g\big(R(X, Y)X, Z\big)\big) \\ &= -X\left(g\left(\frac{3B^2}{A}Y, Z\right)\right) \\ &= -\frac{3B^2}{A}X\left(-\frac{4B^3}{A}x\big(x^2 + y^2\big)\right) \\ &= \frac{12B^5}{A^2}(\partial_1 - By\partial_3)\big(x\big(x^2 + y^2\big)\big) \\ &= \frac{12B^5}{A^2}\big(3x^2 + y^2\big),\end{aligned}$$

from which we have that

$$X\big(R(X, Y, Z, X)\big)_o = 0. \tag{†††}$$

On the other hand, one has that

$$\begin{aligned}R(X, \nabla_X Y, Z, X)_o &= \frac{1}{2}R\big(X, [X, Y]_{\mathfrak{m}}, Z, X\big)_o = BR(X, Z, Z, X)_o \\ &= -\frac{B^3}{A}, \qquad (\diamond) \\ R(X, Y, \nabla_X Z, X)_o &= \frac{1}{2}R\big(X, Y, [X, Z]_{\mathfrak{m}}, X\big)_o = -\frac{B}{A}R(X, Y, Y, X)_o \\ &= -\frac{3B^3}{A}. \qquad (\diamond\diamond)\end{aligned}$$

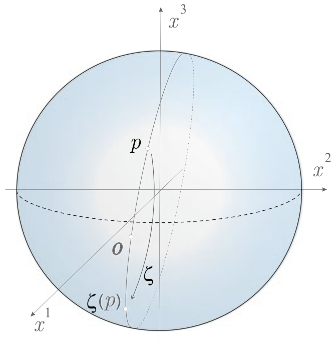


Fig. 6.11 The symmetry of S^2 at a point o

Using now (†††), (◇) and (◇◇), we obtain that

$$(\nabla_X R)(X, Y, Z, X)_o = \frac{4B^3}{A},$$

which being nonzero permits us to conclude that M is not a symmetric space.

Problem 6.101 As the unit sphere in $\mathbb{R}^{n+1}$, $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a symmetric space, with symmetry ζ at $o = (1, 0, \ldots, 0)$ given (see Fig. 6.11) by

$$\left(t^0, t^1, \ldots, t^n\right) \mapsto \left(t^0, -t^1, \ldots, -t^n\right).$$

For the symmetric space S^n, find:

(i) The involutive automorphism σ of $\mathrm{SO}(n+1)$ such that

$$\mathrm{SO}(n+1)_0^\sigma \subset \mathrm{SO}(n) \subset \mathrm{SO}(n+1)^\sigma,$$

where $\mathrm{SO}(n+1)^\sigma$ denotes the closed subgroup of $\mathrm{SO}(n+1)$ of fixed points of σ, and $\mathrm{SO}(n+1)_0^\sigma$ its identity component.

(ii) The subspace

$$\mathfrak{m} = \left\{X \in \mathfrak{so}(n+1) : \sigma_* X = -X\right\}.$$

(iii) The $\mathrm{Ad}(\mathrm{SO}(n))$-invariant inner product on $\mathfrak{m}$.
(iv) The geodesics.
(v) The isomorphism $p_* \colon \mathfrak{m} \cong T_o S^n$.
(vi) The linear isotropy action.
(vii) The curvature.

The relevant theory is developed, for instance, in O'Neill [26, Chap. 11].

Solution

(i) As $\zeta = \mathrm{diag}(1, -1, \ldots, -1)$, for $A \in \mathrm{SO}(n+1)$ we have (see [26, Lemma 28, p. 315]):

$$\sigma(A) = \zeta A \zeta$$

$$= \left(\begin{array}{c|c} a_{00} & -a_{01} \cdots -a_{0n} \\ \hline -a_{10} & \\ \vdots & (a_{ij}) \\ -a_{n0} & \end{array}\right), \quad 1 \leqslant i, j \leqslant n.$$

So $\mathrm{SO}(n+1)^\sigma$ is $S(\mathrm{O}(1) \times \mathrm{O}(n))$, and $\mathrm{SO}(n+1)^\sigma_0$ is the isotropy group $1 \times \mathrm{SO}(n) \cong \mathrm{SO}(n)$.

(ii) As $\zeta = \zeta^{-1}$, we have $\sigma(A) = \zeta A \zeta^{-1}$, so that σ is conjugation by ζ. Thus, σ_* is also conjugation by ζ on the Lie algebra $\mathfrak{so}(n+1)$. Hence

$$\mathfrak{m} = \left\{ X = \begin{pmatrix} 0 & -{}^t x \\ x & 0 \end{pmatrix} \right\},$$

where x denotes any column vector, regarded as an element of $\mathbb{R}^n$. Write $X \leftrightarrow x$ for the resulting correspondence between $\mathfrak{m}$ and $\mathbb{R}^n$.

(iii) Under $X \leftrightarrow x$, the dot product $x \cdot y$ on $\mathbb{R}^n$ corresponds to

$$B(X, Y) = -\frac{1}{2} \operatorname{tr} XY = \frac{1}{2} X \cdot Y$$

on $\mathfrak{m}$, where $X \cdot Y$ denotes the scalar product in $\mathbb{R}^{(n+1)^2}$. B is thus a multiple of the Killing form on $\mathfrak{so}(n+1)$ (see table in p. 557). One has $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ and the Killing form is $\mathrm{Ad}(\mathrm{SO}(n+1))$-invariant. It follows from (v) below that the corresponding metric tensor on S^n is the usual one. In fact,

$$B\left(\begin{pmatrix} 0 & -{}^t x \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & -{}^t y \\ y & 0 \end{pmatrix} \right) = \sum_{i=1}^n x^i y^i = x \cdot y.$$

(iv) Let γ be a geodesic of S^n starting at o. Since S^n is symmetric, we have (see [26, Proposition 31, p. 317])

$$\gamma(t) = \exp(tX)o$$

for some $X \in \mathfrak{m}$.

It is easily seen that

$$\exp tX = \begin{pmatrix} 1 - \frac{t^2}{2} x \cdot x + \cdots & * \\ tx - \frac{t^3}{6}(x \cdot x)x + \cdots & * \end{pmatrix} = \begin{pmatrix} \cos|x|t & * \\ (\sin|x|t)\frac{x}{|x|} & * \end{pmatrix},$$

where $\binom{*}{*}$ stands for an $((n+1) \times n)$-matrix which does not matter for our purpose. Thus,

$$(\exp tX)o = (\exp tX) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \big(\cos|x|t\big)o + \big(\sin|x|t\big)\frac{x}{|x|}.$$

Hence γ is the great circle parametrisation

$$\big(\cos|x|t\big)o + \big(\sin|x|t\big)\frac{x}{|x|},$$

where $X \leftrightarrow x$.

(v) In (iii), $\mathbb{R}^n$ is assumed to be identified with the last n coordinate subspace of $\mathbb{R}^{n+1}$. Hence the canonical isomorphism identifies T_oS^n with $\mathbb{R}^n$. Then, according to (iv), $x = \gamma'(0)$. But X is the initial velocity vector of the 1-parameter subgroup projecting to γ. Hence, the isomorphism $p_*\colon \mathfrak{m} \cong T_oS^n$ is $X \leftrightarrow x$.

(vi) If $h = \left(\begin{smallmatrix}1 & 0\\ 0 & A\end{smallmatrix}\right) \in H = \mathrm{SO}(n)$ and $X \in \mathfrak{m}$, then

$$\mathrm{Ad}_h X = \begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\begin{pmatrix}0 & -{}^t x\\ x & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & A^{-1}\end{pmatrix} = \begin{pmatrix}0 & -{}^t x\, A^{-1}\\ Ax & 0\end{pmatrix}$$

(which is skew-symmetric since ${}^tA = A^{-1}$). Thus the linear isotropy action of H on T_oS^n is, via the identifications, the usual action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, i.e. $(A, x) \mapsto Ax$.

(vii) In terms of the subspace $\mathfrak{m}$, we have

$$R(X,Y)Z = -\big[[X,Y],Z\big].$$

If x, y, z denote the corresponding vectors in $\mathbb{R}^n \equiv T_oS^n$, we obtain

$$[X,Y] = \begin{pmatrix}0 & 0\\ 0 & -(x^i y^j - x^j y^i)\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -\widetilde{A}\end{pmatrix},$$

where $(\widetilde{A}^i_j) = (x^i y^j - x^j y^i)$, so

$$\begin{aligned} R(X,Y)Z = &-\begin{pmatrix}0 & 0\\ 0 & -(x^i y^j - x^j y^i)\end{pmatrix}\begin{pmatrix}0 & -{}^t z\\ z & 0\end{pmatrix}\\ &+\begin{pmatrix}0 & -{}^t z\\ z & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & -(x^i y^j - x^j y^i)\end{pmatrix} = \begin{pmatrix}0 & -{}^t(\widetilde{A}z)\\ \widetilde{A}z & 0\end{pmatrix}.\end{aligned}$$

Thus, under the identification, $R(X,Y)Z$ corresponds to $\widetilde{A}z$, that is, to

$$(y\cdot z)x - (x\cdot z)y.$$

Hence S^n has constant curvature 1.

Problem 6.102 The complex projective space

$$\mathbb{C}\mathrm{P}^n \cong \mathrm{U}(n+1)/\mathrm{U}(1)\times\mathrm{U}(n) \cong \mathrm{SU}(n+1)/S\big(\mathrm{U}(1)\times\mathrm{U}(n)\big)$$

is a compact simply connected Hermitian symmetric space of dimension $2n$. Find:

(i) The involutive automorphism σ of $\mathrm{U}(n+1)$ such that

$$\mathrm{U}(n+1)_0^\sigma \subset \mathrm{U}(1)\times\mathrm{U}(n) \subset \mathrm{U}(n+1)^\sigma,$$

$\mathrm{U}(n+1)^\sigma$ being the closed subgroup of $\mathrm{U}(n+1)$ of fixed points of σ and $\mathrm{U}(n+1)_0^\sigma$ its identity component.

(ii) The subspace

$$\mathfrak{m} = \left\{X \in \mathfrak{u}(n+1) : \sigma_* X = -X\right\}.$$

(iii) The $\mathrm{Ad}(\mathrm{U}(1)\times\mathrm{U}(n))$-invariant inner product on $\mathfrak{m}$.

(iv) The linear isotropy action.

Moreover, prove:

(v) The scalar multiplication by i on $\mathbb{C}^n \cong \mathfrak{m}$ gives a corresponding complex structure J_0 on $\mathfrak{m}$, which is $\mathrm{Ad}(\mathrm{U}(1)\times\mathrm{U}(n))$-invariant, and so determines an almost complex structure J on $\mathbb{C}\mathrm{P}^n$ making it a Kähler manifold.

(vi) $\mathbb{C}\mathrm{P}^n$ has constant holomorphic sectional curvature.

The relevant theory is developed, for instance, in O'Neill [26, Chap. 11].

Solution

(i) Let $\zeta = \mathrm{diag}(-1, 1, \dots, 1)$. The conjugation

$$\sigma\colon\ A \mapsto \zeta A \zeta^{-1}$$

is an involutive automorphism of $\mathrm{U}(n+1)$ whose fixed point set is $\mathrm{U}(1)\times\mathrm{U}(n)$, thus having

$$S\big(\mathrm{U}(1)\times\mathrm{U}(n)\big) = \mathrm{U}(n+1)_0^\sigma \subset \mathrm{U}(n+1)^\sigma = \mathrm{U}(1)\times\mathrm{U}(n).$$

(ii) The (-1)-eigenspace $\mathfrak{m}$ of σ_* consists of all the elements in $\mathfrak{u}(n+1)$ of the form $X = \left(\begin{smallmatrix} 0 & -{}^t\bar{x} \\ x & 0\end{smallmatrix}\right)$, where x is an $n\times 1$ complex matrix.

(iii) The inner product

$$B(X,Y) = -\frac{1}{2}\,\mathrm{tr}\, XY = \frac{1}{2} X\cdot \bar{Y}$$

is a multiple of the Killing form on $\mathfrak{u}(n+1)$ (see table on p. 557), and hence it is $\mathrm{Ad}(\mathrm{U}(1)\times\mathrm{U}(n))$-invariant. Because of the factor $\frac{1}{2}$, $B|_{\mathfrak{m}}$ corresponds under the identification $\mathfrak{m} \equiv \mathbb{C}^n$ to the real part of the natural Hermitian product $x\cdot\bar{y}$ in $\mathbb{C}^n$.

(iv) We have

$$\mathrm{Ad}_{\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & A\end{smallmatrix}\right)} \begin{pmatrix} 0 & -{}^t\bar{x} \\ x & 0\end{pmatrix} = \begin{pmatrix} 0 & -{}^t\overline{(e^{-i\theta}Ax)} \\ e^{-i\theta}Ax & 0\end{pmatrix}. \qquad (\star)$$

The linear isotropy action of $\mathrm{U}(1) \times \mathrm{U}(n)$ on $\mathfrak{m} \equiv \mathbb{C}^n$ thus corresponds to the action of $\mathrm{U}(1) \times \mathrm{U}(n)$ on $\mathbb{C}^n$ given by

$$\big(\mathrm{e}^{\mathrm{i}\theta}, A\big)x = \mathrm{e}^{-\mathrm{i}\theta} Ax.$$

(v) Scalar multiplication by i in $\mathbb{C}^n \equiv \mathfrak{m}$ gives a complex structure J_0 on $\mathfrak{m}$:

$$\begin{array}{ccc} \mathfrak{m} \equiv \mathbb{C}^n & \stackrel{J_0}{\to} & \mathfrak{m} \\ \begin{pmatrix} 0 & -{}^t\bar{x} \\ x & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} 0 & \mathrm{i}\,{}^t\bar{x} \\ \mathrm{i}\,x & 0 \end{pmatrix}, \end{array}$$

which is $\mathrm{Ad}(\mathrm{U}(n) \times \mathrm{U}(1))$-invariant. In fact, by ($\star$) above we have with the usual notations, for any $X \in \mathfrak{m}$,

$$J_0 \,\mathrm{Ad}_{\left(\begin{smallmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & A \end{smallmatrix}\right)} X = \begin{pmatrix} 0 & -{}^t(\mathrm{i}\,\overline{\mathrm{e}^{-\mathrm{i}\theta} Ax}) \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta} Ax & 0 \end{pmatrix} = \mathrm{Ad}_{\left(\begin{smallmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & A \end{smallmatrix}\right)} J_0 X.$$

Therefore (see [26, Proposition 43, p. 325]), J_0 determines an almost complex structure J on $\mathbb{C}\mathrm{P}^n$ making it a Kähler manifold.

(vi) (a) As $\mathrm{U}(n)$ acts transitively on the complex lines in $\mathbb{C}^n$ (i.e. the holomorphic planes in $T_o(\mathbb{C}\mathrm{P}^n)$), from ($\star$) it follows that for $\theta = 0$, the action of the linear isotropy group is transitive on complex lines.

(b) Let $o \in \mathbb{C}\mathrm{P}^n$ be the point corresponding to $(1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. The holomorphic sectional curvature is constant on $T_o(\mathbb{C}\mathrm{P}^n)$, so by homogeneity it is constant everywhere. In fact, multiplying B by 4, we have $B(X, Y) = -2 \operatorname{tr} XY$. Let $e_1, e_2 \in \mathfrak{m}$ correspond to elements of the natural basis of $\mathbb{C}^n \equiv \mathfrak{m}$. From (a) above, an arbitrary tangent plane on $\mathbb{C}\mathrm{P}^n$ has sectional curvature $K(e_1, Y)$, where $Y = \cos\theta\, Je_1 + \sin\theta\, e_2$. We have

$$e_1 = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad Je_1 = \begin{pmatrix} 0 & \mathrm{i} & 0 & \cdots & 0 \\ \mathrm{i} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, we deduce (see [26, Remark, p. 319]):

$$K(e_1, Y) = \frac{B([e_1, Y], [e_1, Y])}{B(e_1, e_1)B(Y, Y) - B(e_1, Y)^2}$$

$$\begin{aligned}&= \frac{1}{16} B\big(\cos\theta[e_1, Je_1] + \sin\theta[e_1, e_2], \cos\theta[e_1, Je_1]\\&\quad + \sin\theta[e_1, e_2]\big)\\&= \frac{1}{4}\big(1 + 3\cos^2\theta\big).\end{aligned}$$

Hence

$$\frac{1}{4} \leqslant K \leqslant 1.$$

Taking $\theta = 0$, so $Y = Je_1$, shows that $\mathbb{C}\mathrm{P}^n$ has constant holomorphic sectional curvature 1.

Problem 6.103 Let $G = \mathrm{O}(p, q + 1)$ and $H = \mathrm{O}(p, q)$. Show that the homogeneous space $M = G/H$ is symmetric and can be represented as the hyperquadric

$$\begin{aligned}Q = \big\{x = \big(x^1, \dots, x^{p+q+1}\big) \in \mathbb{R}^{p+q+1} :\\ \big(x^1\big)^2 + \cdots + \big(x^p\big)^2 - \big(x^{p+1}\big)^2 - \big(x^{p+q+1}\big)^2 = -1\big\}.\end{aligned}$$

Hint Show that the map

$$\begin{aligned}\sigma\colon \mathrm{O}(p, q + 1) &\to \mathrm{O}(p, q + 1)\\ a &\mapsto \zeta\, a\, \zeta^{-1},\end{aligned}$$

where ζ is the matrix $\zeta = \left(\begin{smallmatrix} -I_{p+q} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the canonical basis of $\mathbb{R}^{p+q+1}$, is an involutive automorphism (i.e. $\sigma^2 = \mathrm{id}$) of $\mathrm{O}(p, q + 1)$.

Solution Since $\zeta^{-1} = \zeta$, the map σ is an involutive automorphism of $\mathrm{O}(p, q + 1)$. The closed subgroup of $\mathrm{O}(p, q + 1)$ of fixed points of σ is $G^\sigma = \mathrm{O}(p, q + 1)^\sigma = \mathrm{O}(p, q)$, and thus

$$\mathrm{SO}(p, q) = \mathrm{O}(p, q + 1)^\sigma_0 \subset \mathrm{O}(p, q) \subset \mathrm{O}(p, q + 1)^\sigma = \mathrm{O}(p, q),$$

where $\mathrm{O}(p, q + 1)^\sigma_0$ denotes the identity component of $\mathrm{O}(p, q + 1)^\sigma$. Hence $M = G/H$ is a symmetric space.

The map

$$\begin{aligned}\varphi\colon \mathrm{O}(p, q + 1)/\mathrm{O}(p, q) &\to Q\\ a \cdot \mathrm{O}(p, q) &\mapsto a \cdot x_0,\end{aligned}$$

where $a \in \mathrm{O}(p, q + 1)$ and $x_0 = (0, \dots, 0, 1)$, is a diffeomorphism of M with the orbit of x_0 under $\mathrm{O}(p, q + 1)$, which is the hyperquadric Q, since $\mathrm{O}(p, q + 1)$ is the group of linear transformations leaving invariant the quadratic form

$$\mathrm{q}(x) = \sum_{i=1}^{p} \big(x^i\big)^2 - \sum_{j=p+1}^{p+q+1} \big(x^j\big)^2.$$

Problem 6.104 Find the involutive automorphism of $\mathrm{SL}(n,\mathbb{R})$ making the homogeneous space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ into an affine symmetric space. Write the decomposition involving the corresponding Lie algebras.

Solution The usual definition $\mathrm{O}(n)=\{A\in \mathrm{GL}(n,\mathbb{R}) : {}^tAA=I\}$ suggests us to take the involutive automorphism σ given by

$$\begin{aligned}\sigma\colon \mathrm{SL}(n,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R})\\ B &\mapsto {}^tB^{-1}\end{aligned}$$

for then, the closed subgroup of $\mathrm{SL}(n,\mathbb{R})$ of fixed points of σ,

$$\mathrm{SL}(n,\mathbb{R})^\sigma=\bigl\{B\in \mathrm{SL}(n,\mathbb{R}) : \sigma(B)=B\bigr\}=\mathrm{SO}(n),$$

and its identity component $\mathrm{SL}(n,\mathbb{R})_0^\sigma$, satisfy

$$\mathrm{SL}(n,\mathbb{R})_0^\sigma=\mathrm{SO}(n)=\mathrm{SL}(n,\mathbb{R})^\sigma,$$

so $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is an affine symmetric space.

As for the Lie algebras, the differential

$$\sigma_*\colon \mathfrak{sl}(n,\mathbb{R})\to\mathfrak{sl}(n,\mathbb{R}),\qquad \sigma_*X=-{}^tX,$$

induces the decomposition in (±1)-eigenspaces

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{R})&=\sigma_{*+}\oplus\sigma_{*-}=\bigl\{X\in\mathfrak{sl}(n,\mathbb{R}) : \sigma_*X=X\bigr\}\oplus\bigl\{X\in\mathfrak{sl}(n,\mathbb{R}) : \sigma_*X=-X\bigr\}\\ &=\mathfrak{o}(n)\oplus\mathfrak{sym}(n,\mathbb{R}),\end{aligned}$$

where $\mathfrak{sym}(n,\mathbb{R})$ denotes the subset of traceless symmetric matrices in $\mathfrak{gl}(n,\mathbb{R})$.

6.12 Spaces of Constant Curvature

Problem 6.105 Let λ be any positive real number and let M be the subset of $\mathbb{R}^n$ such that $x^n>0$. Prove, using Cartan's structure equations, that the Riemannian metric on M given by $g_{ij}(x)=(\lambda^2/(x^n)^2)\delta_{ij}$ has constant curvature $K=-1/\lambda^2$.

Hint Take as connection forms

$$\tilde{\omega}^i_j=\frac{1}{\lambda}\bigl(\delta_{ni}\tilde{\theta}^j-\delta_{nj}\tilde{\theta}^i\bigr),$$

where $\tilde{\theta}^i=\lambda\,\mathrm{d}x^i/x^n$, for $i,j=1,\dots,n$.

The relevant theory is developed, for instance, in Wolf [36].

Solution The frame

$$\sigma = \left(X_1 = \frac{x^n}{\lambda} \frac{\partial}{\partial x^1}, \dots, X_n = \frac{x^n}{\lambda} \frac{\partial}{\partial x^n} \right)$$

is an orthonormal moving frame, with dual moving coframe

$$\left(\tilde{\theta}^1 = \lambda \frac{\mathrm{d}x^1}{x^n}, \dots, \tilde{\theta}^n = \lambda \frac{\mathrm{d}x^n}{x^n} \right).$$

The forms $\widetilde{\omega}^i_j$ in the hint satisfy the conditions $\mathrm{d}\widetilde{\omega}^i = -\widetilde{\omega}^i_j \wedge \tilde{\theta}^j$ and $\widetilde{\omega}^i_j + \widetilde{\omega}^j_i = 0$. In fact,

$$\mathrm{d}\tilde{\theta}^i = -\frac{\lambda}{(x^n)^2} \mathrm{d}x^n \wedge \mathrm{d}x^i = -\sum_j \frac{1}{x^n} \big(\delta_{ni} \, \mathrm{d}x^j - \delta_{nj} \, \mathrm{d}x^i \big) \wedge \frac{\lambda}{x^n} \mathrm{d}x^j,$$

and the other condition is obvious. Thus, the forms $\widetilde{\omega}^i_j$ must be the connection forms relative to σ, since these are determined uniquely by the first structure equation. The second structure equation is

$$\begin{aligned}
\widetilde{\Omega}^i_j &= \mathrm{d}\widetilde{\omega}^i_j + \sum_k \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j \\
&= \frac{1}{\lambda} \big(\delta_{ni} \, \mathrm{d}\tilde{\theta}^j - \delta_{nj} \, \mathrm{d}\tilde{\theta}^i \big) + \sum_k \frac{1}{\lambda} \big(\delta_{ni} \tilde{\theta}^k - \delta_{nk} \tilde{\theta}^i \big) \wedge \frac{1}{a} \big(\delta_{nk} \tilde{\theta}^j - \delta_{nj} \tilde{\theta}^k \big) \\
&= -\frac{1}{\lambda^2} \left(\sum_k \delta_{nk} \delta_{nk} \right) \tilde{\theta}^i \wedge \tilde{\theta}^j = -\frac{1}{\lambda^2} \tilde{\theta}^i \wedge \tilde{\theta}^j.
\end{aligned}$$

Problem 6.106 Let (M, g) be a Riemannian manifold of constant curvature K. We define a metric $\widetilde{g}$ on $M \times M$ by

$$\widetilde{g}\big((X_1, Y_1), (X_2, Y_2) \big) = g(X_1, X_2) \circ \mathrm{pr}_1 + g(Y_1, Y_2) \circ \mathrm{pr}_2,$$

where pr_1 and pr_2 denote the projection map onto the first and the second factor, respectively. Is $(M \times M, \widetilde{g})$ a space of constant curvature?

Solution Let $(U, x^1, \dots, x^n)$ and $(V, x^{n+1}, \dots, x^{2n})$ be coordinate systems on the first and second factor of $M \times M$, respectively. Hence, $(U \times V, x^1, \dots, x^{2n})$ is a coordinate system of $M \times M$. If $g = (g_{ij}(x))$ on U and $g = (g_{i+n, j+n}(y))$ on V, then $\widetilde{g}$ has matrix

$$\big(\widetilde{g}_{AB}(x, y) \big) = \begin{pmatrix} g_{ij}(x) & 0 \\ 0 & g_{i+n\, j+n}(y) \end{pmatrix}, \quad A, B = 1, \dots, 2n.$$

Computing the Christoffel symbols

$$\widetilde{\Gamma}^A_{BC} = \frac{1}{2}\sum_{D=1}^{2n} \widetilde{g}^{AD}\left(\frac{\partial \widetilde{g}_{DB}}{\partial x^C} + \frac{\partial \widetilde{g}_{DC}}{\partial x^B} - \frac{\partial \widetilde{g}_{BC}}{\partial x^D}\right), \qquad A, B, C = 1, \dots, 2n,$$

it is easy to see that all of them vanish except perhaps $\widetilde{\Gamma}^i_{jk}(x, y) = \Gamma^i_{jk}(x)$ and $\widetilde{\Gamma}^{i+n}_{j+n,k+n}(x, y) = \Gamma^i_{jk}(y)$. Therefore, as one can easily compute, all the components of the curvature tensor field

$$\widetilde{R}^A_{BCD} = \frac{\partial \widetilde{\Gamma}^A_{BD}}{\partial x^C} - \frac{\partial \widetilde{\Gamma}^A_{BC}}{\partial x^D} + \sum_E \left(\widetilde{\Gamma}^A_{EC}\widetilde{\Gamma}^E_{BD} - \widetilde{\Gamma}^A_{ED}\widetilde{\Gamma}^E_{BC}\right)$$

vanish except perhaps

$$\widetilde{R}^i_{jkl}(x, y) = R^i_{jkl}(x), \qquad \widetilde{R}^{i+n}_{j+n,k+n,l+n}(x, y) = R^i_{jkl}(y).$$

Now, if $(M \times M, \widetilde{g})$ is a space of constant curvature, say $\widetilde{K}$, we have

$$\widetilde{R}^i_{jkl} = \widetilde{K}\left(\delta^i_k g_{jl} - g_{jk}\delta^i_l\right).$$

Hence, by the considerations above we deduce that, in particular,

$$0 = \widetilde{R}^{k+n}_{j,k+n,j} = \widetilde{K} g_{jj}.$$

Hence $(M \times M, \tilde{g})$ does not have constant curvature except when $\widetilde{K} = 0$.

Problem 6.107 Prove that a Riemannian manifold of constant curvature K is an Einstein manifold.

Solution Let g denote the Riemannian metric, $\mathbf{r}$ the Ricci tensor and (e_i), $i = 1, \dots, n$, a local orthonormal frame. Given any $X, Y \in \mathfrak{X}(M)$, one has

$$\begin{aligned}\mathbf{r}(X, Y) &= \sum_{i=1}^n g\big(R(e_i, Y)X, e_i\big) = \sum_{i=1}^n X^j Y^k g\big(R(e_i, e_k)e_j, e_i\big)\\ &= \sum_{i=1}^n X^j Y^k R_{ijik} = \sum_i X^j Y^k K(\delta_{ii}\delta_{jk} - \delta_{ik}\delta_{ji}) = K(n-1)g(X, Y).\end{aligned}$$

Problem 6.108 Prove that a 3-dimensional Einstein manifold (M, g) is a space of constant curvature.

Solution Suppose $\mathbf{r} = \lambda g$. Choose any plane $P \in T_pM$, and any orthonormal basis $\{e_1, e_2, e_3\}$ for T_pM such that $P = \langle e_1, e_2\rangle$. Denote by P_{ij} the plane spanned by e_i

and e_j for $i \neq j$, so that $P_{ij} = P_{ji}$. Then

$$\mathbf{r}(e_i, e_i) = \sum_{j \neq i} K(P_{ij}),$$

where $K(P_{ij})$ stands for the sectional curvature determined by P_{ij}. Thus we have

$$\mathbf{r}(e_1, e_1) + \mathbf{r}(e_2, e_2) - \mathbf{r}(e_3, e_3) = 2K(P).$$

As $\mathbf{r}(e_i, e_i) = \lambda g(e_i, e_i) = \lambda$, we obtain $K(P) = \frac{1}{2}\lambda$. As P is arbitrary, we conclude.

6.13 Gradient, Divergence, Codifferential, Curl, Laplacian, and Hodge Star Operator on Riemannian Manifolds

Problem 6.109 Let (M, g) be a Riemannian manifold, T_pM the tangent space at $p \in M$ and T_p^*M its dual space. The *musical isomorphisms* $\flat$ and $\sharp$ are defined (see Problem 6.40) by

$$\flat \colon T_pM \to T_p^*M, \qquad X \mapsto X^\flat, \qquad X^\flat(Y) = g(X, Y),$$

and its inverse $\omega \mapsto \omega^\sharp$, respectively. The *gradient* of a function $f \in C^\infty M$ is defined as

$$\operatorname{grad} f = (\mathrm{d}f)^\sharp.$$

(i) Prove that $g(\operatorname{grad} f, X) = Xf$, $X \in \mathfrak{X}(M)$.

Given local coordinates $\{x^i\}$:

(ii) Compute $(\partial/\partial x^i)^\flat$.
(iii) Calculate $(\mathrm{d}x^i)^\sharp$.
(iv) Write $\operatorname{grad} f$ in local coordinates.
(v) Verify that in the particular case of $\mathbb{R}^3$ equipped with the Euclidean metric, we recover the classical expression of $\operatorname{grad} f$.

Solution

(i) $g(\operatorname{grad} f, X) = g((\mathrm{d}f)^\sharp, X) = \mathrm{d}f(X) = Xf$.
(ii) Since

$$\left(\frac{\partial}{\partial x^i}\right)^\flat \left(\frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_k g_{ij} = g_{ik}\,\mathrm{d}x^k\left(\frac{\partial}{\partial x^j}\right),$$

we have

$$\left(\frac{\partial}{\partial x^i}\right)^\flat = \sum_k g_{ik}\,\mathrm{d}x^k.$$

(iii) From (ii) we have $\frac{\partial}{\partial x^i} = \sum_k g_{ik}(\mathrm{d}x^k)^\sharp$, since $\flat$ and $\sharp$ are inverse maps. So we obtain

$$\left(\mathrm{d}x^j\right)^\sharp = \sum_i g^{ji}\frac{\partial}{\partial x^i}.$$

(iv)

$$\operatorname{grad} f = (\mathrm{d}f)^\sharp = \left(\sum_i \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\right)^\sharp = \sum_i \frac{\partial f}{\partial x^i}\left(\mathrm{d}x^i\right)^\sharp = \sum_{i,j} g^{ji}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}.$$

(v) In this case,

$$\operatorname{grad} f = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}.$$

Problem 6.110 Let M be a C^∞ manifold equipped with a linear connection ∇. Let $\{X_1, \ldots, X_n\}$ be a basis of T_pM, and $\{\theta^1, \ldots, \theta^n\}$ its dual basis. The *divergence* of $Z \in \mathfrak{X}(M)$ is defined by

$$(\operatorname{div} Z)(p) = \sum_i \theta^i(\nabla_{X_i} Z).$$

(i) Prove that $(\operatorname{div}\, Z)(p)$ does not depend on the chosen basis.
(ii) Show that the divergence of a C^∞ field on $\mathbb{R}^3$ is the same as the definition given in Advanced Calculus.

Solution

(i) Given another basis $\{\widetilde{X}_j = \sum_i a_j^i X_i\}$, its dual basis is given by $\{\widetilde{\theta}^j = \sum_i b_i^j \theta^i\}$, so that $\sum_i b_i^j a_k^i = \sum a_i^j b_k^i = \delta_k^j$. Thus,

$$\begin{aligned}\widetilde{\theta}^j(\nabla_{\widetilde{X}_j} Z) &= \sum_{i,j,h} b_i^j \theta^i(\nabla_{a_j^h X_h} Z) = \sum_{i,j,h} a_j^h b_i^j \theta^i(\nabla_{X_h} Z)\\ &= \sum_{i,h} \delta_i^h \theta^i(\nabla_{X_h} Z) = \sum_i \theta^i(\nabla_{X_i} Z).\end{aligned}$$

(ii) Given the basis $\{(\partial/\partial x^i)_p\}$ of $T_p\mathbb{R}^3$, its dual basis is $\{(\mathrm{d}x^i)_p\}$, $i = 1, 2, 3$, and we have for $Z = \sum_i Z^i \partial/\partial x^i \in \mathfrak{X}(\mathbb{R}^3)$, since the Christoffel symbols of the flat connection on $\mathbb{R}^3$ vanish, that

$$\begin{aligned}(\operatorname{div} Z)(p) &= \sum_{i,j}\left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\left(Z^j\frac{\partial}{\partial x^j}\right)\right) = \sum_{i,j}\left(\mathrm{d}x^i\right)_p\frac{\partial Z^j}{\partial x^i}(p)\left.\frac{\partial}{\partial x^j}\right|_p\\ &= \sum_i \frac{\partial Z^i}{\partial x^i}(p).\end{aligned}$$

Problem 6.111 Let (M, g) be a Riemannian manifold, and let:

(a) ∇ be the Levi-Civita connection of g.
(b) $\operatorname{grad} f$ be the gradient of the function $f \in C^\infty M$.
(c) $\operatorname{div} X$ be the divergence (see the previous problem) of the vector field $X \in \mathfrak{X}(M)$. For a local field of orthonormal frames (e_i), $i = 1, \ldots, n$, we have $\operatorname{div} X = \sum_i g(\nabla_{e_i} X, e_i)$.
(d) H^f be the *Hessian* of $f \in C^\infty M$, defined as the second covariant differential $\nabla(\nabla f)$, that is,

$$H^f(X, Y) = XYf - (\nabla_X Y)f, \quad X, Y \in \mathfrak{X}(M).$$

(e) Δf be the *Laplacian* of $f \in C^\infty M$, defined by

$$\Delta f = \operatorname{div} \operatorname{grad} f.$$

Moreover, suppose $\dim M = 3$.

Prove the following formulas for $f, h \in C^\infty M$, $X, Y \in \mathfrak{X}(M)$:

1. $\operatorname{grad}(fh) = f \operatorname{grad} h + h \operatorname{grad} f$.
2. $\operatorname{div}(fX) = Xf + f \operatorname{div} X$.
3. $H^{fh} = fH^h + hH^f + \mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f$.
4. $\Delta(fh) = f\Delta h + h\Delta f + 2g(\operatorname{grad} f, \operatorname{grad} h)$.

Solution

1.

$$\begin{aligned} g(\operatorname{grad} fh, X) = X(fh) = (Xf)h + fXh &= g(\operatorname{grad} f, X)h + g(\operatorname{grad} h, X)f \\ &= g(h \operatorname{grad} f + f \operatorname{grad} h, X). \end{aligned}$$

2.

$$\begin{aligned} \operatorname{div}(fX) &= \sum_i g(\nabla_{e_i} fX, e_i) = \sum_i g\big((e_i f)X + f\nabla_{e_i} X, e_i\big) \\ &= \sum_i g(X, e_i)e_i f + f \sum_i g(\nabla_{e_i} X, e_i) = Xf + f \operatorname{div} X. \end{aligned}$$

3.

$$\begin{aligned} H^{fh}(X, Y) = XYfh - (\nabla_X Y)fh &= X\big((Yf)h + fYh\big) \\ &\quad - \big((\nabla_X Y)f\big)h - f(\nabla_X Y)h \\ &= (XYf)h + (Yf)(Xh) + (Xf)(Yh) + fXYh \\ &\quad - \big((\nabla_X Y)f\big)h - f(\nabla_X Y)h \\ &= \big(fH^h + hH^f + \mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f\big)(X, Y). \end{aligned}$$

4.

$$\begin{aligned}\Delta f h &= \operatorname{div}\operatorname{grad} f h = \operatorname{div}(f \operatorname{grad} h + h \operatorname{grad} f)\\ &= (\operatorname{grad} h) f + f \Delta h + (\operatorname{grad} f) h + h \Delta f\\ &= f \Delta h + h \Delta f + 2g(\operatorname{grad} f, \operatorname{grad} h).\end{aligned}$$

Problem 6.112 Consider on $\mathbb{R}^n$ the metric $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$ and the volume element $v = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

(i) Prove that given a form Ω_k of degree k there is only one form $\star\Omega_k$, of degree $n-k$, such that

$$(\star\Omega_k)(X_1, \dots, X_{n-k})v = \Omega_k \wedge X_1^\flat \wedge \cdots \wedge X_{n-k}^\flat.$$

(ii) The *Hodge star operator*

$$\star\colon \Lambda^k\mathbb{R}^n \to \Lambda^{n-k}\mathbb{R}^n$$

is defined by the previous formula. Prove that this operator satisfies the following equalities:

$$\star^2 = (-1)^{k(n-k)}, \qquad \star^{-1} = (-1)^{k(n-k)}\star, \qquad \Omega_k \wedge (\star\Theta_k) = \Theta_k \wedge (\star\Omega_k).$$

(iii) The codifferential $\delta\colon \Lambda^k\mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^n$ is defined by

$$\delta = (-1)^{n(k+1)+1} \star \mathrm{d} \star.$$

Prove that δ satisfies $\delta^2 = 0$.

(iv) The *Hodge–de Rham Laplacian*, or simply *Laplacian*, $\Delta\colon \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ is defined by

$$\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}.$$

Prove that if $f \in C^\infty\mathbb{R}^n$, then

$$\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}.$$

Solution

(i) We only have to prove the above properties for a basis of the exterior algebra. Let $\{X_1, \dots, X_n\}$ be an orthonormal basis of vector fields on $\mathbb{R}^n$ and $\{\theta^1, \dots, \theta^n\}$ its dual basis. Consider multi-indexes $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$. We have

$$\bigl\{\star\bigl(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\bigr)\bigr\}(X_{j_1}, \dots, X_{j_{n-k}})v = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}, \quad (\diamond)$$

which vanishes if $(j_1, \ldots, j_{n-k})$ is not the complement of $(i_1, \ldots, i_k)$ in $(1, 2, \ldots, n)$. Denoting by $(j_1, \ldots, j_{n-k})$ such ordered complement, we have

$$\star\big(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\big) = \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})\, \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}} \qquad (\diamond\diamond)$$

(where sgn denotes the sign of a permutation). In fact, from $(\diamond)$ above we deduce

$$\big\{\star\big(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\big)\big\}(X_{j_1}, \ldots, X_{j_{n-k}})v = \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})\, v.$$

(ii) From $(\diamond\diamond)$ above we have

$$\begin{aligned}
\star\big\{\star\big(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\big)\big\} &= \star\big\{\operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})\, \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}\big\} \\
&= \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \star \big(\theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}\big) \\
&= \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \\
&\quad \times \operatorname{sgn}(j_1, \ldots, j_{n-k}, i_1, \ldots, i_k)\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \\
&= (-1)^{i_1+\cdots+i_k-\frac{k(k+1)}{2}+j_1+\cdots+j_{n-k}-\frac{(n-k)(n-k+1)}{2}} \\
&\quad \times \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \\
&= (-1)^{k(n-k)}\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}.
\end{aligned}$$

From $\star^2 = (-1)^{k(n-k)}$ it follows that $\star = (-1)^{k(n-k)}\star^{-1}$, hence $\star^{-1} = (-1)^{k(n-k)}\star$.

Consider

$$\Omega_k = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}, \qquad \Theta_k = \theta^{j_1} \wedge \cdots \wedge \theta^{j_k}.$$

Then

$$\Omega_k \wedge (\star\Theta_k) = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}, \\ v & \text{if } \{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}. \end{cases}$$

Proceed similarly for $\Theta_k \wedge (\star\Omega_k)$.

(iii) $\delta^2 = \star\mathrm{d}\star\star\mathrm{d}\star = (-1)^{(k-1)(n-k+1)} \star \mathrm{d}^2\star = 0$, since $\mathrm{d}^2 = 0$.

(iv) Since $\delta f = 0$ for $f \in C^\infty\mathbb{R}^n$, we have $\Delta f = \delta\, \mathrm{d} f$, hence

$$\begin{aligned}
\Delta f = \delta\, \mathrm{d} f = - \star \mathrm{d} \star \mathrm{d} f &= - \star \mathrm{d} \star \sum_i \frac{\partial f}{\partial x^i}\, \mathrm{d}x^i \\
&= - \star \mathrm{d}(-1)^{i-1} \sum_i \frac{\partial f}{\partial x^i}\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n
\end{aligned}$$

$$= -\star(-1)^{i-1}\sum_{i,j}\frac{\partial^2 f}{\partial x^i \partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

$$= -\star\left(\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}\right)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = -\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}.$$

Problem 6.113 A differential 2-form F on $\mathbb{R}^4$ is said to be *autodual* if

$$\star F = F,$$

where $\star$ stands for the Hodge star operator. Prove that the curvature 2-form ($\star$) in Problem 5.29 is autodual.

Remark A curvature form F satisfying $\star F = F$ is called an *instanton*. The instantons described in Problem 5.29 are called *Belavin–Polyakov–Schwartz–Tyupkin instantons*.

Solution The Hodge star operator on the 2-forms on the Euclidean space $\mathbb{R}^4 \equiv \mathbb{H}$ is easily seen to be defined from

$$\begin{aligned}
&\star\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2\right) = \mathrm{d}x^3 \wedge \mathrm{d}x^4, &&\star\left(\mathrm{d}x^2 \wedge \mathrm{d}x^3\right) = \mathrm{d}x^1 \wedge \mathrm{d}x^4,\\
&\star\left(\mathrm{d}x^1 \wedge \mathrm{d}x^3\right) = -\mathrm{d}x^2 \wedge \mathrm{d}x^4, &&\star\left(\mathrm{d}x^2 \wedge \mathrm{d}x^4\right) = -\mathrm{d}x^1 \wedge \mathrm{d}x^3,\\
&\star\left(\mathrm{d}x^1 \wedge \mathrm{d}x^4\right) = \mathrm{d}x^2 \wedge \mathrm{d}x^3, &&\star\left(\mathrm{d}x^3 \wedge \mathrm{d}x^4\right) = \mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned}$$

Thus, the basis of autodual 2-forms is

$$\left\{\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4,\ \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4,\ \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3\right\}.$$

Now, due to the identification $\mathbb{R}^4 \equiv \mathbb{H}$, one has

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}\bar{x} &= \left(\mathrm{d}x^1 + \mathrm{d}x^2\,\mathbf{i} + \mathrm{d}x^3\,\mathbf{j} + \mathrm{d}x^4\,\mathbf{k}\right) \wedge \left(\mathrm{d}x^1 - \mathrm{d}x^2\,\mathbf{i} - \mathrm{d}x^3\,\mathbf{j} - \mathrm{d}x^4\,\mathbf{k}\right)\\
&= -2\left\{\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right)\mathbf{i} + \left(\mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4\right)\mathbf{j}\right.\\
&\quad \left. + \left(\mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3\right)\mathbf{k}\right\}.
\end{aligned}$$

Problem 6.114 Define on $\mathbb{R}^3$ equipped with the usual flat metric g:

(a) $\operatorname{div} X = \operatorname{div} X^\flat = -\delta X^\flat = \star\mathrm{d}\star X^\flat,\ X \in \mathfrak{X}(\mathbb{R}^3)$.
(b) $\operatorname{curl} X = (\star\mathrm{d}X^\flat)^\sharp$.

Prove the formulas:

1. $\operatorname{curl}\operatorname{grad} f = 0$.
2. $\operatorname{div}\operatorname{curl} X = 0$.
3. $\Delta\omega = -(\operatorname{grad}\operatorname{div}\omega^\sharp + \operatorname{curl}\operatorname{curl}\omega^\sharp)^\flat$, $\omega \in \Lambda^1\mathbb{R}^3$.

4. $\operatorname{curl}(fX) = (\operatorname{grad} f) \times X + f \operatorname{curl} X$, where $\times$ denotes the usual vector product in $\mathbb{R}^3$.
5. $\operatorname{div}(fX) = (\operatorname{grad} f) \cdot X + f \operatorname{div} X$.
6. $$\big(\star(\operatorname{curl} X)^\flat\big)(Y, Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y).$$
7. Prove that $\operatorname{curl} X$ coincides with its classical expression and then
$$\operatorname{div}(X \times Y) = X \cdot \operatorname{curl} Y + (\operatorname{curl} X) \cdot Y,$$
where the dot denotes the usual scalar product in $\mathbb{R}^3$.

Solution

1. $$\operatorname{curl} \operatorname{grad} f = \operatorname{curl}(\mathrm{d}f)^\sharp = \big(\star\mathrm{d}\big((\mathrm{d}f)^\sharp\big)^\flat\big)^\sharp = (\star\mathrm{d}\,\mathrm{d}f)^\sharp = 0.$$
2. $$\begin{aligned}\operatorname{div} \operatorname{curl} X &= \operatorname{div}\big(\star\mathrm{d}X^\flat\big)^\sharp = \operatorname{div}\big(\big(\star\mathrm{d}X^\flat\big)^\sharp\big)^\flat = -\delta \star \mathrm{d}X^\flat = \star\mathrm{d}\star\star\mathrm{d}X^\flat \\ &= (-1)^{2(3-2)} \star \mathrm{d}\,\mathrm{d}X^\flat = 0.\end{aligned}$$
3. $$\begin{aligned}\Delta\omega &= (\mathrm{d}\delta + \delta\mathrm{d})\omega = \mathrm{d}(-\operatorname{div}\omega) + \delta\,\mathrm{d}\omega = -\mathrm{d}\operatorname{div}\omega^\sharp - \star\mathrm{d}\star\,\mathrm{d}\omega \\ &= -\big(\big(\mathrm{d}\operatorname{div}\omega^\sharp\big)^\sharp\big)^\flat - \star\mathrm{d}\big(\big(\star\mathrm{d}\big(\omega^\sharp\big)^\flat\big)^\sharp\big)^\flat = -\big(\operatorname{grad}\operatorname{div}\,\omega^\sharp\big)^\flat - \star\mathrm{d}\big(\operatorname{curl}\,\omega^\sharp\big)^\flat \\ &= -\big(\operatorname{grad}\operatorname{div}\omega^\sharp + \operatorname{curl}\operatorname{curl}\omega^\sharp\big)^\flat.\end{aligned}$$
4. $$\begin{aligned}\operatorname{curl}(fX) &= \big(\star\mathrm{d}\big(fX^\flat\big)\big)^\sharp = \big(\star\big(\mathrm{d}f \wedge X^\flat + f\,\mathrm{d}X^\flat\big)\big)^\sharp \\ &= \big(\star\big(\mathrm{d}f \wedge X^\flat\big)\big)^\sharp + f \operatorname{curl} X \\ &= (\operatorname{grad} f) \times X + f \operatorname{curl}\, X,\end{aligned}$$
since
$$\begin{aligned}\big(\star\big(\mathrm{d}f \wedge X^\flat\big)\big)^\sharp &= \left(\star\left(\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \wedge \big(X_j\,\mathrm{d}x^j\big)\right)\right)^\sharp \\ &= \left(\star\left\{\left(\frac{\partial f}{\partial x^1}X_2 - \frac{\partial f}{\partial x^2}X_1\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots\right\}\right)^\sharp \\ &= \left(\left\{\left(\frac{\partial f}{\partial x^1}X_2 - \frac{\partial f}{\partial x^2}X_1\right)\mathrm{d}x^3 + \cdots\right\}\right)^\sharp\end{aligned}$$

$$= \left(\frac{\partial f}{\partial x^2}X_3 - \frac{\partial f}{\partial x^3}X_2\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial f}{\partial x^3}X_1 - \frac{\partial f}{\partial x^1}X_3\right)\frac{\partial}{\partial x^2}$$
$$+ \left(\frac{\partial f}{\partial x^1}X_2 - \frac{\partial f}{\partial x^2}X_1\right)\frac{\partial}{\partial x_3} = \mathrm{grad}\, f \times X.$$

5.

$$\begin{aligned}\mathrm{div}(fX) &= \mathrm{div}(fX)^\flat = -\delta(fX)^\flat = \star\mathrm{d}\star(fX)^\flat = \star\mathrm{d}\big(f\big(\star X^\flat\big)\big)\\ &= \star\big(\mathrm{d}f \wedge \big(\star X^\flat\big) + f\,\mathrm{d}\big(\star X^\flat\big)\big) = \star\big(X^\flat \wedge (\star \mathrm{d}f)\big) + f\ \star\mathrm{d}\star X^\flat\\ &= \star\big((\star\mathrm{d}f)\wedge X^\flat\big) + f\,\mathrm{div}\ X.\end{aligned}$$

Now,

$$\begin{aligned}\star\big((\star\mathrm{d}f)\wedge X^\flat\big) &= \star\left(\left(\star\sum_i \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\right)\wedge\left(\sum_j X_j\,\mathrm{d}x^j\right)\right)\\ &= \star\left(\left(\frac{\partial f}{\partial x^1}\,\mathrm{d}x^2\wedge\mathrm{d}x^3 - \frac{\partial f}{\partial x^2}\,\mathrm{d}x^1\wedge\mathrm{d}x^3 + \frac{\partial f}{\partial x^3}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\right)\right.\\ &\qquad \left.\wedge\left(\sum_j X_j\mathrm{d}x^j\right)\right)\\ &= \star\left(\left(\sum_i \frac{\partial f}{\partial x^i}X_i\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\right) = \mathrm{grad}\ f\cdot X.\end{aligned}$$

6.

$$\begin{aligned}\big(\star(\mathrm{curl}\ X)^\flat\big)(Y,Z) &= \big(\star\star\,\mathrm{d}X^\flat\big)(Y,Z) = \big(\mathrm{d}X^\flat\big)(Y,Z)\\ &= YX^\flat(Z) - ZX^\flat(Y) - X^\flat\big([Y,Z]\big)\\ &= Yg(X,Z) - Zg(X,Y) - g(X,\nabla_Y Z) + g(X,\nabla_Z Y)\\ &= g(\nabla_Y X, Z) - g(\nabla_Z X, Y).\end{aligned}$$

7.

$$\begin{aligned}\mathrm{curl}\ X &= \mathrm{curl}\left(\sum_i X_i\frac{\partial}{\partial x^i}\right) = \big(\star\,\mathrm{d}\big(X_i\,\mathrm{d}x^i\big)\big)^\sharp\\ &= \left(\star\left\{\left(\frac{\partial X_1}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial X_1}{\partial x^3}\,\mathrm{d}x^3\right)\wedge\mathrm{d}x^1 + \cdots\right\}\right)^\sharp\\ &= \left(\star\left\{\left(-\frac{\partial X_1}{\partial x^2} + \frac{\partial X_2}{\partial x^1}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2 + \cdots\right\}\right)^\sharp\end{aligned}$$

$$= \left(\left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right) \mathrm{d}x^3 + \cdots\right)^\sharp$$

$$= \left(\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3}\right)\frac{\partial}{\partial x^1} - \left(\frac{\partial X_3}{\partial x^1} - \frac{\partial X_1}{\partial x^3}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$

From this, the formula

$$\operatorname{div}(X \times Y) = X \cdot \operatorname{curl} Y + (\operatorname{curl} X) \cdot Y$$

follows.

Problem 6.115 Let g and $\widetilde{g}$ be conformally equivalent metrics on the C^∞ n-manifold M, that is, such that $\widetilde{g} = \mathrm{e}^{2f} g$, $f \in C^\infty M$. Find the relation between:

(i) $\widetilde{\nabla}_X Y$ and $\nabla_X Y$, where $\widetilde{\nabla}$ and ∇ denote, respectively, the Levi-Civita connections of $\widetilde{g}$ and g, and $X, Y \in \mathfrak{X}(M)$.
(ii) $\operatorname{div}_{\widetilde{g}} X$ and $\operatorname{div}_g X$, $X \in \mathfrak{X}(M)$.

Solution

(i) The Levi-Civita connection of $\widetilde{g}$ is given by the Koszul formula in Theorem 6.4. Thus,

$$\begin{aligned} 2\mathrm{e}^{2f} g(\widetilde{\nabla}_X Y, Z) &= X\mathrm{e}^{2f} g(Y,Z) + Y\mathrm{e}^{2f} g(Z,X) - Z\mathrm{e}^{2f} g(X,Y) \\ &\quad + \mathrm{e}^{2f} g\big([X,Y],Z\big) - \mathrm{e}^{2f} g\big([Y,Z],X\big) + \mathrm{e}^{2f} g\big([Z,X],Y\big) \\ &= 2\mathrm{e}^{2f}\big\{g(\nabla_X Y, Z) + (Xf)g(Y,Z) + (Yf)g(Z,X) \\ &\quad - (Zf)g(X,Y)\big\}. \end{aligned}$$

Hence

$$\widetilde{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - g(X,Y)\operatorname{grad} f.$$

(ii) Let (E_i) be a local g-orthonormal frame. Then the frame $(\mathrm{e}^{-f} E_i)$ is a $\widetilde{g}$-orthonormal local frame and we have locally, by definition of divergence and by (i):

$$\begin{aligned} \operatorname{div}_{\widetilde{g}} X &= \sum_i \widetilde{g}\big(\widetilde{\nabla}_{\mathrm{e}^{-f} E_i} X, \mathrm{e}^{-f} E_i\big) = \sum_i \mathrm{e}^{-2f}\widetilde{g}(\widetilde{\nabla}_{E_i} X, E_i) = \sum_i g(\widetilde{\nabla}_{E_i} X, E_i) \\ &= \sum_i \big(g(\nabla_{E_i} X, E_i) + (E_i f)g(X, E_i) + (Xf)g(E_i, E_i) \\ &\quad - (E_i f)g(E_i, X)\big) \\ &= \operatorname{div}_g X + nXf. \end{aligned}$$

Problem 6.116 Prove that the *Hodge–de Rham Laplacian* $\Delta = \delta d + \mathrm{d}\delta$ and the Hodge star operator $\star$ on an oriented Riemannian manifold commute:

$$\Delta\star = \star\Delta.$$

Remark Recall that the codifferential δ, defined as the opposite of the divergence, satisfies

$$\delta\omega = (-1)^{n(k+1)+1}\star \mathrm{d}\star\omega, \qquad \omega \in \Lambda^k M^n.$$

Solution Suppose $\dim M = n$ and $\omega \in \Lambda^k M$, then

$$\begin{aligned}\Delta\star\omega &= (\delta\mathrm{d}+\mathrm{d}\delta)\star\omega = (-1)^{n(n-k+2)+1}\star\mathrm{d}\star\mathrm{d}\star\omega + (-1)^{n(n-k+1)+1}\mathrm{d}\star\mathrm{d}\star\star\omega\\ &= (-1)^{n(n-k)+1}\star\mathrm{d}\star\mathrm{d}\star\omega + (-1)^{n(n-k+1)+1+k(n-k)}\mathrm{d}\star\mathrm{d}\omega,\\ \star\Delta\omega &= \star(\delta\mathrm{d}+\mathrm{d}\delta)\omega = \star(-1)^{n(k+2)+1}\star\mathrm{d}\star\mathrm{d}\omega + (-1)^{n(k+1)+1}\star\mathrm{d}\star\mathrm{d}\star\omega\\ &= (-1)^{nk+1+(n-k)k}\mathrm{d}\star\mathrm{d}\omega + (-1)^{n(k+1)+1}\star\mathrm{d}\star\mathrm{d}\star\omega.\end{aligned}$$

Now, $(-1)^{n(n-k)+1} = (-1)^{n^2-nk+1} = (-1)^{n+nk+1} = (-1)^{n(k+1)+1}$, and on the other hand

$$(-1)^{n(n-k+1)+1+k(n-k)} = (-1)^{1-k} = (-1)^{nk+1+nk-k}.$$

Hence $\Delta\star = \star\Delta$.

Problem 6.117 Prove that a parallel differential form on a Riemannian manifold (M, g) is harmonic.

Solution Let $\alpha \in \Lambda^* M$ be parallel, that is, if ∇ stands for the Levi-Civita connection of g, we have $\nabla\alpha = 0$.

Therefore, α is closed. In fact, if $\alpha \in \Lambda^r M$, one has in general

$$\mathrm{d}\alpha(X_0,\dots,X_r) = \sum_{j=0}^{r}(-1)^j(\nabla_{X_j}\alpha)(X_0,\dots,\widehat{X}_j,\dots,X_r), \qquad X_j \in \mathfrak{X}(M). \tag{$\star$}$$

Moreover, α is coclosed ($\delta\alpha = 0$). In fact, we have in general

$$\begin{aligned}(\delta\alpha)_p(v_1,\dots,v_{r-1}) &= -(\operatorname{div}\alpha)_p(v_1,\dots,v_{r-1}) \quad \text{(by Definition 6.11)}\\ &= -\sum_i(\nabla_{e_i}\alpha)(e_i,v_1,\dots,v_{r-1}),\end{aligned} \tag{$\star\star$}$$

where $\{e_i\}$ is an orthonormal basis for T_pM, and $v_1,\dots,v_{r-1} \in T_pM$. Since $\Delta\alpha = (\mathrm{d}\delta + \delta\mathrm{d})\alpha$, we conclude.

Problem 6.118 If the Riemannian n-manifold M is compact, prove:

(i) The codifferential δ is adjoint of the differential d with respect to the inner product of integration, that is,

$$\int_M \langle \delta\alpha, \beta\rangle v = \int_M \langle \alpha, \mathrm{d}\beta\rangle v, \quad \alpha, \beta \in \Lambda^r M,\ r \in \{0, \dots, n\},$$

where v denotes the volume form on the Riemannian manifold.

(ii) The Laplacian $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ on M is self-adjoint with respect to the inner product of integration, that is,

$$\int_M \langle \Delta\alpha, \beta\rangle v = \int_M \langle \alpha, \Delta\beta\rangle v, \quad \alpha, \beta \in \Lambda^r M,\ r \in \{0, \dots, n\}.$$

Solution

(i) We have

$$\begin{aligned} 0 &= \int_M \mathrm{d}(\alpha \wedge \star\beta) && \text{(by Stokes' Theorem)} \\ &= \int_M \big(\mathrm{d}\alpha \wedge \star\beta + (-1)^r \alpha \wedge \mathrm{d}\star\beta\big) \\ &= \int_M \mathrm{d}\alpha \wedge \star\beta - \int_M \alpha \wedge \star\delta\beta && \text{(by definition of } \delta). \end{aligned}$$

By the definition of inner product of integration, we conclude.

(ii) By (i) above,

$$\begin{aligned} \int_M \langle \Delta\alpha, \beta\rangle v &= \int_M \big\langle (\mathrm{d}\delta + \delta\mathrm{d})\alpha, \beta\big\rangle v = \int_M \big(\langle \delta\alpha, \delta\beta\rangle + \langle d\alpha, \mathrm{d}\beta\rangle\big) v \\ &= \int_M \big(\langle \alpha, \mathrm{d}\delta\beta\rangle + \langle \alpha, \delta\mathrm{d}\beta\rangle\big) v = \int_M \langle \alpha, \Delta\beta\rangle v. \end{aligned}$$

Problem 6.119 Prove that if a compact Riemannian n-manifold M admits a metric of constant positive curvature, then

$$H^r_{dR}(M, \mathbb{R}) = 0, \quad r = 1, \dots, n-1.$$

Hint Use: (i) Weitzenböck's formula for the Laplacian on a Riemannian manifold $(M, \langle\cdot\,,\cdot\rangle)$ of constant sectional curvature c; (ii) the formula

$$\int_M \Delta f v = 0, \quad f \in C^\infty M,$$

which follows from (ii) in Problem 6.118, taking $\alpha = f$ and β to be a constant function.

The relevant theory is developed, for instance, in Poor [28, Chap. 4].

Solution Integrating the two members of the Weitzenböck formula, we have in general:

$$\int_M \langle \Delta\alpha, \alpha\rangle v = \int_M \left(\frac{1}{2}\Delta|\alpha|^2 + |\nabla\alpha|^2 + r(n-r)c|\alpha|^2\right)v, \quad \alpha \in \Lambda^r M,$$

where v stands for the volume form on (M, g).

Let α be the harmonic representative of a class in $H^r_{dR}(M, \mathbb{R})$. Then $\Delta\alpha = 0$. Moreover, by (ii) in the hint, $\int_M \Delta|\alpha|^2 = 0$. Hence

$$0 = \int_M \left(|\nabla\alpha|^2 + r(n-r)c|\alpha|^2\right)v.$$

If $r \neq 0, n$, from $c > 0$, it follows that $\alpha = 0$. Thus $H^r_{dR}(M, \mathbb{R}) = 0$, $r = 1, \dots, n-1$.

Problem 6.120 Let α and β be n-forms on a compact oriented Riemannian n-manifold M such that

$$\int_M \alpha = \int_M \beta.$$

Prove that α and β differ by an exact form.

Hint Use:

(i) Hodge's Decomposition Theorem 6.28.
(ii) Stokes' Theorem 3.6.

Solution Denote here the degree r of a differential form by the subindex r. By Hodge's Decomposition Theorem, each r-form ω_r over such a manifold is decomposed in a unique way as

$$\omega_r = \mathrm{d}\omega_{r-1} + \delta\omega_{r+1} + \theta_r,$$

where θ_r is harmonic. In our case, the decomposition reduces to

$$\alpha - \beta = \mathrm{d}\omega_{n-1} + \theta_n.$$

Applying Stokes' Theorem, we have

$$0 = \int_M \alpha - \beta = \int_M \mathrm{d}\omega_{n-1} + \int_M \theta_n = \int_M \theta_n.$$

As the n-form θ_n is harmonic and each cohomology class has a unique harmonic representative, from $\int_M \theta_n = 0$ it follows that $\theta_n = 0$. Thus $\alpha - \beta = \mathrm{d}\omega_{n-1}$.

Problem 6.121 Let $(M, \langle\cdot\,,\cdot\rangle)$ be a compact oriented Riemannian manifold *without boundary*. Then $\lambda \in \mathbb{R}$ is called an *eigenvalue of the Laplacian* if there exists $f \in C^\infty M$, not identically zero, such that

$$\Delta f = \lambda f.$$

In this case, f is called an *eigenfunction* corresponding to λ.

(i) Prove that 0 is an eigenvalue of Δ, and that other values are strictly negative.
(ii) A function f is said to be *harmonic* if $\Delta f = 0$. Prove that the only harmonic functions are the constants.
(iii) If f and h are eigenfunctions corresponding to distinct eigenvalues, show that

$$\int_M fhv = 0,$$

where v stands for the Riemannian volume element.

Hint (to (i)) Use the first Green identity (see, for instance, Strauss [33, p. 176]) for manifolds without boundary,

$$\int_M \bigl(f\,\Delta h + \langle \operatorname{grad} f, \operatorname{grad} h\rangle\bigr)v = 0. \tag{$\star$}$$

The relevant theory is developed, for instance, in Lee [21].

Solution

(i) Taking $f = h$ in ($\star$) above, we have

$$\int_M f\,\Delta f = -\int_M |\operatorname{grad} f|^2 v. \tag{$\star\star$}$$

If f is an eigenfunction corresponding to a nonzero eigenvalue λ, then it follows that

$$\lambda \int_M f^2 v = -\int_M |\operatorname{grad} f|^2 v.$$

Compactness of M then implies $\lambda \leqslant 0$. If f is a constant function then $|\operatorname{grad} f|^2 = 0$ and $\lambda = 0$.
(ii) Also from ($\star\star$) it follows that if f is harmonic, it is necessarily a constant function.
(iii) Let λ and μ be the eigenvalues of f and h, respectively. From Green's identity ($\star$), we get

$$0 = \int_M (f\,\Delta h - h\,\Delta f)v = (\mu - \lambda)\int_M fhv.$$

As $\lambda \neq \mu$, the conclusion follows.

Problem 6.122 Let (M, g) be an n-dimensional Riemannian manifold, let ∇ be the Levi-Civita connection, and let R be the curvature tensor. Denote by a dot the *Clifford multiplication on forms* (see Definition 6.12).

(i) Prove Leibniz's rule for ∇:

$$\nabla_X(\alpha \cdot \beta) = (\nabla_X \alpha) \cdot \beta + \alpha \cdot \nabla_X \beta, \quad \alpha \in \Lambda^r M, \ \beta \in \Lambda^s M.$$

(ii) Prove Leibniz's rule for R:

$$R(X, Y)(\alpha \cdot \beta) = \big(R(X, Y)\alpha\big) \cdot \beta + \alpha \cdot R(X, Y)\beta, \quad \alpha \in \Lambda^r M, \ \beta \in \Lambda^s M.$$

(iii) Let $\{e_i\}$, $i = 1, \ldots, n$, be any local basis of vector fields and $\{\theta^i\}$ its metrically dual basis of 1-forms. Then prove that the square of the *Dirac operator on forms* D (see Definition 6.13), can be written as

$$D^2\omega = \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i, e_j}\omega, \quad \omega \in \Lambda^r M.$$

(iv) Prove the *Weitzenböck formula* for the square of the Dirac operator on forms, that is, that D^2 may be written in terms of the rough Laplacian $\nabla^*\nabla$ as

$$D^2\omega = \nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot R(e_i, e_j)\omega, \quad \omega \in \Lambda^r M.$$

(v) Prove that

$$D\omega = \mathrm{d}\omega + \delta\omega, \quad \omega \in \Lambda^r M,$$

that is, the Dirac operator on forms is the sum of the exterior differential and the codifferential.

Hint (to (i)) Prove first that

$$\nabla_X(\iota_{\alpha^\sharp}\beta) = \iota_{\nabla_X \alpha^\sharp}\beta + \iota_{\alpha^\sharp}(\nabla_X \beta), \quad \alpha \in \Lambda^1 M, \ \beta \in \Lambda^r M.$$

Hint (to (ii)) Use (i) above.

Hint (to (iii)) First consider that

$$\nabla^2_{e_i, e_j} = \nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i} e_j}$$

is tensorial in both e_i and e_j, so the right-hand side in (iii) is invariantly defined. Then apply (i), and that

$$\nabla_{e_i}\theta^j = \sum_{k=1}^{n} g^{-1}\big(\nabla_{e_i}\theta^j, \theta^k\big)\theta^k = \sum_{k=1}^{n} g(\nabla_{e_i} e_j, e_k)\theta^k.$$

Hint (to (iv)) First note that everything is invariant. Then use (iii), the properties

$$\theta^i \cdot \theta^i = -1, \qquad \theta^i \cdot \theta^j = -\theta^j \cdot \theta^i,$$

and recall that the rough Laplacian is defined by $\nabla^*\nabla = -\operatorname{tr} \nabla^2$, that is,

$$\nabla^*\nabla\alpha = \sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i}\alpha - \nabla_{\nabla_{e_i}e_i}\alpha), \quad \alpha \in \Lambda^r M.$$

Hint (to (v)) Recall that the differential exterior and the covariant derivative of a form $\alpha \in \Lambda^r M$ are related (see Problem 5.44 or formula (7.6)) by

$$\mathrm{d}\alpha(X_0, \ldots, X_r) = \sum_{j=0}^{r}(-1)^j(\nabla_{X_j}\alpha)(X_0, \ldots, \widehat{X}_j, \ldots, X_r), \quad X_j \in \mathfrak{X}(M). \tag{$\star$}$$

The relevant theory is developed in Petersen [27, 7.4].

Solution

(i) Suppose first as in the hint that $\alpha \in \Lambda^1 M$ and $\beta \in \Lambda^r M$. Then we have that

$$\begin{aligned}
&\big(\nabla_X(\iota_{\alpha^\sharp}\beta)\big)(Y_1, \ldots, Y_{r-1}) \\
&= \nabla_X\big((\iota_{\alpha^\sharp}\beta)(Y_1, \ldots, Y_{r-1})\big) - \sum_{k=1}^{r-1}(\iota_{\alpha^\sharp}\beta)(Y_1, \ldots, \nabla_X Y_k, \ldots, Y_{r-1}) \\
&= \nabla_X\big(\beta\big(\alpha^\sharp, Y_1, \ldots, Y_{r-1}\big)\big) - \sum_{k=1}^{r-1}\beta\big(\alpha^\sharp, Y_1, \ldots, \nabla_X Y_k, \ldots, Y_{r-1}\big) \\
&= (\nabla_X\beta)\big(\alpha^\sharp, Y_1, \ldots, Y_{r-1}\big) + \beta\big(\nabla_X\alpha^\sharp, Y_1, \ldots, Y_{r-1}\big) \\
&\quad + \sum_{k=1}^{r-1}\beta\big(\alpha^\sharp, Y_1, \ldots, \nabla_X Y_k, \ldots, Y_{r-1}\big) \\
&\quad - \sum_{k=1}^{r-1}\beta\big(\alpha^\sharp, Y_1, \ldots, \nabla_X Y_k, \ldots, Y_{r-1}\big) \\
&= \big(\iota_{\alpha^\sharp}(\nabla_X\beta) + \iota_{\nabla_X\alpha^\sharp}\beta\big)(Y_1, \ldots, Y_{r-1}).
\end{aligned}$$

Also

$$(\nabla_X\alpha)^\sharp = \nabla_X\big(\alpha^\sharp\big). \tag{$\diamond$}$$

Indeed, by the Koszul formula for the Levi-Civita connection in Theorem 6.4,

$$2g\big(\nabla_X\big(\alpha^\sharp\big), Y\big) = Xg\big(\alpha^\sharp, Y\big) + \alpha^\sharp g(Y, X) - Yg\big(X, \alpha^\sharp\big)$$

$$
\begin{aligned}
&\quad + g\big(\big[X, \alpha^\sharp\big], Y\big) - g\big(\big[\alpha^\sharp, Y\big], X\big) + g\big([Y, X], \alpha^\sharp\big), \\
2g\big((\nabla_X \alpha)^\sharp, Y\big) &= 2(\nabla_X \alpha)(Y) = 2\nabla_X\big(\alpha(Y)\big) - 2\alpha(\nabla_X Y) \\
&= 2Xg\big(\alpha^\sharp, Y\big) - 2g\big(\nabla_X Y, \alpha^\sharp\big) \\
&= 2Xg\big(\alpha^\sharp, Y\big) - \big[Xg\big(Y, \alpha^\sharp\big) + Yg\big(\alpha^\sharp, X\big) - \alpha^\sharp g(X, Y) \\
&\quad + g\big([X, Y], \alpha^\sharp\big) - g\big(\big[Y, \alpha^\sharp\big], X\big) + g\big(\big[\alpha^\sharp, X\big], Y\big)\big].
\end{aligned}
$$

So (i) easily follows from the definition of Clifford multiplication, as

$$
\begin{aligned}
\nabla_X(\alpha \cdot \beta) &= \nabla_X(\alpha \wedge \beta - \iota_{\alpha^\sharp}\beta) \\
&= (\nabla_X \alpha) \wedge \beta + \alpha \wedge (\nabla_X \beta) - \iota_{\alpha^\sharp}(\nabla_X \beta) - \iota_{\nabla_X \alpha^\sharp}\beta \\
&= (\nabla_X \alpha) \cdot \beta + \alpha \cdot (\nabla_X \beta).
\end{aligned}
$$

The extension to any pair of forms is easy, due to Definition 6.12.

(ii) It follows from (i) that

$$
\begin{aligned}
R(X, Y)(\alpha \cdot \beta) &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(\alpha \cdot \beta) \\
&= \nabla_X\big((\nabla_Y \alpha) \cdot \beta + \alpha \cdot (\nabla_Y \beta)\big) - \nabla_Y\big((\nabla_X \alpha) \cdot \beta + \alpha \cdot (\nabla_X \beta)\big) \\
&\quad - (\nabla_{[X,Y]}\alpha) \cdot \beta - \alpha \cdot (\nabla_{[X,Y]}\beta) \\
&= (\nabla_X \nabla_Y \alpha) \cdot \beta + (\nabla_Y \alpha) \cdot (\nabla_X \beta) + (\nabla_X \alpha) \cdot (\nabla_Y \beta) \\
&\quad + \alpha \cdot \nabla_X \nabla_Y \beta \\
&\quad - (\nabla_Y \nabla_X \alpha) \cdot \beta - (\nabla_X \alpha) \cdot (\nabla_Y \beta) \\
&\quad - (\nabla_Y \alpha) \cdot (\nabla_X \beta) - \alpha \cdot \nabla_Y \nabla_X \beta \\
&\quad - (\nabla_{[X,Y]}\alpha) \cdot \beta - \alpha \cdot (\nabla_{[X,Y]}\beta) \\
&= \big(R(X, Y)\alpha\big) \cdot \beta + \alpha \cdot R(X, Y)\beta.
\end{aligned}
$$

(iii) Using invariance, we need only to prove the formula at a point $p \in M$, where the frame is assumed to be orthonormal; and normal, that is, we have

$$(\nabla e_i)_p = 0 \quad \text{hence} \quad \big(\nabla \theta^i\big)_p = 0.$$

We can then compute at p,

$$
\begin{aligned}
D^2\omega &= \sum_{i,j=1}^{n} \theta^i \cdot \big(\nabla_{e_i}\big(\theta^j \cdot \nabla_{e_j}\omega\big)\big) \\
&= \sum_{i,j=1}^{n} \big(\theta^i \cdot \big(\nabla_{e_i}\theta^j\big) \cdot \nabla_{e_j}\omega + \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega\big) \quad \text{(by (i) above)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,j,k=1}^{n} \theta^i \cdot g^{-1}\big(\nabla_{e_i}\theta^j, \theta^k\big)\theta^k \cdot \nabla_{e_j}\omega + \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \\
&= \sum_{i,j,k=1}^{n} \theta^i \cdot g(\nabla_{e_i}e_j, e_k)\theta^k \cdot \nabla_{e_j}\omega \\
&\quad + \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \quad \text{(by ($\diamond$) above)} \\
&= - \sum_{i,j,k=1}^{n} \theta^i \cdot g(e_j, \nabla_{e_i}e_k)\theta^k \cdot \nabla_{e_j}\omega \\
&\quad + \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \quad \text{($\{e_i\}$ orthon.)} \\
&= - \sum_{i,j,k=1}^{n} \theta^i \cdot \theta^k \cdot \nabla_{g(e_j,\nabla_{e_i}e_k)e_j}\omega + \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \\
&= - \sum_{i,k=1}^{n} \theta^i \cdot \theta^k \cdot \nabla_{\nabla_{e_i}e_k}\omega + \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega \\
&= \sum_{i,j=1}^{n} \big(-\theta^i \cdot \theta^j \cdot \nabla_{\nabla_{e_i}e_j}\omega + \theta^i \cdot \theta^j \cdot \nabla_{e_i}\nabla_{e_j}\omega\big) \\
&= \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega.
\end{aligned}
$$

(iv) Due to invariance, we can choose a frame orthonormal and normal at $p \in M$, and then compute at p,

$$
\begin{aligned}
D^2\omega &= \sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega \quad \text{(by (iii) above)} \\
&= -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}\omega + \sum_{i \neq j} \theta^i \cdot \theta^j \cdot \nabla^2_{e_i,e_j}\omega \quad \text{(as $\theta^i \cdot \theta^i = -1$)} \\
&= -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}\omega + \sum_{i<j} \theta^i \cdot \theta^j \cdot \big(\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i}\big)\omega \quad \text{(as $\theta^i \cdot \theta^j = -\theta^i \cdot \theta^j$)} \\
&= -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}\omega + \sum_{i<j} \theta^i \cdot \theta^j \cdot R(e_i, e_j)\omega \quad \text{(by Ricci identity)}
\end{aligned}
$$

$$= -\sum_{i=1}^{n} \nabla^2_{e_i,e_i}\omega + \frac{1}{2}\sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot R(e_i, e_j)\omega$$

$$= \nabla^*\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} \theta^i \cdot \theta^j \cdot R(e_i, e_j)\omega \quad (\text{since } \nabla^*\nabla = -\operatorname{tr}\nabla^2).$$

(v) With the previous notations we have that

$$D\omega = \sum_{i,j=1}^{n} \theta^i \cdot \nabla_{e_i}\omega = \sum_{i=1}^{n}\bigl(\theta^i \wedge \nabla_{e_i}\omega - \iota_{e_i}(\nabla_{e_i}\omega)\bigr)$$

$$= \sum_{i=1}^{n}\bigl(\theta^i \wedge \nabla_{e_i}\omega - (\nabla_{e_i}\omega)(e_i, \cdot, \dots, \cdot)\bigr)$$

$$= \sum_{i=1}^{n}\theta^i \wedge \nabla_{e_i}\omega - \operatorname{div}\omega. \qquad \text{(See p. 592.)}$$

Now,

$$\sum_{i=1}^{n}\bigl(\theta^i \wedge \nabla_{e_i}\omega\bigr)(X_0, X_1, \dots, X_r)$$

$$= \sum_{i,j=1}^{n}\sum_{r=0}^{n}(-1)^j X^i_j (\nabla_{e_i}\omega)(X_0, \dots, \widehat{X}_j, \dots, X_r)$$

$$= \sum_{j=0}^{r}(-1)^j (\nabla_{X_j}\omega)(X_0, \dots, \widehat{X}_j, \dots, X_r)$$

$$= \mathrm{d}\omega(X_0, \dots, X_r), \quad (\text{by } (\star) \text{ in the hint})$$

for all $X_0, \dots, X_r \in \mathfrak{X}(M)$, where $X_j = \sum_i X^i_j e_i$. Hence, applying Definition 6.11,

$$D\,\omega = \mathrm{d}\omega - \operatorname{div}\omega = \mathrm{d}\omega + \delta\omega.$$

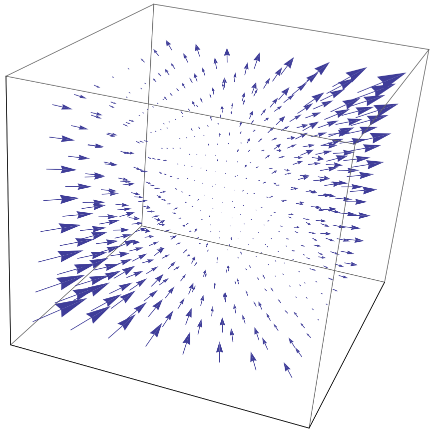
6.14 Affine, Killing, Conformal, Projective, Jacobi, and Harmonic Vector Fields

Problem 6.123 Find a non-affine projective vector field X on $\mathbb{R}^3$.

Hint Let ∇ be the Levi-Civita connection of the Euclidean metric of $\mathbb{R}^3$. The vector field X is projective if

$$(L_X\nabla)(Y, Z) = \theta(Y)Z + \theta(Z)Y, \quad Y, Z \in \mathfrak{X}\bigl(\mathbb{R}^3\bigr), \qquad (\star)$$

Fig. 6.12 A non-affine projective vector field on $\mathbb{R}^3$



for some differential 1-form $\theta \in \Lambda^1\mathbb{R}^3$, where

$$(L_X\nabla)(Y, Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X, Z].$$

Moreover, one has $\mathrm{d}(\operatorname{div} X) = (\dim \mathbb{R}^3 + 1)\theta = 4\theta$.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution Let (see Fig. 6.12)

$$X = 2x(x+y+z)\frac{\partial}{\partial x} + 2y(x+y+z)\frac{\partial}{\partial y} + 2z(x+y+z)\frac{\partial}{\partial z}.$$

Then

$$\theta = \frac{1}{4}\,\mathrm{d}(\operatorname{div} X) = 2(\mathrm{d}x + \mathrm{d}y + \mathrm{d}z).$$

Because of the symmetry of the vector field X and the differential form θ, it suffices to prove the formula ($\star$) for a couple of coordinate vector fields, for example, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. We have

$$\theta\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} + \theta\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = 2\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x},$$

and

$$\begin{aligned}(L_X\nabla)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= -\nabla_{\frac{\partial}{\partial x}}\left[X, \frac{\partial}{\partial y}\right] \\ &= -\nabla_{\frac{\partial}{\partial x}}\left(-2x\frac{\partial}{\partial x} + (-2x - 4y - 2z)\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}\right) \\ &= 2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}.\end{aligned}$$

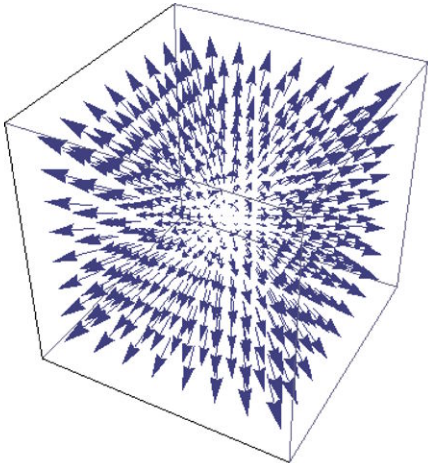


Fig. 6.13 A non-Killing affine vector field on $\mathbb{R}^3$

Problem 6.124 Prove that the vector field $X = \sum_{i=1}^{3} x^i \partial/\partial x^i$ on $\mathbb{R}^3$ with the Euclidean metric is affine but not Killing. Is X a vector field of homotheties?

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution

$$
\begin{aligned}
L_X g &= \sum_{i,j} L_{x^i \frac{\partial}{\partial x^i}} \left(\mathrm{d}x^j \otimes \mathrm{d}x^j\right) \\
&= \sum_{i,j} \left(\mathrm{d}\left(x^i \frac{\partial x^j}{\partial x^i}\right) \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}\left(x^i \frac{\partial x^j}{\partial x^i}\right)\right) = 2g.
\end{aligned}
$$

Hence X is not Killing (see Fig. 6.13).

Note that X is a conformal vector field, with the function $h \in C^\infty \mathbb{R}^n$, such that $L_X g = 2hg$, equal to 1, i.e. it is a constant function. It is said that a conformal vector field with $h = \text{const}$ is a vector field of homotheties.

Let us see if X is affine. As the Levi-Civita connection is torsionless and the curvature vanishes, the condition is $\nabla_Y \nabla X = 0$, $Y \in \mathfrak{X}(\mathbb{R}^n)$. Now, as

$$
\nabla_Y X = \sum_{i=1}^{3} Y\left(x^i\right) \frac{\partial}{\partial x^i} = Y,
$$

we have $\nabla X = I$, hence any covariant derivative under ∇ of ∇X vanishes. Thus X is affine.

Problem 6.125 Let (M, g) be a Riemannian manifold. Prove that $X \in \mathfrak{X}(M)$ is a Killing vector field if and only if $L_X g = 0$.

Solution A vector field X is Killing if $\varphi_t^* g = g$ for every t, where φ_t is the local 1-parameter group generated by X. Hence

$$L_X g = \lim_{t\to 0} \frac{g - \varphi_t^* g}{t} = 0.$$

Conversely, assume $L_X g = 0$. For any tensor field K we know (see Proposition 2.10) that

$$\varphi_s \cdot (L_X K) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t \cdot K)\right)_{t=s}.$$

Hence, by virtue of the hypothesis, we have

$$0 = \varphi_s^*(L_X g) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t^* g\big)\right)_{t=s},$$

and consequently, $\varphi_t^* g$ does not depend on t. Therefore, $\varphi_t^* g = \varphi_0^* g = g$.

Problem 6.126 Show that the set of Killing vector fields of the Euclidean metric $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$ is the real Lie algebra generated by the vector fields

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}, \qquad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \qquad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution Let $X = \sum_{i=1}^3 \lambda^i \partial/\partial x^i$, where λ^i is a function of $x^1 = x$, $x^2 = y$ and $x^3 = z$. Then one has

$$L_X g = \sum_{i,j=1}^{3} \left(\frac{\partial \lambda^j}{\partial x^i} + \frac{\partial \lambda^i}{\partial x^j}\right) \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

If X is Killing, that is, $L_X g = 0$, we deduce:

$$\text{(i)}\quad \frac{\partial \lambda^1}{\partial x^1} = 0, \qquad \text{(ii)}\quad \frac{\partial \lambda^2}{\partial x^2} = 0, \qquad \text{(iii)}\quad \frac{\partial \lambda^3}{\partial x^3} = 0,$$

$$\text{(iv)}\quad \frac{\partial \lambda^1}{\partial x^2} + \frac{\partial \lambda^2}{\partial x^1} = 0, \qquad \text{(v)}\quad \frac{\partial \lambda^1}{\partial x^3} + \frac{\partial \lambda^3}{\partial x^1} = 0, \qquad \text{(vi)}\quad \frac{\partial \lambda^2}{\partial x^3} + \frac{\partial \lambda^3}{\partial x^2} = 0.$$

From (i), (ii) and (iii), it follows that $\lambda^1 = \lambda^1(x^2, x^3)$, $\lambda^2 = \lambda^2(x^1, x^3)$, and $\lambda^3 = \lambda^3(x^1, x^2)$. Thus, from (iv) and (v), one has

$$\frac{\partial^2 \lambda^1}{\partial x^2 \partial x^2} = 0, \qquad \frac{\partial^2 \lambda^1}{\partial x^3 \partial x^3} = 0,$$

from which

$$\lambda^1 = a_1 x^2 x^3 + b_1 x^2 + c_1 x^3 + d_1.$$

Similarly,

$$\lambda^2 = a_2 x^1 x^3 + b_2 x^3 + c_2 x^1 + d_2, \qquad \lambda^3 = a_3 x^1 x^2 + b_3 x^1 + c_3 x^2 + d_3.$$

On account of (iv), (v) and (vi) above, these formulae reduce to

$$\lambda^1 = -c_2 x^2 + c_1 x^3 + d_1, \qquad \lambda^2 = -c_3 x^3 + c_2 x^1 + d_2,$$
$$\lambda^3 = -c_1 x^1 + c_3 x^2 + d_3.$$

Hence the generators are indeed the ones in the statement. By using the property $L_{[X,Y]} = [L_X, L_Y]$, it is easily checked that $\{X : L_X g = 0\}$ is a Lie algebra.

Problem 6.127 Calculate the divergence of a Killing vector field on a Riemannian manifold.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $X \in \mathfrak{X}(M)$ be a Killing vector field. Since $L_X g = 0$, $\nabla g = 0$, and ∇ is torsionless, we have for any $Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned}0 &= (L_X g)(Y, Z) = Xg(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z)\\ &= Xg(Y, Z) - g\big([X, Y], Z\big) - g\big(Y, [X, Z]\big) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).\end{aligned}$$

Hence, for any $p \in M$, and any orthonormal basis $\{e_i\}$ of $T_p M$, one has

$$(\operatorname{div} X)(p) = \sum_i g(\nabla_{e_i} X, e_i) = 0,$$

that is, $\operatorname{div} X = 0$.

Problem 6.128 Prove that a vector field X on a Riemannian manifold (M, g) is Killing if and only if the *Kostant operator* A defined by

$$A_X = L_X - \nabla_X,$$

where ∇ stands for the Levi-Civita connection of g, satisfies

$$g(A_X Y, Z) + g(Y, A_X Z) = 0, \qquad Y, Z \in \mathfrak{X}(M).$$

Remark Notice that as ∇ is torsionless, $A_X Y = -\nabla_Y X$.

Solution As $\nabla_X g = 0$ for all $X \in \mathfrak{X}(M)$, the condition $L_X g = 0$ is equivalent to $A_X g = 0$. Since A_X is the difference of two derivations of the algebra of tensor fields that commute with contractions, one has

$$A_X\big(g(Y, Z)\big) = (A_X g)(Y, Z) + g(A_X Y, Z) + g(Y, A_X Z), \qquad Y, Z \in \mathfrak{X}(M).$$

On the other hand,

$$A_X f = L_X f - \nabla_X f = Xf - Xf = 0, \quad f \in C^\infty M,$$

thus $A_X(g(Y,Z)) = 0$. Hence $(A_X g)(Y,Z) = 0$ if and only if

$$g(A_X Y, Z) + g(Y, A_X Z) = 0,$$

as wanted.

Problem 6.129 Consider $\mathbb{R}^2$ equipped with the metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$.

(i) Show that the vector field

$$X = (ax - by)\frac{\partial}{\partial x} + (bx + ay)\frac{\partial}{\partial y}, \quad a, b \in \mathbb{R},$$

is a conformal vector field.

(ii) Let $\mathbb{R}^3 \setminus \{0\}$ with the usual metric and let v denote the volume form. Write $L_Y v$, $Y \in \mathfrak{X}(\mathbb{R}^3 \setminus \{0\})$, in cylindrical coordinates.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution

(i)

$$\begin{aligned}
&L_{(ax-by)\frac{\partial}{\partial x}+(bx+ay)\frac{\partial}{\partial y}}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) \\
&\quad = \mathrm{d}(ax - by) \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}(ax - by) + \mathrm{d}(bx + ay) \otimes \mathrm{d}y \\
&\qquad + \mathrm{d}y \otimes \mathrm{d}(bx + ay) \\
&\quad = 2ag
\end{aligned}$$

(see Fig. 6.14).

(ii) One has

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \qquad z = z, \quad \rho > 0, \ \theta \in (0, 2\pi).$$

Hence, the volume form is $v = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}z$. Let

$$Y = F\frac{\partial}{\partial \rho} + G\frac{\partial}{\partial \theta} + H\frac{\partial}{\partial z},$$

where F, G, H are functions of ρ, θ, and z. Therefore,

$$\begin{aligned}
L_Y v &= \frac{1}{2} L_Y\big(\mathrm{d}(\rho^2) \wedge \mathrm{d}\theta \wedge \mathrm{d}z\big) \\
&= \frac{1}{2}\left\{\mathrm{d}\left(F\frac{\partial}{\partial \rho}\rho^2\right) \wedge \mathrm{d}\theta \wedge \mathrm{d}z + \mathrm{d}(\rho^2) \wedge \mathrm{d}\left(G\frac{\partial}{\partial \theta}\theta\right) \wedge \mathrm{d}z\right.
\end{aligned}$$

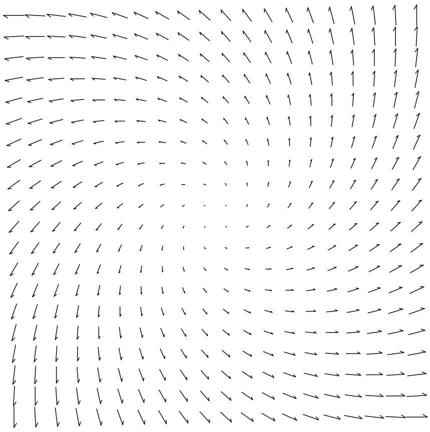


Fig. 6.14 The conformal vector field $(x-y)\partial/\partial x+(x+y)\partial/\partial y$

$$+\,\mathrm{d}\big(\rho^2\big)\wedge \mathrm{d}\theta\wedge \mathrm{d}\bigg(H\frac{\partial}{\partial z}z\bigg)\bigg\}$$
$$=\bigg(\frac{F}{\rho}+\frac{\partial F}{\partial \rho}+\frac{\partial G}{\partial \theta}+\frac{\partial H}{\partial z}\bigg)v.$$

Problem 6.130 Consider the 1-parameter group φ_t, $t\in\mathbb{R}$, of automorphisms of $\mathbb{R}^2$ defined by the equations

$$x(t)=x-\cos t+y\sin t,\qquad y(t)=-x\sin t+y-\cos t.$$

(i) Compute the infinitesimal generator X of φ_t.
(ii) If $g=\mathrm{d}x^2+\mathrm{d}y^2$ and $\omega=\mathrm{d}x\wedge\mathrm{d}y$, find the vector field Y on $\mathbb{R}^2\setminus\{(0,0)\}$ defined by

$$g(Y,Y)=1,\qquad g(X,Y)=0,\qquad \omega(X,Y)>0,$$

and prove that $[X,Y]=0$.
(iii) Calculate $L_X g$, $L_Y g$, $L_X\omega$, and $L_Y\omega$.
(iv) Compute the first integrals of X and Y.
(v) Prove that in a certain neighbourhood of any point different from the origin there is a local coordinate system (u,v) such that

$$X=\frac{\partial}{\partial u},\qquad Y=\frac{\partial}{\partial v}.$$

Solution

(i) Since

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=y(t),\qquad \frac{\mathrm{d}y(t)}{\mathrm{d}t}=-x(t),$$

one has

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(ii) The vector field

$$Y = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

is a unit vector field with respect to g, which is normal to the circles with centre at the origin, so $g(X,Y)=0$. Moreover, $\omega(X,Y)=\sqrt{x^2+y^2}>0$ if $(x,y)\neq(0,0)$. It is easily checked that $[X,Y]=0$.

(iii) Let $\rho=\sqrt{x^2+y^2}$. Then:

$$\begin{aligned}
L_X g &= L_{y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y)=0,\\
L_Y g &= L_{\frac{x}{\rho}\frac{\partial}{\partial x}+\frac{y}{\rho}\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y)\\
&= \frac{2}{\rho^3}\big(y^2\,\mathrm{d}x\otimes\mathrm{d}x - x\,y(\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x)+x^2\,\mathrm{d}y\otimes\mathrm{d}y\big),\\
L_X\omega &= L_{y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}y-\mathrm{d}y\otimes\mathrm{d}x)=0,\\
L_Y\omega &= L_{\frac{x}{\rho}\frac{\partial}{\partial x}+\frac{y}{\rho}\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}y-\mathrm{d}y\otimes\mathrm{d}x)=\frac{1}{\rho}\omega.
\end{aligned}$$

(iv) The first integrals of X and Y are, respectively, $f(u^1)$, where $u^1=x^2+y^2$ and $f(v^1)$, where $v^1=y/x$.

(v) By (iv), we have

$$X=\lambda\frac{\partial}{\partial v^1},\qquad Y=\mu\frac{\partial}{\partial u^1}.$$

If moreover $X=\frac{\partial}{\partial u}$, we would have $Xu=1=\lambda\frac{\partial u}{\partial v^1}$, thus $\frac{\partial u}{\partial v^1}=\frac{1}{\lambda}$. Let us compute λ. One has

$$\lambda = Xv^1=\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\frac{y}{x}=-\frac{y^2}{x^2}-1=-\big(1+\big(v^1\big)^2\big),$$

that is,

$$\frac{\partial u}{\partial v^1}=-\frac{1}{1+(v^1)^2},$$

so $u=-\arctan v^1=-\theta$ (in polar coordinates). Hence

$$u=-\arctan\frac{x}{y}.$$

Similarly, if $Y = \frac{\partial}{\partial v}$, we have $Yv = 1 = \mu \frac{\partial v}{\partial u^1}$. Let us calculate μ. We have

$$\mu = Yu^1 = 2\rho = 2\sqrt{u^1}.$$

Thus

$$\frac{\partial v}{\partial u^1} = \frac{1}{\mu} = \frac{1}{2\sqrt{u^1}},$$

and $v = \sqrt{u^1}$. That is, $v = \sqrt{x^2 + y^2}$.

Problem 6.131 Find two linearly independent harmonic vector fields on the 2-torus T^2 with its usual embedding in $\mathbb{R}^3$ as a surface of revolution.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution Let us see if there exist $f(\varphi, \theta)\partial/\partial\varphi$ and $h(\varphi, \theta)\partial/\partial\theta$ harmonic, φ and θ being the parameters of the usual parametrisation (see Remark 1.4)

$$x = (R + r\cos\varphi)\cos\theta, \qquad y = (R + r\cos\varphi)\sin\theta, \qquad z = r\sin\varphi,$$

$\varphi, \theta \in (0, 2\pi)$, $R > r$, and $f(\varphi, \theta)$, $h(\varphi, \theta)$ functions of these parameters. Such vector fields would obviously be linearly independent. If $j \colon T^2 \to \mathbb{R}^3$ denotes the usual embedding, the metric is

$$j^*\big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big) = r^2\,\mathrm{d}\varphi^2 + (R + r\cos\varphi)^2\,\mathrm{d}\theta^2.$$

If M is compact, as in our case, in order for a vector field Z to be harmonic (see Definition 6.27) it suffices to have $\mathrm{d}\,Z^\flat = 0$, $\delta Z^\flat = 0$.

Putting

$$X = f(\varphi, \theta)\frac{\partial}{\partial\varphi}, \qquad Y = h(\varphi, \theta)\frac{\partial}{\partial\theta},$$

one has

$$X^\flat = r^2 f(\varphi, \theta)\,\mathrm{d}\varphi, \qquad Y^\flat = (R + r\cos\varphi)^2 h(\varphi, \theta)\,\mathrm{d}\theta.$$

Thus, from

$$\mathrm{d}X^\flat = r^2\frac{\partial f}{\partial\theta}\,\mathrm{d}\theta \wedge \mathrm{d}\varphi = 0,$$

we have $f = f(\varphi)$. Suppose similarly $h = h(\varphi)$. Then

$$\mathrm{d}Y^\flat = \frac{\partial((R + r\cos\varphi)^2 h(\varphi))}{\partial\varphi}\,\mathrm{d}\varphi \wedge \mathrm{d}\theta = 0$$

implies

$$h(\varphi) = \frac{A}{(R + r\cos\varphi)^2}.$$

Hence, for

$$X = f(\varphi)\frac{\partial}{\partial\varphi}, \qquad Y = \frac{A}{(R + r\cos\varphi)^2}\frac{\partial}{\partial\theta}, \tag{$\star$}$$

we have $\mathrm{d}X^\flat = \mathrm{d}Y^\flat = 0$.

To compute $\delta X^\flat = -\operatorname{div} X^\flat$ and $\delta Y^\flat = -\operatorname{div} Y^\flat$ we use the formula, valid for any oriented manifold M,

$$L_Z v = (\operatorname{div} Z)v, \quad Z \in \mathfrak{X}(M),$$

where v denotes the volume element on M, which in our case is

$$v = \sqrt{g_{11}g_{22} - g_{12}^2}\,\mathrm{d}\varphi \wedge \mathrm{d}\theta = r(R + r\cos\varphi)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

Applying moreover the general formula

$$L_Z(f\,\mathrm{d}\varphi \wedge \mathrm{d}\theta) = (Zf)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta + f\,\mathrm{d}(Z\varphi) \wedge \mathrm{d}\theta + f\,\mathrm{d}\varphi \wedge \mathrm{d}(Z\theta),$$

$f \in C^\infty M$, to X and Y in ($\star$), we obtain

$$\begin{aligned}(\operatorname{div} X)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta &= L_{f(\varphi)\frac{\partial}{\partial\varphi}} r(R + r\cos\varphi)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta \\ &= \left(-f(\varphi)r^2\sin\varphi + r(R + r\cos\varphi)\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi}\right)\mathrm{d}\varphi \wedge \mathrm{d}\theta, \\ (\operatorname{div} Y)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta &= L_{\frac{A}{(R+r\cos\varphi)^2}\frac{\partial}{\partial\theta}} r(R + r\cos\varphi)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta = 0.\end{aligned}$$

Hence, $\delta Y^\flat = 0$. And $\delta X^\flat = 0$ if

$$-f(\varphi)r\sin\varphi + (R + r\cos\varphi)\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\big(f(\varphi)(R + r\cos\varphi)\big) = 0,$$

that is, if

$$f(\varphi) = \frac{B}{R + r\cos\varphi}.$$

In this case,

$$\Delta X^\flat = (\mathrm{d}\delta + \delta\mathrm{d})X^\flat = 0, \qquad \Delta Y^\flat = (\mathrm{d}\delta + \delta\mathrm{d})Y^\flat = 0,$$

that is, $X^\flat$ and $Y^\flat$ are harmonic forms, and

$$X = \frac{B}{R + r\cos\varphi}\frac{\partial}{\partial\varphi}, \qquad Y = \frac{A}{(R + r\cos\varphi)^2}\frac{\partial}{\partial\theta}$$

satisfy the conditions in the statement.

Notice that in order to compute $\delta X^\flat$ and $\delta Y^\flat$, we can instead use the definition $\operatorname{div} Z = \operatorname{tr} \nabla Z$ and thus the Christoffel symbols of g, as follows. Taking $x^1 = \varphi$, $x^2 = \theta$, since

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos\varphi)^2 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/(R + r\cos\varphi)^2 \end{pmatrix},$$

we deduce that the non-vanishing Christoffel symbols are

$$\varGamma^1_{22} = \frac{1}{b}(R + r\cos\varphi)\sin\varphi, \qquad \varGamma^2_{12} = \varGamma^2_{21} = -\frac{r\sin\varphi}{R + r\cos\varphi}.$$

Let us calculate $\delta X^\flat$ and $\delta Y^\flat$:

$$\begin{aligned}
\delta X^\flat &= -\operatorname{div} X^\flat = -\operatorname{div} X \\
&= -g\left(\nabla_{\frac{1}{r}\frac{\partial}{\partial\varphi}} f(\varphi)\frac{\partial}{\partial\varphi}, \frac{1}{r}\frac{\partial}{\partial\varphi}\right) - g\left(\nabla_{\frac{1}{R+r\cos\varphi}\frac{\partial}{\partial\theta}} f(\varphi)\frac{\partial}{\partial\varphi}, \frac{1}{R + r\cos\varphi}\frac{\partial}{\partial\theta}\right) \\
&= -\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} - f(\varphi)\left(-\frac{r\sin\varphi}{R + r\cos\varphi}\right),
\end{aligned}$$

since $\varGamma^1_{11} = \varGamma^2_{11} = \varGamma^1_{21} = 0$.

$$\begin{aligned}
\delta Y^\flat = -\operatorname{div} Y = &-g\left(\nabla_{\frac{1}{r}\frac{\partial}{\partial\varphi}} \frac{A}{(R+r\cos\varphi)^2}\frac{\partial}{\partial\theta}, \frac{1}{r}\frac{\partial}{\partial\varphi}\right) \\
&- g\left(\nabla_{\frac{1}{R+r\cos\varphi}\frac{\partial}{\partial\theta}} \frac{A}{(R + r\cos\varphi)^2}\frac{\partial}{\partial\theta}, \frac{1}{R + r\cos\varphi}\frac{\partial}{\partial\theta}\right) = 0,
\end{aligned}$$

since $\varGamma^1_{12} = \varGamma^2_{22} = 0$. That is, we obtain the same expressions as above.

Problem 6.132 Let (M, g) be a Riemannian manifold. Prove that if $X \in \mathfrak{X}(M)$ is Killing and $Y \in \mathfrak{X}(M)$ is harmonic, then $g(X, Y)$ is a harmonic function.

Hint Apply the following results:

(i) If $Z \in \mathfrak{X}(M)$ is Killing, then

$$g\big(\operatorname{tr} \nabla^2 Z, W\big) = -\mathbf{r}(Z, W), \qquad W \in \mathfrak{X}(M),$$

where $\mathbf{r}$ denotes the Ricci tensor.

(ii) $Z \in \mathfrak{X}(M)$ is harmonic if and only if $g(\operatorname{tr} \nabla^2 Z, W) = \mathbf{r}(Z, W)$.

(iii) Let K be a symmetric (i.e. self-adjoint) transformation of an inner product space $(E, \langle\,,\,\rangle)$, and let L be skew-symmetric. Then we have $\langle K, L\rangle = 0$.

The relevant theory is developed, for instance, in Poor [28, Chap. 5].

Solution Let (e_i) be an orthonormal frame on a neighbourhood of the point $p \in M$. Then if ∇ denotes the Levi-Civita connection of g, we have

$$\begin{aligned}
\big(\Delta g(X,Y)\big)(p) &= \big(\delta\, \mathrm{d}g(X,Y)\big)(p) = -\big(\operatorname{div} \mathrm{d}(X,Y)\big)(p) \\
&= -\sum_i \big(\big(\nabla_{e_i} \mathrm{d}g(X,Y)\big)(e_i)\big)(p) \\
&= -\sum_i \big\{\nabla_{e_i}\big(\mathrm{d}g(X,Y)(e_i)\big) - \big(\mathrm{d}g(X,Y)\big)(\nabla_{e_i} e_i)\big\}(p) \\
&= -\sum_i \big\{\nabla_{e_i} e_i\, g(X,Y) - (\nabla_{e_i} e_i) g(X,Y)\big\}(p) \\
&= -\sum_i \big\{e_i\, g(\nabla_{e_i} X, Y) + e_i\, g(X, \nabla_{e_i} Y) - g(\nabla_{\nabla_{e_i} e_i} X, Y) \\
&\quad - g(X, \nabla_{\nabla_{e_i} e_i} Y)\big\}(p) \\
&= -\sum_i \big\{g(\nabla_{e_i}\nabla_{e_i} X, Y) + g(\nabla_{e_i} X, \nabla_{e_i} Y) + g(\nabla_{e_i} X, \nabla_{e_i} Y) \\
&\quad + g(X, \nabla_{e_i}\nabla_{e_i} Y) - g(\nabla_{\nabla_{e_i} e_i} X, Y) - g(X, \nabla_{\nabla_{e_i} e_i} Y)\big\}(p) \\
&= -\sum_i \big\{2g(\nabla_{e_i} X, \nabla_{e_i} Y) + g\big((\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})X, Y\big) \\
&\quad + g\big(X, (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})Y\big)\big\}(p) \\
&= \big\{-2g(\nabla X, \nabla Y) - g\big(\mathrm{tr}\, \nabla^2 X, Y\big) - g\big(X, \mathrm{tr}\, \nabla^2 Y\big)\big\}(p) \\
&= \big\{-2g(\nabla X, \nabla Y) - \mathbf{r}(X,Y) + \mathbf{r}(X,Y)\big\}(p) \\
&= \big(-2g(\nabla X, \nabla Y)\big)(p).
\end{aligned}$$

Now, since X is Killing, ∇X is skew-symmetric, i.e.

$$g(\nabla_Z X, W) + g(Z, \nabla_W X) = 0$$

(see Problem 6.127) and as Y is harmonic, ∇Y is symmetric, i.e. $g(\nabla_Z Y, W) = g(Z, \nabla_W Y)$, each one with respect to g. Hence

$$g(\nabla_Z X, \nabla_W Y) = -g(Z, \nabla_{\nabla_W Y} X) = -g(\nabla_W Y, \nabla_Z X),$$

for $Z, W \in \mathfrak{X}(M)$, and we conclude that $\Delta\, g(X,Y) = 0$.

Problem 6.133 Prove that the cohomology group $H^3(G, \mathbb{R})$ of a compact Lie group G of dimension greater than 2 is not zero.

Hint In the non-Abelian case, consider the bi-invariant metric $\langle \cdot, \cdot \rangle$ on G which is the product of a Euclidean inner product on the centre $\mathfrak{z}$ and minus the Killing form on the (semi-simple) derived group $[\mathfrak{g}, \mathfrak{g}]$, and define $\alpha \in \Lambda^3 G$ by

$$\alpha(X, Y, Z) = \frac{1}{2}\langle [X, Y], Z\rangle.$$

Solution Suppose first that G is Abelian. Then it is a torus

$$T^n = S^1 \times \cdots \times S^1, \quad n \geqslant 3,$$

so the result is immediate.

If G is non-Abelian, as it is compact, its Lie algebra $\mathfrak{g}$ decomposes into the direct sum of its centre $\mathfrak{z}$ and its derived algebra $[\mathfrak{g}, \mathfrak{g}]$, which is semi-simple, compact, and has dimension greater than or equal to 3.

Consider the bi-invariant metric $\langle \cdot , \cdot \rangle$ on G which is the product of a Euclidean inner product on the centre $\mathfrak{z}$ and the opposite of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. Define $\alpha \in \Lambda^3 G$ by

$$\alpha(X, Y, Z) = \frac{1}{2}\langle [X, Y], Z\rangle.$$

As $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple then α is non-zero. Moreover, as (see Problem 6.90)

$$\nabla_X Y = [X, Y], \quad X, Y \in \mathfrak{g},$$

by applying the Jacobi identity, we have

$$\begin{aligned}
(\nabla_X \alpha)(Y, Z, W) &= 0 - \alpha(\nabla_X Y, Z, W) - \alpha(Y, \nabla_X Z, W) - \alpha(Y, Z, \nabla_X W) \\
&= -\langle [\nabla_X Y, Z], W\rangle - \langle [Y, \nabla_X Z], W\rangle - \langle [Y, Z], \nabla_X W\rangle \\
&= -\frac{1}{2}\{\langle \big[[X, Y], Z\big], W\rangle + \langle \big[Y, [X, Z]\big], W\rangle + \langle [Y, Z], [X, W]\rangle\} \\
&= -\frac{1}{2}\{\langle \big[[X, Y], Z\big], W\rangle + \langle \big[Y, [X, Z]\big], W\rangle - \langle \big[X, [Y, Z]\big], W\rangle\} \\
&= -\frac{1}{2}\{\langle \big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big], W\rangle\} = 0.
\end{aligned}$$

So α is parallel, and G being compact, α is in particular harmonic, so that it is a nonzero representative of $H^3(G, \mathbb{R})$.

Problem 6.134 Determine the Jacobi fields on $\mathbb{R}^n$ with the Euclidean metric g.

The relevant theory is developed, for instance, in O'Neill [26, Chap. 8].

Solution The geodesics of $(\mathbb{R}^n, g)$ are the straight lines. Since the curvature vanishes, the Jacobi equation reduces to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = 0.$$

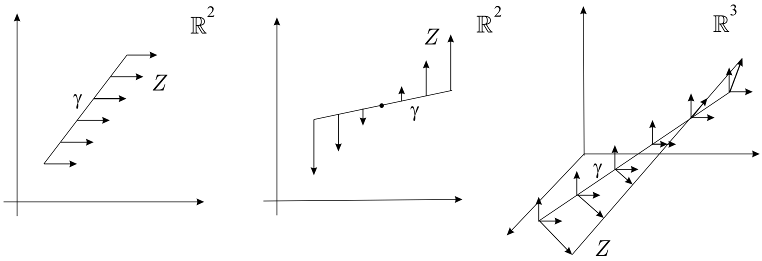


Fig. 6.15 Some simple Jacobi fields

The Jacobi fields along a straight line γ are the fields of the form $X = tY + Z$, where Y and Z are constant vector fields along γ (see Fig. 6.15).

Problem 6.135 Let

$$\varphi(u, v) = \big(u \cos v, u \sin v, f(u)\big), \quad u > 0, \; v \in (0, 2\pi),$$

be a parametric surface of revolution in $\mathbb{R}^3$ (see Remark 1.4), and let

$$Y_v|_{\varphi(u,v)} = \left.\frac{\partial}{\partial v}\right|_{\varphi(u,v)}.$$

Prove:

(i) Y_v is a Jacobi field along meridians.
(ii) If g denotes the metric and s the arc length, then

$$\frac{\mathrm{d}^2|Y_v|}{\mathrm{d}s^2} = -K|Y_v|,$$

where K stands for the Gauss curvature.

Hint For such a surface of revolution, one has

$$K = \frac{f' f''}{u(1 + (f')^2)^2}.$$

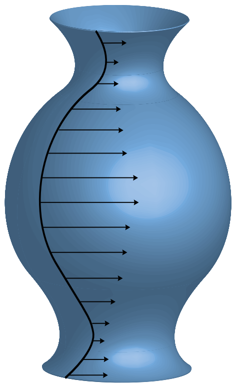
The relevant theory is developed, for instance, in O'Neill [26, Chap. 8].

Solution

(i) The vector fields Y and γ' in the torsionless case of Definition 6.19 are here $Y = \frac{\partial}{\partial v}$ and

$$\gamma' = \frac{\partial}{\partial u} \Big/ \left|\frac{\partial}{\partial u}\right| = \frac{1}{\sqrt{1 + (f'(u))^2}} \frac{\partial}{\partial u}.$$

Fig. 6.16 A Jacobi field on a surface of revolution



We must prove that

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_Y\nabla_{\gamma'}\gamma' - \nabla_{\gamma'}\nabla_Y\gamma' - \nabla_{[Y,\gamma']}\gamma' = 0.$$

Now, since ∇ is torsionless we have $\nabla_Y\gamma' - \nabla_{\gamma'}Y = [Y, \gamma']$; but it is immediate that in the present case $[Y, \gamma'] = 0$. On the other hand, $\nabla_{\gamma'}\gamma' = 0$, as γ' is the tangent vector field to a geodesic curve. So we are done.

(ii) We have $|Y_v| = u$. Moreover, the Gauss curvature of a surface of revolution is given by the expression in the hint, with $f' = \mathrm{d}f/\mathrm{d}u$.

The arc length $s(u)$ along the meridian is given, since v is constant, by

$$s(u) = \int_0^u \sqrt{g\left(\frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial u}\right)}\,\mathrm{d}u = \int_0^u \sqrt{1 + \left(f'(u)\right)^2}\,\mathrm{d}u,$$

and thus

$$\begin{aligned}\frac{\mathrm{d}^2|Y_v|}{\mathrm{d}s^2} &= \frac{\mathrm{d}^2u}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\sqrt{1+\left(f'\right)^2}} = \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{\sqrt{1+\left(f'\right)^2}}\\ &= -\frac{1}{\sqrt{1+(f')^2}}\frac{f'f''}{(1+(f')^2)^{3/2}} = -Ku = -K|Y_v|.\end{aligned}$$

Notice that the lengths for the vector field Y_v are larger where the distance between the given geodesics (the meridians) grows, and are lower where that distance decreases (see Fig. 6.16).

Problem 6.136 Let (M, g) be an n-dimensional space of constant curvature c. Let $\dot{\gamma}, X_1, \dots, X_{n-1}$ be an orthonormal frame invariant by parallelism along a geodesic γ with unit tangent vector field $\dot{\gamma}$.

Prove that the vector fields

(i) $\dot{\gamma}$, $s\dot{\gamma}$, $Y_i = \sin(\sqrt{c}\,s)X_i$, $Z_i = \cos(\sqrt{c}\,s)X_i$,

(ii) $\dot{\gamma}, s\dot{\gamma}, Y_i = \sinh(\sqrt{-c}\, s)X_i, Z_i = \cosh(\sqrt{-c}\, s)X_i,$
(iii) $\dot{\gamma}, s\dot{\gamma}, X_i, sX_i,$

$i = 1, \dots, n-1$, where s denotes the arc length, are a basis of the space of Jacobi vector fields along the geodesic, for $c > 0$ in case (i), $c < 0$ in case (ii), and $c = 0$ in case (iii).

The relevant theory is developed in O'Neill [26, Chap. 8].

Solution That such $\dot{\gamma}$ and $s\dot{\gamma}$ are Jacobi fields is a general fact for Riemannian manifolds, and its proof is immediate from the Jacobi equation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} Y + R(Y, \dot{\gamma})\dot{\gamma} = 0.$$

In cases (i) and (ii), since (M, g) is a space of constant curvature c, we have

$$R(Y_i, \dot{\gamma})\dot{\gamma} = c\big(g(\dot{\gamma}, \dot{\gamma})Y_i - g(\dot{\gamma}, Y_i)\dot{\gamma}\big) = cY_i.$$

In case (i), we have, on the other hand,

$$\begin{aligned} \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} Y_i &= -c\sin(\sqrt{c}\, s)X_i + \sqrt{c}\cos(\sqrt{c}\, s)\nabla_{\dot{\gamma}} X_i \\ &\quad + \sqrt{c}\cos(\sqrt{c}\, s)\nabla_{\dot{\gamma}} X_i + \sin(\sqrt{c}\, s)\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} X_i \qquad (\star) \\ &= -cY_i \quad (\text{as } X_i \text{ is parallel}). \end{aligned}$$

Hence the Jacobi equation for a torsionless connexion,

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} Y_i + R(Y_i, \dot{\gamma})\dot{\gamma} = 0,$$

is satisfied, as wanted. The proof for Z_i (and Y_i, Z_i in the case (ii)) is similar.

The case (iii) is trivially true as $R = 0$ for $c = 0$.

Problem 6.137 Determine the conjugate points and their orders for a point on an n-sphere of constant curvature c.

Solution From Problem 6.136, it follows that the only point conjugate to the point corresponding to $s = 0$ along a geodesic $\gamma(s)$ is the point corresponding to $s = \pi/\sqrt{c}$, with order $n-1$, as a basis of the Jacobi fields vanishing at $s = 0$ and $s = \pi/\sqrt{c}$ is given by the vector fields $Y_i = \sin(\sqrt{c}\, s)X_i$.

Problem 6.138 Show that if M has non-positive sectional curvature, then there are no conjugate points.

Solution Let Y be a Jacobi vector field along a geodesic $\gamma(t)$. From the Jacobi equation

$$\nabla_{\gamma'}\nabla_{\gamma'} Y + R\big(Y, \gamma'\big)\gamma' = 0,$$

we obtain by virtue of the hypothesis that

$$g(\nabla_{\gamma'}\nabla_{\gamma'}Y, Y) = g\big(R(\gamma', Y)\gamma', Y\big) = -R\big(\gamma', Y, \gamma', Y\big) \geqslant 0,$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\nabla_{\gamma'}Y, Y) = g(\nabla_{\gamma'}\nabla_{\gamma'}Y, Y) + |\nabla_{\gamma'}Y|^2 \geqslant 0. \tag{$\star$}$$

The function $g(\nabla_{\gamma'}Y, Y)$ is thus monotonically increasing (strictly if $\nabla_{\gamma'}Y \neq 0$). If $Y(0) = Y(t_0) = 0$ for certain $t_0 > 0$, then $g(\nabla_{\gamma'}Y, Y)$ also vanishes at these points, hence it must vanish along the interval $[0, t_0]$. Thus, we have $Y(0) = (\nabla_{\gamma'}Y)(0) = 0$ by ($\star$), so that Y vanishes identically, as Y is a solution of a second-order differential equation.

Problem 6.139 Prove that the multiplicity of two points p and q conjugate along a geodesic γ in a manifold M is less than the dimension of the manifold.

Solution Let $\dim M = n$. Then the Jacobi vector fields vanishing on a given point $p \in M$ constitute a space of dimension n, but $(t - t_1)\dot{\gamma}$, where $p = \gamma(t_1)$, is a Jacobi vector field vanishing at p but not at q.

6.15 Submanifolds. Second Fundamental Form

Problem 6.140 Prove:

1. Every strictly conformal map is an immersion.
2. If M is connected, then a strictly conformal map

$$f : (M, g) \to (\bar{M}, \bar{g})$$

   of ratio λ transforms the Levi-Civita connection ∇ of g into the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$, if and only if $\lambda = \text{const}$ and the second fundamental form of the immersed submanifold $f(M)$ vanishes.
3. If $\lambda = 1$, that is, f is an isometry, and the second fundamental form of $f(M)$ vanishes, then if R and $\bar{R}$ stand for the Riemann–Christoffel curvature tensors of M and $\bar{M}$, respectively, one has $f_* R = \bar{R}|_{f(M)}$.

Solution

1. Let $X \in T_p M$ such that $f_* X = 0$. Then

$$0 = \bar{g}(f_* X, f_* X) = \lambda(p) g(X, X).$$

   As $\lambda(p) > 0$ for all $p \in M$, we have $X = 0$; that is, $\ker f_{*p} = 0$ for all $p \in M$.

2. As M is connected, we only need to prove that λ is locally constant. Thus we can assume that f is a diffeomorphism from M onto a submanifold $f(M)$ of $\bar{M}$. Denoting by $\bar{X}$ the vector field image $f \cdot X$ on $f(M)$ of $X \in \mathfrak{X}(M)$, we have that $X \mapsto \bar{X}$ is an isomorphism. Hence if f transforms ∇ into $\bar{\nabla}$, it follows that

$$\begin{aligned}\bar{X}\bar{g}(\bar{Y},\bar{Z}) &= \bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\bar{Z}) + \bar{g}(\bar{Y},\bar{\nabla}_{\bar{X}}\bar{Z}) = \bar{g}(\overline{\nabla_X Y},\bar{Z}) + \bar{g}(\bar{Y},\overline{\nabla_X Z}) \\ &= \lambda g(\nabla_X Y, Z) + \lambda g(Y, \nabla_X Z) = \lambda X g(Y,Z).\end{aligned}$$

On the other hand,

$$\bar{X}\bar{g}(\bar{Y},\bar{Z}) = X\lambda g(Y,Z) = (X\lambda) g(Y,Z) + \lambda X g(Y,Z).$$

Hence $X\lambda = 0$ for all X. As M is connected, we deduce that λ is a constant function. Furthermore, as $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y}$, it follows that $\bar{\nabla}_{\bar{X}}\bar{Y}$ is tangent to the submanifold $f(M)$, thus the second fundamental form of $f(M)$ vanishes.

Conversely, if we define on $f(M)$ the connection $\bar{\nabla}$ by $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y}$, and prove that $\bar{\nabla}$ parallelises the metric of $f(M)$ and has no torsion, then it will coincide with the Levi-Civita connection of the metric on $f(M)$. Let k be the constant function λ. One has:

(i)

$$\begin{aligned}\bar{X}\bar{g}(\bar{Y},\bar{Z}) = X\big(kg(Y,Z)\big) &= k\big\{g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\big\} \\ &= \bar{g}(\overline{\nabla_X Y},\bar{Z}) + \bar{g}(\bar{Y},\overline{\nabla_X Z}) \\ &= \bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\bar{Z}) + \bar{g}(\bar{Y},\bar{\nabla}_{\bar{X}}\bar{Z}).\end{aligned}$$

(ii)

$$\bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X} - [\bar{X},\bar{Y}] = \overline{\nabla_X Y} - \overline{\nabla_Y X} - \overline{[X,Y]} = \overline{T_\nabla(X,Y)} = 0.$$

3. Since f is an isometry, it transforms the Riemann–Christoffel curvature tensor of M into that of $f(M)$ (see Problem 6.85); but this one coincides with the restriction in $f(M)$ of the Riemann–Christoffel curvature tensor of $\bar{M}$, as the second fundamental form of $f(M)$ vanishes.

Problem 6.141 Let M be an n-dimensional $(n \geqslant 3)$, totally umbilical submanifold of a $2m$-dimensional complex space form $(\widetilde{M}, g, J)$ of holomorphic sectional curvature $c \neq 0$. Prove that M is one of the following submanifolds:

(i) A complex space form holomorphically immersed in $\widetilde{M}$ as a totally geodesic submanifold.
(ii) A real space form (i.e. a not necessarily simply connected space of constant curvature) immersed in $\widetilde{M}$ as a totally real and totally geodesic submanifold.
(iii) A real space form immersed in $\widetilde{M}$ as a totally real submanifold with non-vanishing parallel mean curvature vector.

The relevant theory is developed, for instance, in Chen and Ogiue [7].

Solution As M is a totally umbilical submanifold, with the usual notations we have

$$\alpha(X, Y) = g(X, Y)\xi, \quad X, Y \in \mathfrak{X}(M).$$

Thus the covariant derivative appearing in the Codazzi equation

$$(\widehat{\nabla}_X\alpha)(Y, Z) = \nabla^{\perp}_X\big(\alpha(Y, Z)\big) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z),$$

reduces to

$$(\widehat{\nabla}_X\alpha)(Y, Z) = g(Y, Z)\nabla^{\perp}_X\xi,$$

so the Codazzi equation is written as

$$\nu\widetilde{R}(X, Y)Z = g(Y, Z)\nabla^{\perp}_X\xi - g(X, Z)\nabla^{\perp}_Y\xi. \tag{$\star$}$$

Since $\dim M \geqslant 3$, for each $X \in \mathfrak{X}(M)$ one can choose a unit vector field $Y \in \mathfrak{X}(M)$ orthogonal to X and JX. For such a Y, from ($\star$) one has

$$\nu\widetilde{R}(X, Y)Y = \nabla^{\perp}_X\xi.$$

On the other hand, since $\widetilde{M}$ has constant holomorphic sectional curvature $c \neq 0$, we have

$$\begin{aligned}\widetilde{R}(X, Y)Z = \frac{c}{4}\big\{&g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY \\ &+ 2g(X, JY)JZ\big\},\end{aligned}$$

from which we deduce $\widetilde{R}(X, Y)Y = \frac{c}{4}X$, so $\nu\widetilde{R}(X, Y)Y = 0$, hence

$$\nabla^{\perp}_X\xi = 0, \quad X \in \mathfrak{X}(M).$$

From ($\star$) we then obtain

$$\nu\widetilde{R}(X, Y)Z = 0, \quad X, Y, Z \in \mathfrak{X}(M).$$

Thus, by Proposition 6.36, M is either a complex or a totally real submanifold of $\widetilde{M}$. If M is a complex submanifold, then M is minimal, hence totally geodesic in $\widetilde{M}$. Therefore, from Gauss' equation

$$R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) + g\big(\alpha(X, Z), \alpha(Y, W)\big) - g\big(\alpha(Y, Z), \alpha(X, W)\big),$$

we obtain

$$R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W).$$

That is, M is a complex space form of constant holomorphic sectional curvature c.

If M is a totally real submanifold, from Gauss' equation and from $\alpha(X,Y) = g(X,Y)\xi$, it follows that

$$R(X,Y,Z,W) = \left(\frac{c}{4} + g(\xi,\xi)\right)\big(g(X,Z)g(Y,W) - g(X,W), g(Y,Z)\big),$$

that is, M is a real space form of constant (ordinary) sectional curvature $\frac{c}{4} + g(\xi,\xi)$.

Problem 6.142 Consider the flat torus $T^2 = \mathbb{R}^2/\Gamma$ defined by the lattice

$$\Gamma = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2, \quad v_1 = (-\pi,\pi),\ v_2 = (0,2\pi).$$

(i) Prove that

$$f(u,v) = (\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v)$$

for $u, v \in (0, 2\pi)$ (see Remark 1.4), is an isometric embedding of T^2 into the unit sphere S^3 of $\mathbb{R}^4$.

(ii) Prove that the total embedded curvature of $f(T^2)$ in S^3 is a constant.

Hint Use the Generalised Gauss Theorema Egregium 6.30.

Solution

(i) We have $f(u,v) \in S^3$ since $|f(u,v)| = 1$. Moreover, $f(u,v) = f(u',v')$ if and only if $(u',v') - (u,v) \in \Gamma$. In fact, the previous equality is equivalent to

$$\sin(u+v) = \sin\big(u'+v'\big), \tag{$\star$}$$

$$\sin(u-v) = \sin\big(u'-v'\big), \tag{$\dagger$}$$

$$\cos(u+v) = \cos\big(u'+v'\big), \tag{$\star\star$}$$

$$\cos(u-v) = \cos\big(u'-v'\big). \tag{$\dagger\dagger$}$$

From ($\star$), ($\star\star$), and from ($\dagger$), ($\dagger\dagger$), we obtain

$$u'+v' = u+v+2k_1\pi, \qquad u'-v' = u-v+2k_2\pi, \quad k_1,k_2 \in \mathbb{Z},$$

respectively, from which

$$u' = u + h_1\pi, \qquad v' = v + h_2\pi, \quad h_1,h_2 \in \mathbb{Z}.$$

Now,

$$\begin{aligned} &f(u+h_1\pi, v+h_2\pi) \\ &\quad = (-1)^{h_1+h_2}(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v). \end{aligned}$$

So, $f(u,v) = f(u',v')$ if and only if $h_1 + h_2 = 2h$, thus

$$(u,v) \sim \bigl(u',v'\bigr) \quad \Leftrightarrow \quad u' - u = k\pi, \qquad v' - v = (2h-k)\pi.$$

Hence

$$\begin{aligned}\bigl(u',v'\bigr) &= (u,v) + \bigl(k\pi, (2h-k)\pi\bigr) = (u,v) + k(\pi,-\pi) + h(0,2\pi)\\ &= (u,v) + kv_1 + hv_2.\end{aligned}$$

On the other hand, f is an immersion, as the rank of the Jacobian matrix

$$\begin{pmatrix} -\sin u - \cos v & -\cos u \sin v \\ -\sin u \sin v & \cos u - \cos v \\ \cos u - \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u - \cos v \end{pmatrix}$$

is equal to 2, as it is easily seen.

Let j be the inclusion of S^3 in $\mathbb{R}^4$. Then the metric induced on T^2 by the embedding f, if $\widetilde{g}$ denotes the Euclidean metric on $\mathbb{R}^4$, is $f^* j^* \widetilde{g} = f^* \widetilde{g}$.

If (x,y,z,t) denote the coordinates on $\mathbb{R}^4$, then the Euclidean metric is

$$\widetilde{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2$$

and the metric induced on T^2 is $\mathrm{d}u^2 + \mathrm{d}v^2$. Hence $f : T^2 \to f(T^2) \subset S^3$ is an isometric embedding. In fact, as we have seen, it is an isometric immersion, and as T^2 is compact, it is homeomorphic to its image with the induced topology of S^3.

(ii) The Generalised Gauss Theorema Egregium applies in our case to $M = f(T^2)$ and $\widetilde{M} = S^3$. Now, since $\widetilde{M} = S^3$ has constant curvature equal to 1, one has $\widetilde{K}(P) = 1$. And as the metric on $f(T^2)$ is flat, we have $K(P) = 0$. So the equation

$$\widetilde{K}(P) = K(P) - \det L,$$

where L stands for the Weingarten map, reduces to $\det L = -1$. Hence the total embedded curvature is equal to -1.

Problem 6.143 Let M be a Riemannian n-manifold endowed with the metric

$$g = \sum_{i,j=1}^{n-1} g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{nn}\,\mathrm{d}x^n \otimes \mathrm{d}x^n,$$

with the condition $\partial g_{ij}/\partial x^n = 0$. Show that any geodesic in the hypersurface $x^n =$ const is a geodesic in M.

Solution The metric given on M induces the metric

$$\widetilde{g} = \sum_{i,j=1}^{n-1} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

on the given hypersurface S. The geodesics in the hypersurface S are the curves having differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n-1} \widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1,\dots,n-1,$$

where $\widetilde{\Gamma}^i_{jk}$ are the Christoffel symbols of $\widetilde{g}$. We have to prove that the functions $x^1 = x^1(t), \dots, x^{n-1} = x^{n-1}(t)$, $x^n = \text{const}$, satisfy the differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n-1} \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1,\dots,n,$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of g. Consider first the case $i = n$ in the equation of the geodesics. As $x^n = \text{const}$, we have $\frac{\mathrm{d}^2x^n}{\mathrm{d}t^2} = 0$. On the other hand, one has $\Gamma^n_{jk} = 0$ for $j,k = 1,\dots,n-1$, as

$$g \equiv \left(\begin{array}{ccc|c} & & & 0 \\ & g_{ij} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & g_{nn} \end{array}\right),$$

and moreover $\partial g_{ij}/\partial x^n = 0$ by hypothesis. So, by virtue of the condition $x^n = \text{const}$, the functions defining the geodesics of S satisfy the case $i = n$ of the equation of the geodesics of M.

Consider now the cases $i = 1,\dots,n-1$. For $i,j,k = 1,\dots,n-1$, we have

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n-1} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}),$$

as $g^{in} = 0$; that is, $\Gamma^i_{jk} = \widetilde{\Gamma}^i_{jk}$. Finally, if $k = n$ (equivalently, $j = n$), one has

$$\Gamma^i_{jn} = \frac{1}{2}\sum_{l=1}^{n-1} g^{il}(g_{lj,n} + g_{ln,j} - g_{jn,l}) = 0,$$

by the hypotheses. We conclude that the geodesics in S are also geodesics in M.

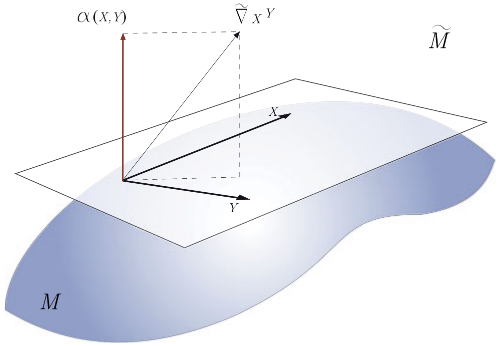


Fig. 6.17 The second fundamental form

Problem 6.144 Let M_1 and M_2 be two hypersurfaces of $\mathbb{R}^n$ and let γ be a common geodesic curve which is not a geodesic of $\mathbb{R}^n$. Prove that M_1 and M_2 are tangent along γ.

Solution Consider the Gauss equations (see Fig. 6.17)

$$\widetilde{\nabla}_X Y = \nabla^i_X Y + \alpha^i(X, Y), \quad i = 1, 2,$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection of the flat metric on $\mathbb{R}^n$, ∇^i the Levi-Civita connection of the metric on the hypersurface M_i, and α^i the second fundamental form of the hypersurface M_i.

Since $\nabla^i_{\gamma'}\gamma' = 0$, $i = 1, 2$, we have that $\widetilde{\nabla}'_\gamma \gamma'$ is normal to both M_1 and M_2. So, at any $p \in \gamma$ we have

$$T_p\mathbb{R}^n = T_p M_1 \oplus \widetilde{\nabla}_{\gamma'}\gamma' = T_p M_2 \oplus \widetilde{\nabla}_{\gamma'}\gamma',$$

hence $T_p M_1 = T_p M_2$.

Problem 6.145 Let x, y, z be the standard coordinate system on Euclidean space $\mathbb{R}^3$. Let a, b, c, d, e be the standard coordinate system on $\mathbb{R}^5$. Let

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^5$$

be the map defined by

$$a = \frac{1}{\sqrt{3}} yz, \qquad b = \frac{1}{\sqrt{3}} zx, \qquad c = \frac{1}{\sqrt{3}} xy,$$
$$d = \frac{1}{2\sqrt{3}}\bigl(x^2 - y^2\bigr), \qquad e = \frac{1}{6}\bigl(x^2 + y^2 - 2z^2\bigr).$$

Show:

(i) The mapping Φ maps the sphere $S^2(\sqrt{3})$ of radius $\sqrt{3}$ isometrically into the sphere $S^4(1)$ of radius 1.

(ii) By observing that (x, y, z) and $(-x, -y, -z)$ map into the same point of $S^4(1)$, show that the induced map Ψ is an embedding of the real projective plane $\mathbb{R}\mathrm{P}^2$ into $S^4(1)$.

(iii) The space $\mathbb{R}\mathrm{P}^2$ is embedded into $S^4(1)$ as a minimal surface.

Hint (to (ii)) Let $\Phi : M \to N$ be a one-to-one immersion. If M is compact, then Φ is an embedding ([20, Chap. 6, Proposition 6.5]).

Hint (to (iii)) Take the Gauss equation for an m-dimensional Riemannian submanifold (M, g) of an n-dimensional space $(N, \tilde{g})$ of constant curvature K (formula (7.9) below), and compute from it the scalar curvature of (M, g). Then consider the present particular case.

The relevant theory is developed, for instance, in Willmore [35, Chap. 4].

Solution

(i) At it is easily checked,

$$a^2 + b^2 + c^2 + d^2 + e^2 = \frac{1}{9}\big(x^2 + y^2 + z^2\big)^2 = 1.$$

We thus have the diagram

$$\begin{array}{ccc} \mathbb{R}^3 & \hookrightarrow & \mathbb{R}^5 \\ {\scriptstyle j_3}\uparrow & & \uparrow{\scriptstyle j_5} \\ S^2(\sqrt{3}) & \overset{\Phi}{\hookrightarrow} & S^4(1), \end{array}$$

j_3, j_5 being the obvious embeddings.

The restriction $g = j_3^*(g_{\mathbb{R}^3})$ of the flat metric

$$g_{\mathbb{R}^3} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

to $S^2(\sqrt{3})$, this with the parametrisation (see Remark 1.4)

$$\begin{aligned} x &= \sqrt{3}\cos\theta\cos\varphi, \\ y &= \sqrt{3}\cos\theta\sin\varphi, \qquad \theta \in (-\pi/2, \pi/2),\ \varphi \in (0, 2\pi), \\ z &= \sqrt{3}\sin\theta, \end{aligned}$$

is easily seen to be

$$g = 3\big(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\varphi^2\big).$$

On the other hand, the restriction $(j_5 \circ \Phi)^*(g_{\mathbb{R}^5})$ of the flat metric

$$g_{\mathbb{R}^5} = \mathrm{d}a^2 + \mathrm{d}b^2 + \mathrm{d}c^2 + \mathrm{d}d^2 + \mathrm{d}e^2$$

on $\mathbb{R}^5$ to the image $\Phi(S^2(\sqrt{3})) \subset S^4(1)$ gives us, after an easy calculation, the same metric g on $S^2(\sqrt{3})$. Hence Φ is indeed an isometric immersion.

(ii) Let τ be the quotient map $\tau\colon S^2(\sqrt{3}) \to \mathbb{R}\mathrm{P}^2$. As $\Phi(x, y, z) = \Phi(-x, -y, -z)$, the map

$$\Psi : \mathbb{R}\mathrm{P}^2 \to S^4(1), \qquad \Psi[x, y, z] = \Phi(x, y, z) = \Phi(-x, -y, -z),$$

is well-defined. Moreover, if

$$\Phi(x, y, z) = \Phi\big(x', y', z'\big), \quad (x, y, z), \big(x', y', z'\big) \in S^2(\sqrt{3}),$$

then

$$yz = y'z', \tag{$\star$}$$

$$zx = z'x', \tag{$\star\star$}$$

$$x\,y = x'y', \tag{$\star\star\star$}$$

$$x^2 - y^2 = x'^2 - y'^2, \tag{$\diamond$}$$

$$x^2 + y^2 - 2z^2 = x'^2 + y'^2 - 2z'^2. \tag{$\diamond\diamond$}$$

As $x'^2 + y'^2 + z'^2 = 3$, one has $(x', y', z') \neq (0, 0, 0)$. If $x' \neq 0$, then from ($\star\star$) and ($\star\star\star$), one obtains

$$z' = \frac{x\,z}{x'}, \qquad y' = \frac{xy}{x'},$$

and replacing these equations into ($\star$), one deduces $(x'^2 - x^2)yz = 0$. If $x'^2 \neq x^2$, then either $y = 0$ or $z = 0$. If $y = 0$, then from ($\diamond$) one obtains $x^2 = x'^2$, thus leading one to a contradiction; if $z = 0$, then by adding ($\diamond$) and ($\diamond\diamond$) one again deduces $x^2 = x'^2$. Hence $x' = \pm x$. Hence, from ($\star\star$) and ($\star\star\star$) one obtains $z' = \pm z$, $y' = \pm y$. The cases $y' \neq 0$ and $z' \neq 0$ are dealt with similarly. Hence Ψ is one-to-one.

On the other hand, since $\Psi \circ \tau = \Phi$ and τ is a local diffeomorphism, Ψ is C^∞. Furthermore, we have

$$\Psi_* = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \\ x & -y & 0 \\ \frac{1}{\sqrt{3}}\,x & \frac{1}{\sqrt{3}}\,y & -\frac{2}{\sqrt{3}}\,z \end{pmatrix}.$$

Hence $\operatorname{rank} \Psi_* \neq 3$ implies $x = y = z = 0$, but $(0, 0, 0) \notin S^2(\sqrt{3})$ hence $\tau(0, 0, 0) \notin \mathbb{R}\mathrm{P}^2$, so Ψ is an immersion. Since Ψ is a one-to-one immersion, it follows from the compactness of $\mathbb{R}\mathrm{P}^2$ that it is an embedding.

(iii) According to the hint, we calculate the scalar curvature $\mathbf{s}$ of such an (M, g). In terms of orthonormal bases, $\{e_i\}$, $i = 1, \ldots, m$, of T_pM and ξ_r, $r = 1, \ldots, n-m$, of $(T_pM)^\perp$, respectively, we have

$$\begin{aligned}
\mathbf{s} &= \sum_{i,j=1}^{m} R(e_i, e_j, e_i, e_j) \\
&= \sum_{i,j=1}^{m} K(g_{ii}g_{jj} - g_{ij}g_{ij}) + \sum_{i,j}\sum_{r,s=1}^{n-m} \left(\tilde{g}\left(\alpha^r_{ii}, \alpha^s_{jj}\right) - \tilde{g}\left(\alpha^r_{ij}, \alpha^s_{ij}\right)\right) \\
&= Km(m-1) + \sum_{i,j=1}^{m}\sum_{r=1}^{n-m}\left(\alpha^r_{ii}\alpha^r_{jj} - \alpha^r_{ij}\alpha^r_{ij}\right) = Km(m-1) + m^2|H|^2 - \ell^2,
\end{aligned}$$

ℓ^2 being the length of the second fundamental form α, so that the norm of the mean curvature vector H at $p \in M$ is given in terms of K, $\mathbf{s}$ and ℓ^2, by

$$|H|^2 = \frac{1}{m^2}\left(\mathbf{s} - Km(m-1) + \ell^2\right). \tag{†}$$

In our case, $m = 2$, $K = 1$, and

$$\mathbf{s}_{\mathbb{R}\mathrm{P}^2} = \mathbf{s}_{S^2(\sqrt{3})} = \sum_{i,j=1}^{2} R(e_i, e_j, e_i, e_j) = \frac{2}{3},$$

so (†) is now written as

$$|H|^2 = \frac{1}{4}\left(\ell^2 - \frac{4}{3}\right).$$

Now, we can choose an orthonormal basis $\{e_1, e_2\}$ at p of principal directions, that is, eigenvectors of the second fundamental tensors A_{ξ_r}, $r = 1, 2$, so that ξ_r, $r = 1, 2$, being an orthonormal basis at p of $(T_p(\mathbb{R}\mathrm{P}^2))^\perp$, we have

$$\begin{aligned}
\ell^2 &= \sum_{i,j=1,2} \tilde{g}\left(\alpha(e_i, e_i),\, \alpha(e_j, e_j)\right) \\
&= \sum_{i,j=1,2}\sum_{r,s=1,2} \tilde{g}\left(\alpha^r(e_i, e_i)\xi_r,\, \alpha^s(e_j, e_j)\xi_s\right) \\
&= \sum_{i,j=1,2}\sum_{r,s=1,2} \tilde{g}\left(g(A_{\xi_r}e_i, e_i)\xi_r,\, g(A_{\xi_s}e_j, e_j)\xi_s\right) \\
&= \sum_{i=1,2}\sum_{r=1,2} \left(g(A_{\xi_r}e_i, e_i)\right)^2 \\
&= \sum_{i=1,2} \left\{\left(g(A_{\xi_1}e_i, e_i)\right)^2 + \left(g(A_{\xi_2}e_i, e_i)\right)^2\right\}
\end{aligned}$$

$$= \sum_{i=1,2} \left\{ k_i^2 \big(g(e_i, e_i) \big)^2 + k_i^2 \big(g(e_i, e_i) \big)^2 \right\}$$

$$= k_1^2 + k_2^2 + k_1^2 + k_2^2 = \frac{4}{3},$$

where the last line follows from the fact that

$$k_1 = k_2 = \frac{1}{r}$$

on $S^2(r)$ (recall that the Gauss curvature of $S^2(r)$ is $K = 1/r^2$, and the same values hold for $\mathbb{R}\mathrm{P}^2$).

Hence $|H|^2 = 0$ or, equivalently, $H = 0$, and $\mathbb{R}\mathrm{P}^2$ is indeed embedded in $\mathbb{R}^5$ as a minimal surface.

6.16 Energy of Hopf Vector Fields

In this section we follow Wood [38]. (We suggest the reader interested in the topic to see also [37] and [17].) We shall denote by TS^3 both the tangent bundle over the 3-sphere S^3 and the total space of that bundle, and similarly for the unit tangent bundle $T_1(S^3)$. The context should in principle avoid confusion: For instance, a metric on TS^3 or $T_1(S^3)$ is obviously a metric on the corresponding total space, but a section is one of the relevant bundle.

Problem 6.146 Consider the standard isometric embedding $(S^3, g) \hookrightarrow (\mathbb{C}^2, \langle \cdot, \cdot \rangle)$ of the 3-sphere with the round metric into $\mathbb{C}^2 \equiv \mathbb{R}^4$ with the flat metric.

(i) Prove that the mean curvature vector field H on S^3 is the inward-pointing unit normal field on S^3,

$$H = -z = -\big(x^1, x^2, x^3, x^4\big),$$

x^1, x^2, x^3, x^4 being the coordinates of $\mathbb{R}^4 \equiv \mathbb{C}^2 = \{z^1 = x^1 + \mathrm{i}x^2,\ z^2 = x^3 + \mathrm{i}x^4\}$.

(ii) A *Hopf vector field* on S^3 is a vector field tangent to the fibres of the Hopf fibration $\pi_{\mathbb{C}} \colon S^3 \to S^2$ (see Problem 5.17). The *standard Hopf vector field U* on S^3 is the restriction of the vector field

$$U_z = \mathrm{i}z \in \mathfrak{X}\big(\mathbb{C}^2\big). \qquad (\star)$$

Prove that U coincides with the vector field ($\diamond$) in Problem 5.28, that is, with

$$X^*_{(z^1,z^2)} = -u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2} - u^4 \frac{\partial}{\partial u^3} + u^3 \frac{\partial}{\partial u^4},$$
$$z^1 = u^1 + \mathrm{i}u^2,\ z^2 = u^3 + \mathrm{i}u^4.$$

(iii) Let ω be the restriction to S^3 of the usual Kähler 2-form ω

$$\langle \mathrm{i}X, Y\rangle, \quad X, Y \in \mathfrak{X}(\mathbb{C}^2),$$

on $\mathbb{C}^2$ and denote by ∇ the Levi-Civita connection of g. Prove that

$$(\nabla_X \omega)(Y, Z) = g(X, Z)g(Y, U) - g(X, Y)g(Z, U), \quad X, Y, Z \in \mathfrak{X}(S^3). \tag{$\star\star$}$$

Hint (to (i)) Recall that the mean curvature vector field at any point $p \in S^3$ is

$$H_p = \frac{1}{3}\sum_{j=1}^{3} \alpha(e_j, e_j),$$

where α stands for the second fundamental form of S^3 and e_1, e_2, e_3 is any orthonormal frame on S^3 at p.

Hint (to (iii)) Apply that S^3 is a totally umbilical hypersurface of $\mathbb{C}^2 \equiv \mathbb{R}^4$, so that (see [26, p. 101]) the *shape tensor* (another usual name for the second fundamental form) is given, as here $r = 1$, by

$$\alpha(X, Y) = -g(X, Y)z. \tag{$\star\star\star$}$$

The reader can find the relevant theory developed, for instance, in Wood [38].

Solution

(i) Let $\widetilde{\nabla}$ denote the flat connection on $\mathbb{R}^4$. We have the usual decomposition into tangential and normal parts,

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^{\tan} + (\widetilde{\nabla}_X Y)^{\mathrm{nor}} = \nabla_X Y + \alpha(X, Y), \quad X, Y \in \mathfrak{X}(S^3).$$

Using the identification $\mathbb{C}^2 \equiv \mathbb{R}^4$, we can write the position vector field on $\mathbb{R}^4$ by $z = (z^1, z^2) = (x^1, x^2, x^3, x^4)$. This is normal to S^3 at each point $p \in S^3$, and (it is immediate that) it satisfies

$$\widetilde{\nabla}_X z = X. \tag{$\star\star\star\star$}$$

Hence we have at any $p \in S^3$ that, e_1, e_2, e_3 being an orthonormal frame on S^3 at p,

$$g(H_p, z) = \frac{1}{3}\sum_{j=1}^{3} g\big(\alpha(e_j, e_j), z\big) = \frac{1}{3}\sum_{j=1}^{3} g\big((\widetilde{\nabla}_{e_j} e_j)^{\mathrm{nor}}, z\big) = \frac{1}{3}\sum_{j=1}^{3} g(\widetilde{\nabla}_{e_j} e_j, z)$$

$$= -\frac{1}{3}\sum_{j=1}^{3} g(e_j, \widetilde{\nabla}_{e_j} z) \quad (\text{as } g(e_j, z) = 0)$$

$$= -\frac{1}{3}\sum_{j=1}^{3} g(e_j, e_j) = -1.$$

So indeed $H = -z$.

(ii) Immediate from the identification $\mathbb{C}^2 \equiv \mathbb{R}^4$.

(iii) We have that

$$\begin{aligned}
(\nabla_X \omega)(Y, Z) &= \nabla_X\big(\omega(Y, Z)\big) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) \\
&= X\big(\omega(Y, Z)\big) + (\iota_Z\omega)(\nabla_X Y) - (\iota_Y\omega)(\nabla_X Z) \\
&= X\big(\omega(Y, Z)\big) + (\iota_Z\omega)(\widetilde{\nabla}_X Y) - (\iota_Z\omega)\big(\alpha(X, Y)\big) \\
&\quad - (\iota_Y\omega)(\widetilde{\nabla}_X Z) + (\iota_Y\omega)\big(\alpha(X, Z)\big) \quad (\text{by Gauss eq.}) \\
&= (\widetilde{\nabla}_X\omega)(Y, Z) - (\iota_Z\omega)\big(\alpha(X, Y)\big) + (\iota_Y\omega)\big(\alpha(X, Z)\big) \\
&= -\omega\big(g(X, Y)z, Z\big) - \omega\big(Y, g(X, Z)z\big) \quad (\text{by } \widetilde{\nabla}\omega = 0 \text{ and } (\star\star\star)) \\
&= g(X, Y)\omega(\mathrm{i}U, Z) - g(X, Z)\omega(\mathrm{i}U, Y) \\
&= -g(X, Y)g(U, Z) + g(X, Z)(U, Y).
\end{aligned}$$

Problem 6.147 Let TS^3 be equipped with the Sasaki metric g (see Definition 6.3). Prove that the energy functional of a unit vector field $U \in \mathfrak{X}(S^3)$ is given by

$$E(U) = \frac{1}{2}\int_{S^3} |\nabla U|^2 v + 3\pi^2,$$

where v denotes the Riemannian volume element.

Hint Viewing U as a map of S^3 to $T_1(S^3)$, the differential map U_* splits into vertical and horizontal components, $U_* = U_*^{\mathrm{v}} + U_*^{\mathrm{h}}$, and since the Sasaki metric renders this an orthogonal decomposition, the energy

$$E(U) = \frac{1}{2}\int_{S^3} |U_*|^2 v,$$

splits accordingly.

The reader can find the relevant theory developed, for instance, in Wood [38].

Solution Let $\pi : T_1(S^3) \to S^3$ be the bundle projection, and let

$$T\big(T_1\big(S^3\big)\big) = \mathscr{V} \oplus \mathscr{H}$$

denote the vertical/horizontal splitting induced by the Levi-Civita connection. Furthermore, write

$$TS^3 = \mathscr{U} \oplus \mathscr{U}^{\perp},$$

where $\mathscr{U}$ denotes the line bundle generated by U, and $\mathscr{U}^{\perp}$ is the orthogonal complement. Thinking of U as a map from S^3 to $T_1(S^3)$, the differential map U_* splits into vertical and horizontal components,

$$U_* = U_*^{\mathrm{v}} + U_*^{\mathrm{h}}.$$

Since the Sasaki metric renders this an orthogonal decomposition, there is a corresponding splitting of the energy of U,

$$E(U) = \frac{1}{2}\int_{S^3} |U_*|^2 v = \frac{1}{2}\int_{S^3} \left|U_*^{\mathrm{v}}\right|^2 v + \frac{1}{2}\int_{S^3} \left|U_*^{\mathrm{h}}\right|^2 v.$$

If

$$\kappa\colon TTS^3 \to TS^3$$

denotes the Levi-Civita connection map (see Definition 5.1), then

$$\left|U_*^{\mathrm{v}}\right|^2 = \left|\kappa \circ U_*^{\mathrm{v}}\right|^2 = |\kappa \circ U_*|^2 = |\nabla U|^2.$$

On the other hand, since π is a Riemannian submersion and U a section, then e_1, e_2, e_3 being an orthonormal frame field on S^3, we have

$$\begin{aligned}\left|U_*^{\mathrm{h}}\right|^2 &= \left|\pi_* \circ U_*^{\mathrm{h}}\right|^2 = |\pi_* \circ U_*|^2 = |\mathrm{id}_{TS^3}|^2 = \sum_{j=1}^{3} g\bigl(\mathrm{id}(e_j), \mathrm{id}(e_j)\bigr) \\ &= \sum_{j=1}^{3} g(e_j, e_j) = 3,\end{aligned}$$

the dimension of S^3.

Therefore, the energy functional is (for $\mathrm{vol}(S^3(1))$ see the table on p. 581 and take $\lambda = 1$)

$$E(U) = \frac{1}{2}\int_{S^3} |\nabla U|^2 v + \frac{3}{2}\,\mathrm{vol}\bigl(S^3\bigr) = \frac{1}{2}\int_{S^3} |\nabla U|^2 v + 3\pi^2.$$

Problem 6.148 Consider TS^3 equipped with the Sasaki metric.

(i) Prove that a unit vector field $U \in \mathfrak{X}(S^3)$ is a harmonic section (see Remark 1 below) of the unit tangent bundle $T_1(S^3)$ if and only if U satisfies the Euler–Lagrange equations (see, for instance, [4])

$$\nabla^*\nabla U = |\nabla U|^2 U, \tag{†}$$

where $\nabla^*\nabla$ is the *trace Laplacian* $\nabla^*\nabla U = -\operatorname{tr}\nabla^2 U$.

(ii) Prove that (†) is equivalent to the similar equation for the 1-form $U^\flat$ metrically dual to U,

$$\nabla^*\nabla U^\flat = \left|\nabla U^\flat\right|^2 U^\flat. \qquad (\dagger\dagger)$$

Hint Let $\{U_t : t \in I\}$ be a one-parameter variation of U through unit vector fields, for some open interval I containing 0, and define

$$\begin{aligned} \Phi\colon S^3 \times I &\longrightarrow T_1\big(S^3\big) \\ (x,t) &\longmapsto \Phi(x,t) = U_t(x). \end{aligned}$$

Let e_1, e_2, e_3 be an orthonormal frame field on S^3, and denote by $\bar{e}_j$ the natural extension of e_j to a vector field on $S^3 \times I$. Then $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \partial/\partial t\}$ is an orthonormal frame field on $S^3 \times I$. Let $\bar{\nabla}$ denote the Levi-Civita connection of $S^3 \times I$, and $\bar{R}$ the curvature tensor; note that $\bar{R}(TS^3, TI) = 0$, since $S^3 \times I$ is a product manifold. Start computing $\frac{\mathrm{d}}{\mathrm{d}t}|\nabla U_t|^2$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}E(U_t)$, $E(U_t)$ being the energy of U_t. Then note that

$$\left.\frac{\mathrm{d}U_t}{\mathrm{d}t}\right|_{t=0} \in \mathfrak{X}\big(S^3\big),$$

and that

$$g\left(\left.\frac{\mathrm{d}U_t}{\mathrm{d}t}\right|_{t=0}, U\right) = 0,$$

since U is a unit vector field.

Remark 1 According to [38], a unit vector field U which is a critical point of the energy functional subject to the present constraints is called simply a *harmonic section of* $T_1(S^3)$. This is weaker than asking for U to be a harmonic section (in the usual sense, i.e. with respect to the Hodge–de Rham Laplacian $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$) of TS^3, as in the latter case U should criticalise energy with respect to variations through all nearby vector fields, not just those of unit length.

Remark 2 The trace Laplacian is also called the *rough* (or *connection) Laplacian*.

The reader can find the relevant theory developed, for instance, in Wood [38].

Solution

(i) According to the hint, we calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\nabla U_t|^2 &= \sum_{j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t} g(\nabla_{e_j} U_t, \nabla_{e_j} U_t) \\ &= \sum_{j=1}^{3} \big(\bar{\nabla}_{\frac{\partial}{\partial t}} g(\bar{\nabla}_{\bar{e}_j}\Phi, \bar{\nabla}_{\bar{e}_j}\Phi)\big)_{t=0} \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{j=1}^{3} g(\bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\bar{e}_j} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi)_{t=0} \\
&= 2\sum_{j=1}^{3} g\big(\bar{\nabla}^2_{\frac{\partial}{\partial t}, \bar{e}_j} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi\big)_{t=0} \quad (\text{as } \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{e}_j = 0) \\
&= 2\sum_{j=1}^{3} g\bigg(\bar{\nabla}^2_{\bar{e}_j, \frac{\partial}{\partial t}} \varPhi + \bar{R}\Big(\frac{\partial}{\partial t}, \bar{e}_j\Big)\varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi\bigg)_{t=0} \quad (\text{by Ricci identity}) \\
&= 2\sum_{j=1}^{3} g\big(\bar{\nabla}^2_{\bar{e}_j, \frac{\partial}{\partial t}} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi\big)_{t=0} \quad (\text{as } \bar{R}(TS^3, TI) = 0).
\end{aligned}
$$

So, according to Problem 6.147, we have that the energy $E(U_t)$ of U_t satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} E(U_t) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \bigg(\frac{1}{2}\int_{S^3} |\nabla U_t|^2\, v + 3\pi^2\bigg) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \bigg(\frac{1}{2}\int_{S^3} |\nabla U_t|^2 v\bigg) \\
&= \frac{1}{2}\int_{S^3} \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |\nabla U_t|^2\bigg) v = \int_{S^3} \sum_{j=1}^{3} g\big(\bar{\nabla}^2_{\bar{e}_j, \frac{\partial}{\partial t}} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi\big)_{t=0} v \\
&= \int_{S^3} \sum_{j=1}^{3} g(\bar{\nabla}_{\bar{e}_j} \bar{\nabla}_{\frac{\partial}{\partial t}} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi)_{t=0} v \\
&= -\int_{S^3} g\bigg(\frac{\partial \varPhi}{\partial t}, \sum_{j=1}^{3} \bar{\nabla}_{\bar{e}_j} \bar{\nabla}_{\bar{e}_j} \varPhi\bigg)_{t=0} v \quad (\text{as } g(\bar{\nabla}_{\frac{\partial}{\partial t}} \varPhi, \bar{\nabla}_{\bar{e}_j} \varPhi) = 0) \\
&= -\int_{S^3} g\bigg(\frac{\mathrm{d}U_t}{\mathrm{d}t}, \sum_{j=1}^{3} \nabla_{e_j} \nabla_{e_j} U_t\bigg)_{t=0} v \\
&= \int_{S^3} g\bigg(\frac{\mathrm{d}U_t}{\mathrm{d}t}, \nabla^* \nabla U_t\bigg)_{t=0} v = \int_{S^3} g\bigg(\frac{\mathrm{d}U_t}{\mathrm{d}t}\bigg|_{t=0}, \nabla^* \nabla U\bigg) v.
\end{aligned}
$$

Now, according to the Fundamental Lemma of the Calculus of Variations, given a connected, compact and oriented manifold M, if

$$
\int_M h_1 h_2\, v = 0, \quad h_1, h_2 \in C^\infty M,
$$

for all h_1, then $h_2 = 0$.

In the present case, given an arbitrary vector field V in the subspace orthogonal to U, there exists a one-parameter variation U_t of U such that

$$
V = \frac{\mathrm{d}U_t}{\mathrm{d}t}\bigg|_{t=0}.
$$

Hence from the formula

$$\int_{S^3} g\big(V, \nabla^*\nabla U\big)v = 0$$

one obtains that $\nabla^*\nabla U$ should be proportional to U. In fact, if this were not so, one could take a V such that the scalar product $g(V, \nabla^*\nabla U)$ would be positive, which is impossible.

Therefore, the Euler–Lagrange equations read

$$\nabla^*\nabla U = fU,$$

for some function f on S^3. Taking $\{e_j\}$ to be at the centre of a system of normal coordinates, it follows that

$$\begin{aligned} f = g\big(\nabla^*\nabla U, U\big) &= -\sum_{j=1}^{3} g(\nabla_{e_j}\nabla_{e_j}U, U) \\ &= \sum_{j=1}^{3}\big(-e_j \cdot g(\nabla_{e_j}U, U) + g(\nabla_{e_j}U, \nabla_{e_j}U)\big) \\ &= |\nabla U|^2 \quad (\text{since } |U|^2 = 1). \end{aligned}$$

(ii) is immediate from (i).

Problem 6.149 Let U be a Hopf vector field on S^3 viewed as a map of Riemannian manifolds

$$U : S^3 \to T_1\big(S^3\big),$$

where $T_1(S^3)$ is endowed with the restriction of the Sasaki metric on TS^3.

Prove that U is a *critical point of the energy functional*, that is, it is a critical point with respect to variations through nearby unit vector fields.

Hint Let ω be the 2-form on S^3 obtained by pulling back the Kähler form of $\mathbb{C}^2$. Prove that $\nabla U^\flat = \omega$ and then that

$$\nabla^*\nabla U^\flat = \delta\omega.$$

The reader can find the relevant theory developed, for instance, in Wood [38].

Solution From Gauss' equation, we have for all $X \in TS^3$ that

$$\begin{aligned} \nabla_X U &= \widetilde{\nabla}_X U - \alpha(X, U) \\ &= \widetilde{\nabla}_X U + g(X, U)z \qquad \text{(by } (\star\star\star) \text{ in Problem 6.146)} \end{aligned}$$

$$= \begin{cases} \widetilde{\nabla}_X U + g(X,U)z & \text{if } X \in \mathscr{U}, \\ \widetilde{\nabla}_X U & \text{if } X \in \mathscr{U}^\perp \end{cases}$$

$$= \begin{cases} \lambda \mathrm{i} \widetilde{\nabla}_U z + \lambda z, & \\ \mathrm{i}\widetilde{\nabla}_X z & \text{if } X \in \mathscr{U}^\perp \end{cases} \qquad \begin{matrix} (\text{as } X = \lambda U) \\ (\text{by } (\star) \text{ in Problem 6.146}) \end{matrix}$$

$$= \begin{cases} \lambda \mathrm{i} U + \lambda z, & \\ \mathrm{i}\widetilde{\nabla}_X z & \text{if } X \in \mathscr{U}^\perp \end{cases} \qquad (\text{see } (\star\star\star\star) \text{ in Problem 6.146})$$

$$= \begin{cases} 0, & \\ \mathrm{i}X & \text{if } X \in \mathscr{U}^\perp. \end{cases}$$

Therefore, if ω denotes the 2-form on S^3 obtained by pulling back the Kähler form of $\mathbb{C}^2$, we have for all $X \in \mathscr{U}^\perp$ that

$$\big(\nabla_X U^\flat\big)(Y) = \nabla_X \langle U, Y\rangle - \langle U, \nabla_X Y\rangle = \langle \nabla_X U, Y\rangle = \langle \mathrm{i}X, Y\rangle = \omega(X,Y).$$

Since moreover both sides of this equation vanish when $X = U$ or $Y = U$ it follows that

$$\nabla U^\flat = \omega.$$

Hence,

$$|\nabla U|^2 = \big|\nabla_X U^\flat\big|^2 = |\omega|^2 = \sum_{i,j=1}^{3} g\big(\omega(e_i,e_j), \omega(e_i,e_j)\big) = 2. \qquad (\diamond)$$

Furthermore, we have

$$\begin{aligned} \big(\nabla^2_{X,Y} U^\flat\big)(Z) &= \big(\nabla_X \nabla_Y U^\flat - \nabla_{\nabla_X Y} U^\flat\big)(Z) \\ &= \nabla_X\big(\big(\nabla_Y U^\flat\big)Z\big) - \big(\nabla_Y U^\flat\big)(\nabla_X Z) - \omega(\nabla_X Y, Z) \\ &= \nabla_X\big(\omega(Y,Z)\big) - \omega(Y, \nabla_X Z) - \omega(\nabla_X Y, Z) = (\nabla_X \omega)(Y,Z), \end{aligned}$$

and consequently,

$$\big(\nabla^* \nabla U^\flat\big)(Z) = -\left(\sum_{j=1}^{3} \nabla_{e_j} \nabla_{e_j} U^\flat\right)(Z) = -\sum_{j=1}^{3} (\nabla_{e_j}\omega)(e_j, Z) = (\delta\omega)(Z). \quad (\diamond\diamond)$$

From formula $(\star\star)$ in Problem 6.146, it follows that

$$\begin{aligned} (\delta\omega)(Z) &= -\sum_{j=1}^{3} (\nabla_{e_j}\omega)(e_j, Z) = -\sum_{j=1}^{3} \big(g(e_j, Z) g(U, e_j) - g(e_j, e_j) g(U, Z)\big) \\ &= 2\, U^\flat(Z). \end{aligned}$$

So from ($\diamond$) and ($\diamond\diamond$), we have

$$\nabla^* \nabla U^\flat = \left|\nabla_X U^\flat\right|^2 U^\flat.$$

Taking into account formula (††) in Problem 6.148, one concludes that U is a harmonic section of $T_1(S^3)$.

6.17 Surfaces in $\mathbb{R}^3$

Problem 6.150 Let

$$S_o^2 = S^2 \setminus \{N \cup S\} \subset \mathbb{R}^3$$

denote the sphere of radius 1 in $\mathbb{R}^3$ excluding the north and south poles. Let g be the metric inherited from the ambient space $\mathbb{R}^3$ and consider on S_o^2 the *Nunes connection*, or *navigator connection*, ∇^{n} defined by the following rule of parallel transport:

A tangent vector at an arbitrary point of S_o^2 is parallel-transported along a curve γ if it determines a vector field on γ such that at any point of γ the angle between the transported vector and the vector tangent to the latitude line passing through that point is constant during the transport.

(i) Prove that ∇^{n} is a metric connection.
(ii) Find the curvature tensor and the torsion tensor of ∇^{n}.

The relevant theory is developed, for instance, in Fernández and Rodrigues [10].

Solution

(i) Consider the spherical coordinates $\psi \in (0, \pi)$, $\varphi \in (0, 2\pi)$, on S^2 (see Remark 1.4). The metric inherited from that of the ambient Euclidean metric on $\mathbb{R}^3$ is given by

$$g = \mathrm{d}\psi \otimes \mathrm{d}\psi + \sin^2 \psi \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi,$$

or, in terms of the basis of differential 1-forms

$$\theta^1 = \mathrm{d}\psi, \qquad \theta^2 = \sin \psi \, \mathrm{d}\varphi,$$

dual to the orthonormal basis of vector fields

$$e_1 = \frac{\partial}{\partial \psi}, \qquad e_2 = \frac{1}{\sin \psi} \frac{\partial}{\partial \varphi},$$

by

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2.$$

The vector fields e_1, e_2 define an orthonormal basis of $T_p(S_o^2)$ for each point $p \in S_o^2$, and ∇^{n} is clearly characterized by

$$\nabla^{\mathrm{n}}_{e_i} e_j = 0.$$

Hence, for all $i, j, k = 1, 2$, we have

$$\begin{aligned} 0 = \nabla^{\mathrm{n}}_{e_k} g(e_i, e_j) &= \big(\nabla^{\mathrm{n}}_{e_k} g\big)(e_i, e_j) + g\big(\nabla^{\mathrm{n}}_{e_k} e_i, e_j\big) + g\big(e_i, \nabla^{\mathrm{n}}_{e_k} e_j\big) \\ &= \big(\nabla^{\mathrm{n}}_{e_k} g\big)(e_i, e_j). \end{aligned}$$

(ii) The possibly nonzero components of the Riemann–Christoffel curvature and torsion tensors are

$$\begin{aligned} R^{\mathrm{n}}(e_1, e_2, e_1, e_2) &= g\big(R^{\mathrm{n}}_{e_1, e_2} e_2, e_1\big) = g\big(\big(\nabla^{\mathrm{n}}_{e_1} \nabla^{\mathrm{n}}_{e_2} - \nabla^{\mathrm{n}}_{e_2} \nabla^{\mathrm{n}}_{e_1} - \nabla^{\mathrm{n}}_{[e_1, e_2]}\big) e_2, e_1\big) \\ &= 0, \\ T^{\mathrm{n}}(e_1, e_2) &= \nabla^{\mathrm{n}}_{e_1} e_2 - \nabla^{\mathrm{n}}_{e_2} e_1 - [e_1, e_2] = (\cot \psi)\, e_2. \end{aligned}$$

That is, ∇^{n} is flat but has non-zero torsion.

Remark P.S. Nunes (1502–1578) discovered the loxodromic curves and advocated the drawing of maps in which loxodromic spirals would appear as straight lines. This led to the celebrated *Mercator projection*, constructed along these recommendations (see Fernández and Rodrigues [10]).

Problem 6.151 Let

$$\mathbf{x} \colon U = (0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2 \to \mathbb{R}^3$$

be the parametrisation of S^2 (see Remark 1.4) given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta).$$

(i) Find the equation of the *loxodromic curves* (that is, the curves meeting the meridians at a constant angle) in the coordinate neighbourhood $V = \mathbf{x}(U)$.

(ii) Prove that a new parametrisation of the coordinate neighbourhood V is given by

$$\mathbf{y}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

Find the expression of the metric on S^2 in terms of the coordinates u, v, and conclude that $\mathbf{y}^{-1} \colon V \subset S^2 \to \mathbb{R}^2$ is a conformal map transforming the meridians and parallels of S^2 into straight lines of the plane. This map is called the *Mercator projection*.

(iii) Consider a triangle on the unit sphere S^2 whose sides are segments of loxodromic curves without any of the poles. Prove that the sum of the internal angles of such a triangle is π.

For the relevant theory of this problem and other similar in this chapter, see do Carmo [5].

Solution

(i) The metric inherited on S^2 from the Euclidean metric on $\mathbb{R}^3$ is given by $g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. A loxodromic curve $\sigma(t)$ can be taken as the image under $\mathbf{x}$ of a curve $(\theta(t), \varphi(t))$ in the plane $\theta\varphi$. At the point $\mathbf{x}(\theta, \varphi)$ where the curve meets the meridian $\varphi = \text{const}$ at the angle, say, β we thus have

$$\cos\beta = \frac{g(\mathbf{x}_\theta, \sigma'(t))}{|\mathbf{x}_\theta||\sigma'(t)|} = \frac{g(\mathbf{x}_\theta, \theta'(t)\mathbf{x}_\theta + \varphi'(t)\mathbf{x}_\varphi)}{|\sigma'(t)|} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta\,\varphi'^2}}.$$

From this one easily obtains $\tan^2\beta = \sin^2\theta\,\varphi'^2/\theta'^2$. Thus

$$\frac{\theta'}{\sin\theta} = \pm\cot\beta\,\varphi'.$$

Integrating, we obtain the equation of the loxodromic curves

$$\log\tan\frac{\theta}{2} = \pm\cot\beta(\varphi + A).$$

The integration constant A is determined when a point in the curve is given.

(ii) It is immediate that the image points belong to S^2. The metric inherited from the Euclidean metric on $\mathbb{R}^3$ is now

$$\operatorname{sech}^2 u\left(\mathrm{d}u^2 + \mathrm{d}v^2\right).$$

The map $\mathbf{y}$ is a diffeomorphism which is clearly conformal. The meridians and parallels are the images of the coordinate lines $v = \text{const}$ and $u = \text{const}$, respectively.

The fact that the Mercator projection $\mathbf{y}^{-1}$ is conformal has been useful in cartography, since the angles are preserved.

(iii) Under the Mercator projection, the meridians are transformed into parallel straight lines of the plane. As the Mercator projection is conformal, the loxodromic curves are also transformed into straight lines. So, the asked sum is the same as that for a plane triangle.

Problem 6.152 Prove that if two families of geodesics on a surface of $\mathbb{R}^3$ cut at a constant angle, the surface is developable.

Solution Consider those families as local coordinate curves (u, v), and let X_u, X_v be the respective coordinate vector fields. Thus $[X_u, X_v] = 0$ and $\nabla_{X_u}X_u = \nabla_{X_v}X_v = 0$, where ∇ denotes the Levi-Civita connection of the metric g on the surface, inherited from the Euclidean metric on $\mathbb{R}^3$. Hence $\nabla_{X_u}X_v = \nabla_{X_v}X_u$. As

$|X_u|, |X_v|$ are constant, it follows by the hypothesis of constant angle, say β, that one has

$$g(X_u, X_v) = |X_u|\,|X_v| \cos\beta = \text{const}.$$

Thus,

$$g(\nabla_{X_u} X_u, X_v) + g(X_u, \nabla_{X_u} X_v) = g(X_u, \nabla_{X_u} X_v) = 0.$$

Similarly $g(X_v, \nabla_{X_u} X_v) = 0$. So ∇ is flat, thus the Gauss curvature is zero, hence the surface is developable.

Problem 6.153 Consider a surface of revolution around the z-axis in $\mathbb{R}^3$, the vector field $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ tangent to the parallels of the surface, and a unit vector field Y on that surface. Show that if $g(X, Y) = \text{const}$, where g denotes the metric on the surface, inherited from the Euclidean metric on $\mathbb{R}^3$, then Y is invariant by X, that is, $L_X Y = 0$.

Solution We have

$$(L_X g)(X, Y) + g\big([X, X], Y\big) + g\big(X, [X, Y]\big) = 0,$$

but $L_X g = 0$ since X is the infinitesimal generator of the group of rotations; so that we have $g(X, [X, Y]) = 0$. On the other hand, as $g(Y, Y) = 1$, one has

$$(L_X g)(Y, Y) + g\big([X, Y], Y\big) + g\big(Y, [X, Y]\big) = 0,$$

that is, $g(Y, [X, Y]) = 0$. Therefore, $[X, Y] = L_X Y = 0$.

Problem 6.154 Consider the following surfaces in $\mathbb{R}^3$:

(a) The *catenoid* C with parametric equations (see Remark 1.4)

$$x = \cos\alpha \cosh\beta, \qquad y = \sin\alpha \cosh\beta, \qquad z = \beta, \quad \alpha \in (0, 2\pi),\ \beta \in \mathbb{R},$$

that is, the surface of revolution obtained rotating the curve $x = \cosh z$ around the z-axis.

(b) The *helicoid* H with parametric equations

$$x = u\cos v, \qquad y = u\sin v, \qquad z = v, \quad u, v \in \mathbb{R},$$

generated by one straight line parallel to the plane xy that intersects with the z-axis and the helix $x = \cos t$, $y = \sin t$, $z = t$ (see Fig. 6.18).

Let $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ be the Euclidean metric of $\mathbb{R}^3$ and denote by $i\colon C \hookrightarrow \mathbb{R}^3$ and $j\colon H \hookrightarrow \mathbb{R}^3$ the respective inclusion maps.

1. Compute i^*g and j^*g.
2. Prove that (C, i^*g) and (H, j^*g) are locally isometric. Are they isometric?

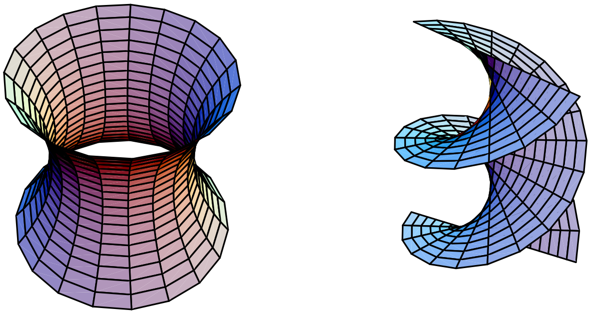


Fig. 6.18 The catenoid (*left*). The helicoid (*right*)

Solution

(i)

$$i^* g = \cosh^2 \beta \left(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2\right), \qquad j^* g = \mathrm{d}u^2 + \left(1 + u^2\right) \mathrm{d}v^2.$$

(ii) The coefficient $1 + u^2$ of $\mathrm{d}v^2$ in $j^* g$ suggests that we try the change $u = \sinh \beta$, $v = \beta$, which is only a local isometry. There is a global isometry of the catenoid with the open submanifold of the helicoid corresponding to any interval $v \in (2k\pi, 2(k+1)\pi)$, $k \in \mathbb{Z}$.

Problem 6.155 Let S be a surface of $\mathbb{R}^3$ with the metric induced from that of $\mathbb{R}^3$. Say if the following statements are true or not:

(i) The geodesics of S are the intersections of S with the planes of $\mathbb{R}^3$, and conversely.
(ii) The geodesics of S are obtained intersecting S with some chosen planes.

Solution

(i) No. For example, the geodesics of S^2 are only obtained when the plane goes through the origin.
(ii) No. For example, the helices in the cylinder are not obtained in such a way.

Problem 6.156 Prove that there is no Riemannian metric on the torus $T^2 = S^1 \times S^1$ with Gauss curvature either $K > 0$ in all points or $K < 0$ in all points.

Hint Use the Gauss–Bonnet Theorem.

Solution The Gauss–Bonnet Theorem establishes that for a connected, compact and oriented 2-dimensional Riemannian manifold, one has

$$\int_M K = 2\pi \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M. On the torus, since $\chi(T^2) = 0$, we have $\int_{T^2} K = 0$, and thus it follows that it is not possible either to be $K > 0$ for all $p \in T^2$, or $K < 0$ for all $p \in T^2$.

Problem 6.157 Determine the volume form for the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^3$ on the unit sphere S^2 in $\mathbb{R}^3$, in terms of spherical coordinates (ρ, θ, φ) with $\rho = 1$. Compute the volume $\operatorname{vol}(S^2)$.

Solution The sphere S^2 with radius $\rho = 1$ can be parametrised (see Remark 1.4) as

$$x = \sin\theta \cos\varphi, \qquad y = \sin\theta \sin\varphi, \qquad z = \cos\theta, \quad \theta \in (0, \pi),\ \varphi \in (0, 2\pi).$$

Hence, the metric induced on S^2 by the metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ of $\mathbb{R}^3$ is $\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$. The volume element is

$$v = \sqrt{\det(g_{ij})}\, \mathrm{d}\theta \wedge \mathrm{d}\varphi = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi,$$

and

$$\operatorname{vol}\big(S^2\big) = \int_{S^2} v = \int_{S^2} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi = \int_0^{2\pi} \left(\int_0^{\pi} \sin\theta\, \mathrm{d}\theta \right) \mathrm{d}\varphi = 4\pi.$$

Problem 6.158 Compute the volume form for the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$ on the torus T^2 on $\mathbb{R}^3$ obtained by rotating a circle with radius a and centre $(b, 0, 0)$, $b > a > 0$, around the z-axis. Determine the volume $\operatorname{vol}(T^2)$.

Solution T^2 can be parametrised (see Remark 1.4) as

$$x = (R + r\cos\varphi)\cos\theta, \qquad y = (R + r\cos\varphi)\sin\theta, \qquad z = r\sin\varphi,$$

$\varphi \in (0, 2\pi)$, $\theta \in (0, 2\pi)$, $R > r$. Hence, the metric induced by the metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$ is

$$g = r^2\, \mathrm{d}\varphi^2 + (R + r\cos\varphi)^2\, \mathrm{d}\theta^2,$$

the volume form is

$$v = \sqrt{g_{11}g_{22} - g_{12}^2}\, \mathrm{d}\varphi \wedge \mathrm{d}\theta = r(R + r\cos\varphi)\, \mathrm{d}\varphi \wedge \mathrm{d}\theta,$$

and the volume is

$$\begin{aligned}\operatorname{vol}\big(T^2\big) &= \int_{T^2} v = r\int_{T^2} (R + r\cos\varphi)\, \mathrm{d}\varphi \wedge \mathrm{d}\theta = r\int_0^{2\pi}\!\!\int_0^{2\pi} (R + r\cos\varphi)\, \mathrm{d}\varphi\, \mathrm{d}\theta \\ &= 4\pi^2 Rr.\end{aligned}$$

Problem 6.159

(i) Consider the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Prove that the map induced on T^2 by the map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$\Phi(x, y) = \frac{1}{2\pi}(\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y),$$

is an isometric embedding of T^2 in $\mathbb{R}^4$.

(ii) Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^4$ be the map

$$\Psi(x, y, z) = \bigl(x^2 - y^2, xy, xz, yz\bigr).$$

Since $\Psi(-x, -y, -z) = \Psi(x, y, z)$, by restricting Ψ to the sphere $S^2 \subset \mathbb{R}^3$ and passing to the quotient, Ψ induces a map from the projective plane $\mathbb{R}\mathrm{P}^2 = S^2/\!\sim$ into $\mathbb{R}^4$. Prove that this map is an embedding.

(iii) Compute the length of the circles $z = \text{const}$ on S^2 with respect to the metric

$$g = \Psi^*\bigl(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^4 \otimes \mathrm{d}x^4\bigr)\big|_{S^2}.$$

(iv) Prove that (S^2, g), where g is the metric in (iii), is not isometric to S^2 with the standard round metric. (Actually, it is not even homothetic.)

Solution

(i) Let $\tau : \mathbb{R}^2 \to T^2$ denote the quotient map. Since

$$\Phi(x + m, y + n) = \Phi(x, y), \qquad m, n \in \mathbb{Z},$$

the map

$$\varphi\colon T^2 \to \mathbb{R}^4, \qquad \varphi(p) = \Phi(q), \quad q \in \tau^{-1}(p),$$

is well-defined. Since $\varphi \circ \tau = \Phi$ and $\tau : \mathbb{R}^2 \to T^2$ is a local diffeomorphism, φ is C^∞. Moreover,

$$\operatorname{rank} \varphi_* = \operatorname{rank} \Phi_* = \operatorname{rank} \begin{pmatrix} -2\pi \sin 2\pi x & 0 \\ 2\pi \cos 2\pi x & 0 \\ 0 & -2\pi \sin 2\pi y \\ 0 & 2\pi \cos 2\pi y \end{pmatrix} = 2.$$

Hence φ is an immersion. Let us see that it is isometric. We have, putting $\mathbb{R}^4 = \{(x^1, x^2, x^3, x^4)\}$,

$$\Phi^*\bigl(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^4 \otimes \mathrm{d}x^4\bigr) = \mathrm{d}x^2 + \mathrm{d}y^2.$$

From the compactness of T^2 it follows that φ is an embedding. Hence, φ is an isometric embedding.

(ii) As $\Psi(p) = \Psi(-p)$, the restriction of Ψ (again denoted by Ψ) to the unit sphere with centre at the origin of $\mathbb{R}^3$ induces a map $\psi\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4$ with $\psi(\tilde{p}) = \Psi(p)$, where $\tilde{p}$ denotes the class of p in $\mathbb{R}\mathrm{P}^2$. Let us see that Ψ (hence ψ) is an immersion. We have

$$\Psi_* = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}.$$

Hence rank $\Psi_* \neq 3$ implies $x = y = 0$. Given any $X \in T_{(0,0,\pm 1)}S^2$, then $X = {}^t(a, b, 0)$ and if $\Psi_* X = \pm\, {}^t(0, 0, a, b) = 0$ then $X = 0$. Thus, if $j\colon S^2 \to \mathbb{R}^3$ denotes the inclusion map, then $\Psi \circ j$ is an immersion.

The tangent bundle $T\mathbb{R}\mathrm{P}^2$ can be defined as the set

$$T\mathbb{R}\mathrm{P}^2 = \big\{\big((q, Y), (-q, -Y)\big),\ q \in S^2,\ Y \in T_q S^2\big\}$$

endowed with the differentiable structure inherited from the usual one of TS^2. Thus, from the diagram

$$\begin{tikzcd} \mathbb{R}\mathrm{P}^2 \arrow[r, "\psi"] & \mathbb{R}^4 \\ S^2 \arrow[u, "\pi"] \arrow[r, "j"] & \mathbb{R}^3 \arrow[u, "\Psi"'] \end{tikzcd}$$

we conclude that ψ is an immersion.

On the other hand, ψ is injective, as it follows from calculation, due to the condition $x^2 + y^2 + z^2 = 1$. From the compactness of $\mathbb{R}\mathrm{P}^2$, it follows that ψ is an embedding.

(iii) Consider the parametrisation of the sphere (see Remark 1.4)

$$\begin{aligned} &x = \cos\theta\cos\varphi, \qquad y = \cos\theta\sin\varphi, \\ &z = \sin\theta, \quad -\pi/2 < \theta < \pi/2,\ 0 < \varphi < 2\pi. \end{aligned}$$

As a simple computation shows, we have

$$\begin{aligned} g &= \big((2x\,\mathrm{d}x - 2y\,\mathrm{d}y)^2 + (x\,\mathrm{d}y + y\,\mathrm{d}x)^2 + (x\,\mathrm{d}z + z\,\mathrm{d}x)^2 + (y\,\mathrm{d}z + z\,\mathrm{d}y)^2\big)\big|_{S^2} \\ &= \left(1 - \frac{3}{4}\sin^2 2\theta\sin^2 2\varphi\right)\mathrm{d}\theta^2 + \frac{3}{2}\cos^2\theta\sin 2\theta\sin 4\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &\quad + \cos^2\theta\big(1 + 3\cos^2\theta\sin^2 2\varphi\big)\,\mathrm{d}\varphi^2. \end{aligned}$$

The length with respect to g of the circle C defined by $\theta = \theta_0$ is

$$l_g(C) = \cos\theta_0 \int_0^{2\pi} \sqrt{1 + 3\cos^2\theta_0\sin^2 2\varphi}\,\mathrm{d}\varphi.$$

Making the change of variables

$$t = 2\varphi - \frac{\pi}{2},$$

we obtain $l_g(C)$ in terms of an elliptic integral of the second kind,

$$l_g(C) = \cos\theta_0\sqrt{1 + 3\cos^2\theta_0} \int_{-\pi/2}^{7\pi/2} \sqrt{1 - k\sin^2 t}\,\mathrm{d}t,$$

where

$$k = \frac{\sqrt{3}\cos\theta_0}{\sqrt{1+3\cos^2\theta_0}}.$$

(iv) The explicit expression of the Gauss curvature $K = K(\theta, \varphi)$ obtained by using the formula for the Gauss curvature of an abstract parametrised surface on p. 597 is rather long, but, as a simple computation shows, K is not constant. In fact, we have

$$K\left(\frac{\pi}{4}, 0\right) = -2\cos\left(\frac{\pi}{4}\right)^2 + 9\cos\left(\frac{\pi}{4}\right)^4 - 3\cos\left(\frac{\pi}{4}\right)^8 + 3\cos\left(\frac{\pi}{4}\right)^{10} = \frac{37}{32},$$
$$K\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1973}{2048}, \qquad K\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = -\frac{16067107}{48234496}.$$

This proves that g is not isometric to the round metric.

6.18 Pseudo-Riemannian Manifolds

Problem 6.160 Consider $M = \mathbb{R}^2 \setminus \{0\}$ equipped with the metric

$$g = \frac{\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x}{x^2 + y^2}.$$

The multiplication by any nonzero real scalar is an isometry of M. Consider, in particular, the isometry $\lambda(x, y) = (2x, 2y)$. The group $\Gamma = \{\lambda^n : n \in \mathbb{Z}\}$ generated by λ acts properly discontinuously. Hence $T = M/\Gamma$ is a Lorentz surface. Topologically, T is the closed ring $1 \leqslant r \leqslant 2$ with the points of the boundary identified by λ. Consequently T is a torus, named the *Clifton–Pohl torus*; in particular, it is compact.

(i) Show that T is not complete. According to [26, p. 202], it suffices to prove that M is not complete. For this, prove that the curve

$$\sigma(t) = \left(\frac{1}{1-t}, 0\right)$$

is a geodesic.

(ii) Find a group of eight isometries and anti-isometries of M.

(iii) Prove that $s \to (\tan s, 1)$ is a geodesic, and deduce that every null geodesic of M and T is incomplete.

(iv) Prove that $X = x\,\partial/\partial x + y\,\partial/\partial y$ is a Killing vector field on M.

(v) If $\sigma(s) = (x(s), y(s))$ is a geodesic, then if $r^2 = x^2 + y^2$, prove that $\dot{x}\dot{y}/r^2$ and $(x\dot{y} + y\dot{x})/r^2$ are constant.

(vi) Show that the curve $\beta\colon s \to (s, 1/s)$ is a pregeodesic of finite length on $[1, \infty)$. (A *pregeodesic* is a curve that becomes a geodesic by a reparametrization.)

Remark This example shows that for pseudo-Riemannian manifolds compactness does not imply completeness.

The relevant theory is developed, for instance, in O'Neill [26, Chaps. 7, 9].

Solution

(i) We have

$$g = \begin{pmatrix} 0 & \frac{1}{x^2+y^2} \\ \frac{1}{x^2+y^2} & 0 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 0 & x^2+y^2 \\ x^2+y^2 & 0 \end{pmatrix},$$

so the only non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = -\frac{2x}{x^2+y^2}, \qquad \Gamma^2_{22} = -\frac{2y}{x^2+y^2},$$

and the differential equations of the geodesics are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{2x}{x^2+y^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{2y}{x^2+y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0,$$

which are easily seen to be satisfied by the given curve. The given geodesic is not defined for $t = 1$, hence M is not complete.

(ii)

$$\begin{array}{lll} (x,y) \mapsto (x,y); & (x,y) \mapsto (-x,-y); & (x,y) \mapsto (-x,y); \\ (x,y) \mapsto (x,-y); & (x,y) \mapsto (y,x); & (x,y) \mapsto (-y,x); \\ (x,y) \mapsto (y,-x); & (x,y) \mapsto (-y,-x). & \end{array}$$

(iii) We have

$$\frac{2\sin s}{\cos^3 s} - \frac{2\tan s}{\tan^2 s + 1}\frac{1}{\cos^4 s} = 0,$$

and the other equation of geodesics is trivially satisfied, for $y = 1$. The geodesic is incomplete because it is not defined for $\pm\pi/2$. The null curves are the ones satisfying

$$\sum_{i,j=1}^{2} g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s} = 0,$$

that is,

$$\frac{2}{x^2+y^2}\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}y}{\mathrm{d}s} = 0,$$

which are the curves $x = \text{const}$ or $y = \text{const}$. Due to the symmetry in x and y of the equations of geodesics, we can suppose $y = \text{const}$. Then the only equation is

$$\frac{\ddot{x}}{\dot{x}} = \frac{2x\dot{x}}{x^2+1},$$

so $\log \dot{x} = \log A(x^2+1)$, thus $\arctan x = As + B$, that is, $x = \tan(As+B)$. As the geodesic

$$s \mapsto \bigl(\tan(As+B), 1\bigr)$$

is a model for the null geodesics, it follows that these are incomplete for M. So they are also incomplete for T.

(iv)

$$L_X g = L_{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}} \frac{1}{x^2+y^2}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) = 0.$$

(v)

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{x}\dot{y}}{r^2} = \frac{1}{2}g(\dot{\sigma},\dot{\sigma}) = \text{const}.$$

As for $(x\dot{y} + y\dot{x})/r^2$, we have on account of the differential equations of the geodesics:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x\dot{y}+y\dot{x}}{r^2}\right) &= \frac{1}{r^4}\bigl(x^3\ddot{y} + x^2\ddot{x}y + x\,y^2\ddot{y} + y^3\ddot{x} - 2x\,y\dot{y}^2 - 2x\,y\dot{x}^2\bigr)\\ &= \frac{1}{r^4}\left(x^3\frac{2y\dot{y}^2}{r^2} + x^2\frac{2x\dot{x}^2}{r^2}y + xy^2\frac{2y\dot{y}^2}{r^2}\right.\\ &\quad\left. + y^3\frac{2x\dot{x}^2}{r^2} - 2x\,y\dot{y}^2 - 2x\,y\dot{x}^2\right) = 0.\end{aligned}$$

(vi) We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{x'y'' - x''y'}{x'^3}.$$

From the equations of the geodesics given in (i) it follows that

$$\bigl(x'y'' - x''y'\bigr)\bigl(x^2+y^2\bigr) = 2x'y'\bigl(yy' - xx'\bigr).$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\bigl(x^2+y^2\bigr) = 2\frac{\mathrm{d}y}{\mathrm{d}x}\left(y\frac{\mathrm{d}y}{\mathrm{d}x} - x\right), \tag{$\star$}$$

which are the equations of the geodesics for any parameter. The condition ($\star$) is satisfied if $y = 1/x$, as it is easily seen.

The tangent vector along the curve is

$$\dot{\beta}(s) = \frac{\partial}{\partial x}\bigg|_{\beta(s)} - \frac{1}{s^2}\frac{\partial}{\partial y}\bigg|_{\beta(s)},$$

hence

$$\left|\dot{\beta}(s)\right| = \sqrt{\left|g\big(\dot{\beta}(s), \dot{\beta}(s)\big)\right|} = \frac{\sqrt{2}}{\sqrt{1+s^4}}.$$

Since $1 + s^{-4} \geqslant 1$ for $s > 0$, we obtain

$$\int_1^\infty \frac{\sqrt{2}\,\mathrm{d}s}{\sqrt{1+s^4}} = \int_1^\infty \frac{\sqrt{2}s^{-2}\,\mathrm{d}s}{\sqrt{1+s^{-4}}} \leqslant \int_1^\infty \sqrt{2}s^{-2}\,\mathrm{d}s = \sqrt{2}.$$

Problem 6.161 Consider on $\mathbb{R}^6$ the scalar product

$$\langle\cdot,\cdot\rangle = \mathrm{d}x^1\otimes\mathrm{d}x^3 + \mathrm{d}x^3\otimes\mathrm{d}x^1 + \mathrm{d}x^2\otimes\mathrm{d}x^4 + \mathrm{d}x^4\otimes\mathrm{d}x^2 + \mathrm{d}x^5\otimes\mathrm{d}x^5 + \mathrm{d}x^6\otimes\mathrm{d}x^6,$$

and the tensor of type $(1, 1)$ given by

$$J = \frac{\partial}{\partial x^1}\otimes\mathrm{d}x^1 + \frac{\partial}{\partial x^2}\otimes\mathrm{d}x^2 - \frac{\partial}{\partial x^3}\otimes\mathrm{d}x^3 - \frac{\partial}{\partial x^4}\otimes\mathrm{d}x^4 + \frac{\partial}{\partial x^6}\otimes\mathrm{d}x^5 - \frac{\partial}{\partial x^5}\otimes\mathrm{d}x^6.$$

(i) Let $W = \langle\frac{\partial}{\partial x^i}\rangle_{i=2,\dots,6}$. Calculate $W^\perp = \{v \in \mathbb{R}^6 : v \perp W\}$.
(ii) Let $W = \langle\frac{\partial}{\partial x^i}\rangle_{i=3,\dots,6}$. Calculate $W^\perp$.
(iii) Do we have $\dim W + \dim W^\perp = 6$ in (i) and (ii)?
(iv) Let $U = \langle\frac{\partial}{\partial x^i}\rangle_{i=1,2}$ and $V = \langle\frac{\partial}{\partial x^i}\rangle_{i=3,4}$. Prove that $JX = X$, $X \in U$ and $JX = -X$, $X \in V$.
(v) Calculate a vector $X \notin U \cup V$ such that $\langle JX, JX\rangle = 0$.

Solution

(i) $W^\perp = \langle\partial/\partial x^3\rangle$.
(ii) $W^\perp = \langle\partial/\partial x^3, \partial/\partial x^4\rangle$.
(iii) Yes.
(iv) Immediate.
(v) Take, for instance, $X = (1, 0, 1, 0, 0, \sqrt{2})$. Then

$$\langle JX, JX\rangle = \big\langle(1, 0, -1, 0, -\sqrt{2}, 0),\ (1, 0, -1, 0, -\sqrt{2}, 0)\big\rangle = 0.$$

Problem 6.162 Consider the *pseudo-Euclidean space* $\mathbb{R}^n_k$, that is, $\mathbb{R}^n$ with the pseudo-Euclidean metric of signature $(k, n-k)$:

$$g = -\sum_{i=1}^k \mathrm{d}x^i\otimes\mathrm{d}x^i + \sum_{i=k+1}^n \mathrm{d}x^i\otimes\mathrm{d}x^i. \qquad (\star)$$

Compute the isometry group $I(\mathbb{R}^n_k)$ of $\mathbb{R}^n_k$. For this prove:

(i) The linear isometries of $\mathbb{R}^n_k$ (i.e. the isometries of $\mathbb{R}^n_k$ which belong to $\mathrm{GL}(n,\mathbb{R})$) form a subgroup $\mathrm{O}(k,n-k)$ of $I(\mathbb{R}^n_k)$.
(ii) The set $T(n)$ of all translations of $\mathbb{R}^n_k$ is an Abelian subgroup of $I(\mathbb{R}^n_k)$ and it is isomorphic to $\mathbb{R}^n$ (under vector addition) via $\tau_x \leftrightarrow x$.
(iii) Each isometry φ of $\mathbb{R}^n_k$ has a unique expression as $\tau_x \circ A$, with $x \in \mathbb{R}^n_k$ and $A \in \mathrm{O}(k,n-k)$.
(iv) The composition law in $I(\mathbb{R}^n_k)$ is

$$(\tau_x \circ A)(\tau_y \circ B) = \tau_{x+Ay} \circ AB.$$

Hint (to (iii)) Suppose first $\varphi(0) = 0$.

The reader can find the relevant theory developed, for instance, in O'Neill [26, Chap. 9].

Solution

(i) The group $\mathrm{O}(k,n-k)$ of linear isometries of $\mathbb{R}^n_k$ can be viewed as the subgroup of matrices of $\mathrm{GL}(n,\mathbb{R})$ which preserve the scalar product

$$\langle v, w\rangle = -\sum_{i=1}^{k} v^i w^i + \sum_{i=k+1}^{n} v^i w^i, \qquad v, w \in \mathbb{R}^n.$$

(ii) Given $x_0 \in \mathbb{R}^n_k$, from ($\star$) one has that the translation τ_{x_0} sending each $v \in \mathbb{R}^n_k$ to $v + x_0$ is an isometry. It is clear that $T(n)$ is an Abelian subgroup of $I(\mathbb{R}^n_k)$ isomorphic to $\mathbb{R}^n$.
(iii) If $\varphi(0) = 0$, then the differential φ_{*0} at 0 is a linear isometry, hence it corresponds under the canonical linear isometry $T_0\mathbb{R}^n_k \cong \mathbb{R}^n_k$ to a linear isometry $A\colon \mathbb{R}^n_k \to \mathbb{R}^n_k$. But then $A_{*0} = \varphi_{*0}$ and thus $\varphi = A$ by Theorem 6.22.
Now, if $\varphi \in I(\mathbb{R}^n_k)$, let $x = \varphi(0) \in \mathbb{R}^n_k$. Thus $(\tau_{-x} \circ \varphi)(0) = 0$, so that by the above results, $\tau_{-x} \circ \varphi$ equals some $A \in \mathrm{O}(k,n-k)$. Hence $\varphi = \tau_x \circ A$.
If $\tau_x \circ A = \tau_y \circ B$, then $x = (\tau_x \circ A)(0) = (\tau_y \circ B)(0) = y$, hence also $A = B$.
(iv) Immediate.

Problem 6.163

(i) Find the exponential map for $\mathbb{R}^n_k$.
(ii) Is $\exp_p$, for $p \in \mathbb{R}^n_k$, a diffeomorphism?
(iii) Is $\exp_p$ an isometry when $T_p\mathbb{R}^n_k$ has the metric induced by the canonical diffeomorphism $T_p\mathbb{R}^n_k \cong \mathbb{R}^n_k$?

The reader can find the relevant theory developed, for instance, in O'Neill [26, Chap. 3].

Solution

(i) The geodesic $\gamma(t)$ through p with initial velocity vector $v_p \in T_p\mathbb{R}^n_k$ is the straight line $\gamma(t) = p + tv$. Thus

$$\begin{array}{rccc} \exp_p \colon & T_p\mathbb{R}^n_k & \to & \mathbb{R}^n_k \\ & v_p & \mapsto & \gamma(1) = p + v. \end{array}$$

(ii) Yes, as $\exp_p$ is the composition of the canonical diffeomorphism $T_p\mathbb{R}^n_k \cong \mathbb{R}^n_k$ and the translation $\tau_p \colon x \mapsto x + p$.

(iii) Yes, since both maps $T_p\mathbb{R}^n_k \cong \mathbb{R}^n_k$ and τ_p are isometries.

Problem 6.164 Consider the open submanifold

$$M = \left\{(x, y) \in \mathbb{R}^2 : x + y > 0\right\}$$

of the 2-dimensional *Minkowski space*

$$\left(\mathbb{R}^2, g = \mathrm{d}x^2 - \mathrm{d}y^2\right),$$

equipped with the inherited metric $g|_M$, with which M is a flat simply connected Lorentz manifold.

1. Prove that $(M, g|_M)$ is a non-complete G-homogeneous pseudo-Riemannian manifold, where G is the non-Abelian group $G = \mathbb{R} \times \mathbb{R}$ with product

$$(u, v)\left(u', v'\right) = \left(u + u'\mathrm{e}^{-v},\ v + v'\right),$$

under the action

$$(u, v) \cdot (x, y) = (x - \cosh v + y \sinh v + u, x \sinh v + y \cosh v - u). \qquad (\star)$$

2. Does act G freely on M?
3. Can we identify M and G?

The relevant theory is developed, for instance, in Besse [2, 7.I].

Solution

1. (i) It is immediate that (M, g) is incomplete since its geodesics are the restrictions of the geodesics of $(\mathbb{R}^2, g)$ to M, and these are the straight lines.
   (ii) On the other hand, G acts on M: Writing $(x', y') = (u, v) \cdot (x, y)$, we have $x' + y' = \mathrm{e}^v(x + y) > 0$, hence $(x', y') \in M$.
   (iii) The action is transitive: Given two points $(x_1, y_1), (x_2, y_2) \in M$, there exists $(u, v) \in G$ such that $(u, v) \cdot (x_1, y_1) = (x_2, y_2)$. In fact, take the parallels to the straight line $x + y = 0$ through (x_1, y_1) and (x_2, y_2), and let (x_1', y_1') and (x_2', y_2') be, respectively, the points of intersection with the branch of the hyperbola $x^2 - y^2 = 1$ passing through $(1, 0)$. Then it suffices to consider the composition of three transformations: The first one from (x_1, y_1)

to (x_1', y_1'), of type $(u_1, 0)$; the second one from (x_1', y_1') to (x_2', y_2') along the branch of hyperbola (with $u = 0$); and the third one from (x_2', y_2') to (x_2, y_2), again of type $(u_2, 0)$.

2. If $(u, v) \cdot (x, y) = (x, y)$, it is clear that we must have $u = v = 0$.
3. Yes, as the action ($\star$) of G on M is simply transitive (that is, transitive and free, see Definition 4.32 and Theorem 4.34).

Problem 6.165 Find, using Cartan's structure equations, the Gauss curvature of $\mathbb{R}^2$ endowed with the pseudo-Riemannian metric

$$g = \frac{4}{c}\left(\cosh^2 2y\,\mathrm{d}x^2 - \mathrm{d}y^2\right), \quad 0 \neq c \in \mathbb{R}.$$

Hint Consider an orthonormal moving frame and Cartan's structure equations on p. 597, that is,

$$\mathrm{d}\widetilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge (\varepsilon_j \widetilde{\theta}_j), \quad \widetilde{\omega}_{ij} + \widetilde{\omega}_{ji} = 0, \qquad \mathrm{d}\widetilde{\omega}_{ij} = -\sum_k \varepsilon_k \widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik} + \widetilde{\Omega}_{ij}.$$

Solution We have the orthonormal moving frame on $\mathbb{R}^2$,

$$\sigma = \left(X_1 = \frac{\sqrt{|c|}}{2}\frac{1}{\cosh 2y}\frac{\partial}{\partial x},\ X_2 = \frac{\sqrt{|c|}}{2}\frac{\partial}{\partial y}\right).$$

That is, $g(X_i, X_i) = \varepsilon_i$, $i = 1, 2$, with $\varepsilon_1 = +1$, $\varepsilon_2 = -1$ if $c > 0$, and $\varepsilon_1 = -1$, $\varepsilon_2 = +1$ if $c < 0$. Its dual moving coframe is

$$\left(\tilde{\theta}^1 = \frac{2}{\sqrt{|c|}}\cosh 2y\,\mathrm{d}x,\ \tilde{\theta}^2 = \frac{2}{\sqrt{|c|}}\,\mathrm{d}y\right).$$

Let $\tilde{\theta}_i = \varepsilon_i \tilde{\theta}^i$ (no sum) and let $\widetilde{\omega}^i_j$ be the connection forms relative to σ. Then

$$\widetilde{\omega}_{ij} = \varepsilon_i \widetilde{\omega}^i_j \quad \text{(no sum)}$$

is the only set of differential 1-forms satisfying the first structure equation

$$\widetilde{\omega}_{ij} + \widetilde{\omega}_{ji} = 0, \qquad \mathrm{d}\tilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge (\varepsilon_j \tilde{\theta}_j).$$

We only have to calculate $\widetilde{\omega}_{12}$. From

$$\begin{aligned}\mathrm{d}(\varepsilon_1\tilde{\theta}_1) = \varepsilon_1\,\mathrm{d}\tilde{\theta}_1 = (\varepsilon_1)^2\,\mathrm{d}\tilde{\theta}^1 &= \frac{4}{\sqrt{|c|}}\sinh 2y\,\mathrm{d}y \wedge \mathrm{d}x = -\varepsilon_1\widetilde{\omega}_{12} \wedge \varepsilon_2\tilde{\theta}_2 \\ &= -\widetilde{\omega}_{12} \wedge \varepsilon_1(\varepsilon_2)^2\left(\frac{2}{\sqrt{|c|}}\,\mathrm{d}y\right) = -\widetilde{\omega}_{12} \wedge \varepsilon_1\left(\frac{2}{\sqrt{|c|}}\,\mathrm{d}y\right),\end{aligned}$$

one obtains that

$$\widetilde{\omega}_{12} = 2\varepsilon_1 \sinh 2y \, \mathrm{d}x.$$

The Gauss curvature of the pseudo-Riemannian manifold $(\mathbb{R}^2, g)$ is the differentiable real-valued function K defined by

$$\mathrm{d}\widetilde{\omega}_{12} = K\tilde{\theta}_1 \wedge \tilde{\theta}_2,$$

that is, by

$$4\varepsilon_1 \cosh 2y \, \mathrm{d}y \wedge \mathrm{d}x = \varepsilon_1\varepsilon_2 K \frac{2}{\sqrt{|c|}} \cosh 2y \, \mathrm{d}x \wedge \frac{2}{\sqrt{|c|}} \mathrm{d}y.$$

Thus, as

$$K = -\varepsilon_2 |c| = \begin{cases} c & \text{if } \varepsilon_2 = -1 \ (c > 0), \\ -|c| = c & \text{if } \varepsilon_2 = 1 \ (c < 0), \end{cases}$$

we obtain that $(\mathbb{R}^2, g)$ has constant Gauss curvature $K = c$.

Problem 6.166 Prove, by using the Koszul formula, that the half-space

$$H = \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 : x^1 > 0 \right\}$$

endowed with the pseudo-Riemannian metric

$$g = \frac{1}{K} \frac{\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4}{(x^1)^2}, \quad 0 \neq K \in \mathbb{R},$$

has constant curvature $-K$.

Solution Applying the Koszul formula in Theorem 6.4 to $e_i = \partial/\partial x^i$, $i = 1, 2, 3, 4$, we obtain, on account of $[e_i, e_j] = 0$, for instance, for $\nabla_{e_1} e_1$:

$$\begin{aligned} 2g(\nabla_{e_1} e_1, e_i) &= 2e_1 g(e_1, e_i) - e_i g(e_1, e_1) = 2e_1 \left(\frac{\delta_{1i}}{K(x^1)^2} \right) - e_i \left(\frac{1}{K(x^1)^2} \right) \\ &= 2\frac{\delta_{1i}}{K} \left(-\frac{2}{(x^1)^3} \right) - \frac{1}{K} \left(-\frac{2\delta_{1i}}{(x^1)^3} \right) = -\frac{2\delta_{1i}}{K(x^1)^3}, \end{aligned}$$

from which $\nabla_{e_1} e_1 = -\frac{1}{x^1} e_1$. Similarly, one obtains:

$$\begin{aligned} &\nabla_{e_1} e_1 = -\nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\frac{1}{x^1} e_1, \qquad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -\frac{1}{x^1} e_2, \\ &\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{x^1} e_3, \qquad \nabla_{e_1} e_4 = \nabla_{e_4} e_1 = -\frac{1}{x^1} e_4, \\ &\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \nabla_{e_2} e_4 = \nabla_{e_4} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_3 = 0. \end{aligned}$$

Thus

$$-R(e_1,e_2)e_2 = R(e_1,e_3)e_3 = R(e_1,e_4)e_4 = \frac{1}{(x^1)^2}e_1,$$

$$R(e_2,e_3)e_3 = R(e_2,e_4)e_4 = \frac{1}{(x^1)^2}e_2, \qquad R(e_3,e_4)e_4 = \frac{1}{(x^1)^2}e_3.$$

So

$$-R_{1212} = R_{1313} = R_{1414} = R_{2323} = R_{2424} = -R_{3434} = \frac{K}{(x^1)^4}.$$

Finally, the sectional curvature $K(P_{ij})$, $1 \leqslant i < j \leqslant 4$, has values

$$K(P_{12}) = K(P_{13}) = K(P_{14}) = K(P_{23}) = K(P_{24}) = K(P_{34}) = -K.$$

Problem 6.167 Let M be a pseudo-Riemannian manifold of dimension $n \geqslant 2$. Show, by using Cartan's structure equations, that if there exist local coordinates x^i on a neighbourhood of each $x \in M$ in which the metric is given by

$$g = \frac{\sum_i \varepsilon_i \,\mathrm{d}x^i \otimes \mathrm{d}x^i}{(1 + \frac{K}{4}\sum_i \varepsilon_i (x^i)^2)^2}, \quad \varepsilon_i = \pm 1,\ i = 1,\dots,n,\ K \in \mathbb{R},$$

then (M,g) has constant curvature K.

Hint See the hint in Problem 6.165.

The relevant theory is developed, for instance, in Wolf [36].

Solution Let $r(x) = (\varepsilon_i (x^i)^2)^{1/2}$ and $A(x) = -\log(1 + (K/4)r^2)$. Then $(\mathrm{e}^{-A}\frac{\partial}{\partial x^i})$ is an orthonormal moving frame, that is,

$$g\left(\mathrm{e}^{-A}\frac{\partial}{\partial x^i}, \mathrm{e}^{-A}\frac{\partial}{\partial x^j}\right) = \begin{cases} \varepsilon_i & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

whose dual moving coframe is $(\tilde\theta^i = \mathrm{e}^A \,\mathrm{d}x^i)$. Therefore,

$$\begin{aligned} \mathrm{d}\tilde\theta^i &= \mathrm{e}^A \,\mathrm{d}A \wedge \mathrm{d}x^i = \sum_j \mathrm{e}^A \frac{\partial A}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i = \sum_j \tilde\theta^j \wedge \frac{\partial A}{\partial x^j}\,\mathrm{d}x^i \\ &= \sum_j \tilde\theta^j \wedge \left(\frac{\partial A}{\partial x^j}\,\mathrm{d}x^i - \varepsilon_i\varepsilon_j \frac{\partial A}{\partial x^i}\,\mathrm{d}x^j\right). \end{aligned}$$

Let $\widetilde\omega^i_j$ denote the term in parentheses. One has

$$\widetilde\omega_{ij} = \varepsilon_i \widetilde\omega^i_j = \varepsilon_i \frac{\partial A}{\partial x^j}\,\mathrm{d}x^i - \varepsilon_j \frac{\partial A}{\partial x^i}\,\mathrm{d}x^j = -\widetilde\omega_{ji},$$

hence $\widetilde{\omega}^i_j$ are the connection forms relative to $(\mathrm{e}^{-A}\frac{\partial}{\partial x^i})$. The second structure equation is thus

$$\begin{aligned}\widetilde{\Omega}_{ij} &= \mathrm{d}\widetilde{\omega}_{ij} + \sum_k \varepsilon_k \widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik} \\ &= \sum_k \left(\varepsilon_i \frac{\partial^2 A}{\partial x^k \partial x^j} \mathrm{d}x^k \wedge \mathrm{d}x^i - \varepsilon_j \frac{\partial^2 A}{\partial x^k \partial x^i} \mathrm{d}x^k \wedge \mathrm{d}x^j \right) \\ &\quad + \sum_k \varepsilon_k \left(\varepsilon_j \frac{\partial A}{\partial x^k} \mathrm{d}x^j - \varepsilon_k \frac{\partial A}{\partial x^j} \mathrm{d}x^k \right) \wedge \left(\varepsilon_i \frac{\partial A}{\partial x^k} \mathrm{d}x^i - \varepsilon_k \frac{\partial A}{\partial x^i} \mathrm{d}x^k \right).\end{aligned}$$

Now, since

$$\frac{\partial A}{\partial x^i} = -\frac{\frac{K}{2}\varepsilon_i x^i}{1+\frac{K}{4}r^2},$$

the three summands at the right hand side can be written, respectively, as

$$\sum_k \varepsilon_i \frac{(1+\frac{K}{4}r^2)(-\frac{K}{2}\varepsilon_j \delta_{jk}) + \frac{K}{2}\varepsilon_j x^j \frac{K}{2}\varepsilon_k x^k}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^k \wedge \mathrm{d}x^i,$$

$$-\sum_k \varepsilon_j \frac{(1+\frac{K}{4}r^2)(-\frac{K}{2}\varepsilon_i \delta_{ik}) + \frac{K}{2}\varepsilon_i x^i \frac{K}{2}\varepsilon_k x^k}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^k \wedge \mathrm{d}x^j,$$

and

$$\begin{aligned}\sum_k \Bigg(&\varepsilon_k \varepsilon_j \varepsilon_i \frac{\frac{K^2}{4}\varepsilon_k \varepsilon_k x^k x^k}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^j \wedge \mathrm{d}x^i - \varepsilon_k \varepsilon_j \varepsilon_k \frac{\frac{K^2}{4}\varepsilon_i \varepsilon_k x^i x^k}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &- \varepsilon_k \varepsilon_k \varepsilon_i \frac{\frac{K^2}{4}\varepsilon_j \varepsilon_k x^j x^k}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^k \wedge \mathrm{d}x^i \Bigg).\end{aligned}$$

Substituting, we obtain

$$\widetilde{\Omega}_{ij} = \frac{\varepsilon_i \varepsilon_j K}{(1+\frac{K}{4}r^2)^2} \mathrm{d}x^i \wedge \mathrm{d}x^j = K \varepsilon_i \tilde{\theta}^i \wedge \varepsilon_j \tilde{\theta}^j = K \tilde{\theta}_i \wedge \tilde{\theta}_j.$$

Problem 6.168 Consider, for each integer $n \geqslant 1$, the Lie group

$$G = \left\{ \begin{pmatrix} sI_n & {}^t v \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1,\mathbb{R}) : s > 0,\ v \in \mathbb{R}^n \right\}.$$

(i) Prove that the Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} sI_n & {}^t v \\ 0 & 0 \end{pmatrix} \in M(n+1,\mathbb{R}) : s \in \mathbb{R},\ v \in \mathbb{R}^n \right\}.$$

(ii) Let $\{E_0, E_1, \dots, E_n\}$ be the basis of $\mathfrak{g}$ given by

$$E_0 = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_i = \begin{pmatrix} 0 & {}^t e_i \\ 0 & 0 \end{pmatrix}, \quad 1 \leqslant i \leqslant n,$$

$\{e_i\}$ being the usual basis of $\mathbb{R}^n$. Consider the left-invariant metric g on G such that

$$g(E_0, E_0) = -c^2, \qquad g(E_0, E_i) = 0, \qquad g(E_i, E_j) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant n.$$

That is, $\{\theta^0, \theta^1, \dots, \theta^n\}$ being the basis dual to $\{e_i\}$,

$$g = -c^2 \theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \cdots + \theta^n \otimes \theta^n, \quad c \neq 0.$$

Prove that this metric is not bi-invariant on G.

(iii) Prove that the Lorentzian Lie group (G, g) is a manifold of positive constant sectional curvature $1/c^2$.

(iv) Show that the given metric is not complete.

Hint (to (iv)) Find a specific geodesic in G, for instance, the geodesic $\gamma(t)$ with initial conditions

$$\gamma(0) = I_{n+1}, \qquad \gamma'(0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E_i|_{I_{n+1}},$$

and show that $t \in (-\infty, +\infty)$ does not hold true.

The relevant theory is developed, for instance, in Poor [28, Chap. 6].

Solution

(i) We have for any element of $\mathfrak{g}$ that

$$\begin{aligned} \exp \begin{pmatrix} s\, I_n & {}^t v \\ 0 & 0 \end{pmatrix} &= I_{n+1} + \begin{pmatrix} s\, I_n & {}^t v \\ 0 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} s^2 I_n & s\, {}^t v \\ 0 & 0 \end{pmatrix} \\ &\quad + \frac{1}{3!} \begin{pmatrix} s^3 I_n & s^2\, {}^t v \\ 0 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} (1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots) I_n & (1 + \frac{s}{2!} + \frac{s^2}{3!} + \cdots)^t v \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^s I_n & {}^t w \\ 0 & 1 \end{pmatrix} \in G. \end{aligned}$$

(ii) One has

$$[E_i, E_j] = \left[\begin{pmatrix} 0 & {}^t e_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & {}^t e_j \\ 0 & 0 \end{pmatrix} \right] = 0, \quad 1 \leqslant i, j \leqslant n,$$

$$[E_i, E_0] = \left[\begin{pmatrix} 0 & {}^te_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\right] = -\begin{pmatrix} 0 & {}^te_i \\ 0 & 0 \end{pmatrix} = -E_i.$$

Hence, for $1 \leqslant i, j \leqslant n$, one has

$$g(\mathrm{ad}_{E_i} E_0, E_j) + g(E_0, \mathrm{ad}_{E_i} E_j) = -\delta_{ij},$$

so g is indeed not bi-invariant.

(iii) As g is left-invariant, the Koszul formula for the Levi-Civita connection ∇ reduces to

$$2\, g(\nabla_X Y, Z) = g\big([X, Y], Z\big) - g\big([Y, Z], X\big) + g\big([Z, X], Y\big),$$

and we obtain that

$$\begin{aligned} &\nabla_{E_0} E_0 = \nabla_{E_0} E_i = 0, \qquad \nabla_{E_i} E_0 = -E_i, \\ &\nabla_{E_i} E_j = -\frac{1}{c^2}\,\delta_{ij}\, E_0, \quad 1 \leqslant i, j \leqslant n. \end{aligned} \tag{$\star$}$$

The curvature tensor field has thus components

$$R(E_0, E_i)E_0 = E_i, \qquad R(E_i, E_j)E_i = -\frac{1}{c^2}\, E_j, \quad 1 \leqslant i \neq j \leqslant n,$$

from which

$$R\left(\frac{1}{c}E_0, E_i, \frac{1}{c}E_0, E_i\right) = \frac{1}{c^2}, \qquad R(E_i, E_j, E_i, E_j) = \frac{1}{c^2},$$

so (G, g) has in fact constant sectional curvature $1/c^2$.

(iv) G may be considered as the semi-direct product solvable Lie group $\mathbb{R}^n \rtimes_\Psi \mathbb{R}^+$ of the additive group $\mathbb{R}^n$ and the multiplicative group of positive real numbers $\mathbb{R}^+$ under the homomorphism $\Psi\colon \mathbb{R}^+ \to \operatorname{Aut}\mathbb{R}^n$, $\Psi(s)(v) = sv$, $s \in \mathbb{R}^+$, $v \in \mathbb{R}^n$, and $\mathfrak{g}$ can be identified with the semi-direct product $\mathbb{R}^n \rtimes_\varphi \mathbb{R}$ with respect to the induced homomorphism $\varphi\colon \mathbb{R} \to \operatorname{End}\mathbb{R}^n$. We can thus write

$$E_0 = (0, 1), \qquad E_i = (e_i, 0), \quad 1 \leqslant i \leqslant n.$$

A curve γ in G is given by

$$\gamma(t) = \begin{pmatrix} v_0(t) & 0 & \dots & 0 & v_1(t) \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & v_0(t) & v_n(t) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix},$$

where $v_0, v_1, \dots, v_n$ are real differentiable functions and v_0 is positive.

Denote local coordinates on G by

$$\big(x = \big(x^1, \dots, x^n\big), x^0\big).$$

To get the equations of a geodesic curve, one must take derivatives with respect to the coordinates and then sum with respect to the coordinate vector fields; but the coordinate vector fields are not left-invariant, i.e. they coincide with the vector fields E_i only at the identity element of G, which is the identity matrix I_{n+1},

$$E_i|_{I_{n+1}} = \left.\frac{\partial}{\partial x^i}\right|_{I_{n+1}}, \quad i = 0, 1, \dots, n.$$

To find the relation between the coordinate vector fields and the left-invariant vector fields at a generic point in $\gamma(t)$, note that due to the product law in the group we have

$$\gamma(t)(x, y) = \big(v(t), v_0(t)\big)\big(x, x^0\big) = \big(v(t) + v_0(t)x,\ v_0(t)x^0\big)$$

so that, L denoting left translation, one has

$$E_i|_{\gamma(t)} = (L_{\gamma(t)*})_{I_{n+1}} \left.\frac{\partial}{\partial x^i}\right|_{I_{n+1}} = v_0(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}, \quad i = 0, 1, \dots, n.$$

Thus the relation we looked for is

$$\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = \frac{1}{v_0(t)} E_i\Big|_{\gamma(t)}, \quad i = 0, 1, \dots, n,$$

and hence, from expressions ($\star$) above, it follows that the equations of geodesics are

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}t^2} - \frac{1}{v_0}\left(\frac{\mathrm{d}v_0}{\mathrm{d}t}\right)^2 - \frac{1}{c^2 v_0}\sum_{i=1}^{n}\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)^2 = 0, \tag{$\star\star$}$$

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} - \frac{2}{v_0}\frac{\mathrm{d}v_0}{\mathrm{d}t}\frac{\mathrm{d}v_i}{\mathrm{d}t} = 0, \quad 1 \leqslant i \leqslant n. \tag{$\star\star\star$}$$

We look for the solution $\gamma(t)$ with initial conditions

$$\gamma(0) = I_{n+1}, \qquad \gamma'(0) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E_i|_{I_{n+1}}. \tag{$\dagger$}$$

To this end, multiplying the equation ($\star\star\star$) by $\frac{\mathrm{d}v_i}{\mathrm{d}t}$, $i = 1, \dots, n$, we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)^2\right) = \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\mathrm{d}^2 v_i}{\mathrm{d}t^2} = \frac{2}{v_0}\frac{\mathrm{d}v_0}{\mathrm{d}t}\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)^2,$$

that is,

$$\frac{1}{2}\frac{\mathrm{d}w_i}{\mathrm{d}t} = \frac{2}{v_0}\frac{\mathrm{d}v_0}{\mathrm{d}t}w_i \quad \left(\text{where } w_i = \left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)^2\right)$$

or equivalently,

$$\frac{\mathrm{d}w_i}{w_i} = 4\frac{\mathrm{d}v_0}{v_0}.$$

Integrating we get

$$\log w_i = \log v_0^4 + a_i, \quad a_i \in \mathbb{R},$$

from which one has

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \exp\left(\frac{a_i}{2}\right)v_0^2.$$

Integrating the previous equation, we obtain

$$v_i = \exp\left(\frac{a_i}{2}\right)\int v_0^2\,\mathrm{d}t + b_i, \quad b_i \in \mathbb{R}. \tag{††}$$

Substituting the previous expression into equations (⋆⋆), we have, after simplifying, that

$$0 = \frac{\mathrm{d}^2 v_0}{\mathrm{d}t^2} - \frac{1}{v_0}\left(\frac{\mathrm{d}v_0}{\mathrm{d}t}\right)^2 - \frac{a}{c^2}v_0^3 \quad \left(\text{where } a = \sum_{i=1}^{n}\exp a_i\right).$$

Taking then

$$v_0 = \sec\left(\frac{t}{c}\right) = \frac{1}{\cos(\frac{t}{c})}, \tag{†††}$$

one gets

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = \frac{1}{c}\frac{\sin(\frac{t}{c})}{\cos^2(\frac{t}{c})}, \qquad \frac{\mathrm{d}^2 v_0}{\mathrm{d}t^2} = \frac{1}{c^2}\frac{1+\sin^2(\frac{t}{c})}{\cos^3(\frac{t}{c})},$$

and hence

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}t^2} - \frac{1}{v_0}\left(\frac{\mathrm{d}v_0}{\mathrm{d}t}\right)^2 = \frac{1}{c^2}\frac{1}{\cos^3(\frac{t}{c})} = \frac{1}{c^2}v_0^3$$

and $a = 1$.

In turn, from (††) and (†††), we have

$$\begin{aligned} v_i &= \exp\left(\frac{a_i}{2}\right)\int \sec^2\left(\frac{t}{c}\right)\mathrm{d}t + b_i \\ &= \exp\left(\frac{a_i}{2}\right)\left(c\tan\left(\frac{t}{c}\right) + c_i\right) + b_i, \quad b_i, c_i \in \mathbb{R}. \end{aligned}$$

Now, the first initial condition in (†) implies

$$v_i = c \exp\left(\frac{a_i}{2}\right) \tan\left(\frac{t}{c}\right), \quad i = 1, \dots, n,$$

and the second initial condition in (†) then implies $\exp a_i = \frac{1}{n}$.

We thus obtain

$$v_0 = \sec\left(\frac{t}{c}\right), \qquad v_i = \frac{c}{\sqrt{n}} \tan\left(\frac{t}{c}\right), \quad i = 1, \dots, n.$$

Since $\cos(\frac{t}{c}) = 0$ for $\frac{t}{c} = \pm\frac{\pi}{2}$, the wanted solution is

$$\gamma(t) = \begin{pmatrix} \sec(\frac{t}{c}) & 0 & \dots & 0 & \frac{c}{\sqrt{n}}\tan(\frac{t}{c}) \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \sec(\frac{t}{c}) & \frac{c}{\sqrt{n}}\tan(\frac{t}{c}) \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad -\frac{c\pi}{2} < t < \frac{c\pi}{2},$$

and the manifold is, in fact, not complete.

Problem 6.169 Consider $\mathbb{R}^4$ as a space-time with coordinates ρ, φ, ψ and t, where the first three are the usual spherical coordinates on $\mathbb{R}^3$, equipped with the metric (see Remark 1.4)

$$g = -\left(1 - \frac{2m}{\rho}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{\rho}\right)^{-1} \mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\psi^2 + \sin^2\psi\, \mathrm{d}\varphi^2\right),$$
$$m \in \mathbb{R}^+, \ \psi \in (0, \pi), \ \varphi \in (0, 2\pi).$$

Prove, by using Cartan's structure equations, that g is a solution (except at the singularity $\rho = 0$) of the empty space Einstein field equations

$$\mathbf{r} - \frac{1}{2}\mathbf{s}\, g = 0.$$

Hint See the hint in Problem 6.165.

Remark This solution, found by Schwarzschild, was the first one known to such field equations, and it is sometimes called *Schwarzschild "black hole" metric*.

Solution The frame

$$\sigma=\left(X_1=\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}\frac{\partial}{\partial t},\ X_2=\left(1-\frac{2m}{\rho}\right)^{-\frac{1}{2}}\frac{\partial}{\partial\rho},\ X_3=\frac{1}{\rho}\frac{\partial}{\partial\psi},\right.$$
$$\left.X_4=\frac{1}{\rho\sin\psi}\frac{\partial}{\partial\varphi}\right)$$

is an orthonormal moving frame, that is,

$$g(X_1,X_1)=\varepsilon_1=-1,\qquad g(X_i,X_i)=\varepsilon_i=1,\quad i=2,3,4,$$

with dual moving coframe

$$\left(\tilde\theta_1=\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}\mathrm{d}t,\tilde\theta_2=\left(1-\frac{2m}{\rho}\right)^{-\frac{1}{2}}\mathrm{d}\rho,\tilde\theta_3=\rho\,\mathrm{d}\psi,\tilde\theta_4=\rho\sin\psi\,\mathrm{d}\varphi\right).$$

The first structure equation,

$$\mathrm{d}\tilde\theta_i=-\sum_j\widetilde\omega_{ij}\wedge(\varepsilon_j\tilde\theta_j),$$

gives us the non-vanishing connection forms relative to σ,

$$\widetilde\omega_{12}=-\widetilde\omega_{21}=\frac{m}{\rho^2}\,\mathrm{d}t,\qquad\widetilde\omega_{23}=-\widetilde\omega_{32}=-\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}\mathrm{d}\psi,$$
$$\widetilde\omega_{24}=-\widetilde\omega_{42}=-\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}\sin\psi\,\mathrm{d}\varphi,\qquad\widetilde\omega_{34}=-\widetilde\omega_{43}=-\cos\psi\,\mathrm{d}\varphi.$$

The second structure equation,

$$\widetilde\Omega_{ij}=\mathrm{d}\widetilde\omega_{ij}+\sum_k\varepsilon_k\widetilde\omega_{jk}\wedge\widetilde\omega_{ik},$$

furnishes the non-vanishing curvature 2-forms relative to σ,

$$\widetilde\Omega_{12}=-\widetilde\Omega_{21}=\frac{2m}{\rho^3}\tilde\theta^1\wedge\tilde\theta^2,\qquad\widetilde\Omega_{13}=-\widetilde\Omega_{31}=-\frac{m}{\rho^3}\tilde\theta^1\wedge\tilde\theta^3,$$
$$\widetilde\Omega_{14}=-\widetilde\Omega_{41}=-\frac{m}{\rho^3}\tilde\theta^1\wedge\tilde\theta^4,\qquad\widetilde\Omega_{23}=-\widetilde\Omega_{32}=-\frac{m}{\rho^3}\tilde\theta^2\wedge\tilde\theta^3,$$
$$\widetilde\Omega_{24}=-\widetilde\Omega_{42}=-\frac{m}{\rho^3}\tilde\theta^2\wedge\tilde\theta^4,\qquad\widetilde\Omega_{34}=-\widetilde\Omega_{43}=\frac{2m}{\rho^3}\tilde\theta^3\wedge\tilde\theta^4.$$

From the equations

$$\widetilde\Omega_{ij}=\sum_{k<l}R_{ijkl}\tilde\theta^k\wedge\tilde\theta^l,\qquad\mathbf{r}_{ij}=\sum_k R_{kikj},$$

one obtains that the Ricci tensor $\mathbf{r}$ vanishes. In fact,

$$\mathbf{r}_{12} = R_{k1k2} = R_{3132} + R_{4142} = 0,$$
$$\mathbf{r}_{22} = R_{k2k2} = R_{1212} + R_{3232} + R_{4242} = \frac{2m}{\rho^3} - \frac{m}{\rho^3} - \frac{m}{\rho^3} = 0.$$

The remaining calculations for the components $\mathbf{r}_{ij}$, $i \neq j$, or $\mathbf{r}_{ii}$, are similar.

Since the scalar curvature is given by $\mathbf{s} = \sum_i \mathbf{r}_{ii}$, empty space Einstein field equations are automatically satisfied.

Problem 6.170

(i) Let V be an $(n+1)$-dimensional vector space, and let V^* be its dual space. We shall write $x+\alpha$, $y+\beta, \dots$, to denote the elements of $V \oplus V^*$. On the space $V \oplus V^*$ there exists a natural non-degenerate bilinear form $\langle \cdot , \cdot \rangle$ given by

$$\langle x+\alpha, y+\beta\rangle = \frac{1}{2}\big(\alpha(y) + \beta(x)\big),$$

and an involutive linear automorphism J_0 given by

$$J_0|_V = \mathrm{id}_V, \qquad J_0|_{V^*} = -\mathrm{id}_{V^*}.$$

The subgroup of the automorphism group $\mathrm{GL}(V \oplus V^*)$ of $V \oplus V^*$ preserving both $\langle \cdot , \cdot \rangle$ and J_0 can be identified to the automorphism group $\mathrm{GL}(V)$ of V. In fact, if $A \in \mathrm{GL}(V)$, we put

$$A(x+\alpha) = Ax + \alpha \cdot A^{-1}.$$

Let us introduce on

$$\big(V \oplus V^*\big)_+ = \big\{x+\alpha \in V \oplus V^* : \langle x+\alpha, x+\alpha\rangle = \alpha(x) > 0\big\}$$

the equivalence relation $\sim$ defined by $x+\alpha \sim ax + b\alpha$ if $0 < a, b \in \mathbb{R}$, and define the *paracomplex projective space* $P(V \oplus V^*)$ by

$$P\big(V \oplus V^*\big) = \big(V \oplus V^*\big)_+ / \sim .$$

Let π denote the natural projection $\pi : (V \oplus V^*)_+ \to P(V \oplus V^*)$. If $a, b \in \mathbb{R}^+$, we have $A(ax+b\alpha) = aAx + b(\alpha \cdot A^{-1})$, and so we can define an action of $\mathrm{GL}(V)$ on $P(V \oplus V^*)$ in such a way that

$$A\big(\pi(x+\alpha)\big) = \pi\big(A(x+\alpha)\big), \quad A \in \mathrm{GL}(V).$$

Then the identity component $\mathrm{GL}_0(V)$ of $\mathrm{GL}(V)$ acts transitively on the pseudosphere in $V \oplus V^*$,

$$S = \big\{x+\alpha \in \big(V \oplus V^*\big)_+ : \langle x+\alpha, x+\alpha\rangle = \alpha(x) = 1\big\}.$$

Prove that $P(V \oplus V^*)$ is a homogeneous space under the action of the group $\mathrm{GL}_0(V)$, for $n \geqslant 1$.

(ii) We have a principal bundle $\pi : S \to P(V \oplus V^*)$ with group $\mathbb{R}^+$. The subgroup $\{aI \in \mathrm{GL}_0(V) : a > 0\}$ of $\mathrm{GL}_0(V)$ acts transitively on the fibres. The quotient of S by that action is $P(V \oplus V^*)$. Consider S equipped with the pseudo-Riemannian metric inherited from that of $V \oplus V^*$. Then, as $\mathrm{GL}_0(V)$ acts on $V \oplus V^*$ by isometries, and preserves S, it also acts on S by isometries.

Now, consider the formula

$$\langle Z, Z\rangle = \left\langle Z^h, Z^h\right\rangle, \quad Z \in T_{\pi(x+\alpha)} P\left(V \oplus V^*\right), \tag{$\star$}$$

where $Z^h \in T_{x+\alpha} S$ is orthogonal to the fibre and satisfies $\pi_* Z^h = Z$. Show that this construction induces on $P(V \oplus V^*)$ a pseudo-Riemannian metric g such that π is a pseudo-Riemannian submersion.

(iii) The group $G = \mathbb{R}^+ \times \mathbb{R}^+$ acts on $(V \oplus V^*)_+$ by

$$(a, b)(x + \alpha) = ax + b\alpha, \quad (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

and $P(V \oplus V^*)$ is the quotient space of this action. Let J_0 be the *almost product structure* (that is, an automorphism such that $J_0^2 = I$) defined on $(V \oplus V^*)_+$ by

$$J_0(v, \omega) = (v, -\omega), \quad (v, \omega) \in T_{x+\alpha}\left(V \oplus V^*\right)_+.$$

Prove that J_0 passes to the quotient and gives an almost product structure J (*a* $(1, 1)$ tensor field with $J^2 = I$) on $P(V \oplus V^*)$ such that this manifold has a *para-Hermitian structure* with the metric in (ii) and J (that is, we have $g(JX, Y) + g(X, JY) = 0$, where $X, Y, \in \mathfrak{X}(V \oplus V^*)$).

(iv) Consider a basis $\{e_0, \dots, e_n\}$ of V and the dual basis $\{\theta_0, \dots, \theta_n\}$ of V^*. We can consider the e_k, $k = 0, \dots, n$, as coordinates on V^* and the θ_k as coordinates on V. Let U_0^+ be the open subset of $P(V \oplus V^*)$ given by

$$U_0^+ = \left\{\pi(x + \alpha) : \theta_0(x) > 0,\ e_0(\alpha) > 0\right\}.$$

Let (x^i, y^i), $i = 1, \dots, n$, be coordinates on U_0^+ given by

$$x^i\left(\pi(x+\alpha)\right) = \frac{\theta_i(x)}{\theta_0(x)}, \qquad y^i\left(\pi(x+\alpha)\right) = \frac{e_i(\alpha)}{e_0(\alpha)}. \tag{$\star\star$}$$

Prove that the metric in terms of these coordinates on U_0^+ has the expression

$$g = \frac{1}{2(1 + \langle x, y\rangle)} \left\{ \sum_{i=1}^{n} \left(\mathrm{d}x^i \otimes \mathrm{d}y^i + \mathrm{d}y^i \otimes \mathrm{d}x^i\right) - \sum_{i,j=1}^{n} \frac{x^i y^j}{1 + \langle x, y\rangle} \left(\mathrm{d}y^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}y^i\right) \right\},$$

where $\langle x, y\rangle = \sum_i x^i y^i$.

(v) Compute the almost product structure J on $P(V \oplus V^*)$ in terms of the coordinates (x^i, y^i).

Remark Since the metric admits locally that expression, it is said that the manifold $(P(V \oplus V^*), g, J)$ is a *para-Kähler manifold* (that is, the Levi-Civita connection of g parallelises J) *of constant paraholomorphic sectional curvature* (equal to 4), which is an analog of the holomorphic sectional curvature.

For such a space, the curvature tensor field R satisfies

$$\begin{aligned} R(X,Y)Z = {} & g(X,Z)Y - g(Y,Z)X + g(X,JZ)JY \\ & - g(Y,JZ)JX + 2g(X,JY)JZ. \end{aligned}$$

The relevant theory is developed, for instance, in [11–13].

Solution

(i) Let $x + \alpha$ be an arbitrarily fixed element of S. Then, for each $y + \beta \in S$, there exists an element $A \in \mathrm{GL}_0(V)$ such that $A(y + \beta) = x + \alpha$. For, given a linearly independent set of elements $y_1, \dots, y_n$ of V such that $\beta(y_i) = 0$, then $\{y, y_1, \dots, y_n\}$ is a basis of V. Similarly, if we have a linearly independent set of elements $x_1, \dots, x_n$ of V such that $\alpha(x_i) = 0$, then $\{x, x_1, \dots, x_n\}$ is a basis of V. Take A such that $Ay = x$, $Ay_i = x_i$. Then

$$\big(\beta \cdot A^{-1}\big)(x) = \beta(y) = 1 = \alpha(x), \qquad \big(\beta \cdot A^{-1}\big)(x_i) = \beta(y_i) = 0 = \alpha(x_i),$$

and hence $\beta \cdot A^{-1} = \alpha$. Taking two bases with the same orientation we have $A \in \mathrm{GL}_0(V)$ as desired.

(ii) Denote by $\mathbf{n}$ and $\mathbf{v}$ the natural vector fields on $V \oplus V^*$ whose values at $x + \alpha$ are $\mathbf{n}_{x+\alpha} = x + \alpha$ and $\mathbf{v}_{x+\alpha} = x - \alpha$. Then one has $\pi_* \mathbf{n} = \pi_* \mathbf{v} = 0$. In fact, $\mathbf{n}_{x+\alpha}$ is the vector tangent at $t = 0$ to the curve $t \mapsto x + \alpha + t(x + \alpha)$ and $\mathbf{v}_{x+\alpha}$ is the vector tangent to the curve $t \mapsto x + \alpha + t(x - \alpha)$. As $\pi((1+t)(x+\alpha)) = \pi(x+\alpha)$ and $\pi((1+t)x + (1-t)\alpha) = \pi(x+\alpha)$ for small t, the claim follows. Thus $\ker \pi_*$ is spanned by $\mathbf{n}$ and $\mathbf{v}$. The vector $\mathbf{v}$ is tangent to the fibre and $\mathbf{n}$ is normal to S in $V \oplus V^*$. The process given in the statement of lifting a vector Z to such a vector Z^h has a unique solution if and only if the subspace orthogonal to the fibres has dimension equal to $2n$ or, equivalently, if and only if the restriction of $\langle \cdot, \cdot \rangle$ to the subspace spanned by $\mathbf{v}_{x+\alpha}$ and $\mathbf{n}_{x+\alpha}$ is non-degenerate; but indeed,

$$\begin{pmatrix} \langle \mathbf{n}, \mathbf{n} \rangle & \langle \mathbf{n}, \mathbf{v} \rangle \\ \langle \mathbf{v}, \mathbf{n} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consequently, we have the desired structure on $P(V \oplus V^*)$, which makes π a pseudo-Riemannian submersion.

(iii) We have

$$\big(J_0 \circ (a,b)_* - (a,b)_* \circ J_0\big)(v,\omega) = J_0(av, b\omega) - (a,b)_*(v,-\omega) = 0.$$

Hence, J_0 passes to the quotient, giving an almost product structure J, which is easily seen to be para-Hermitian.

(iv) After computation we have

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)} &= -e_i(\alpha)\theta_0(x)x + \theta_0(x)\frac{\partial}{\partial e^i}\bigg|_{x+\alpha}, \\ \frac{\partial}{\partial y^i}\bigg|^h_{\pi(x+\alpha)} &= -\theta_i(x)e_0(\alpha)\alpha + e_0(\alpha)\frac{\partial}{\partial e_i}\bigg|_{x+\alpha}. \end{aligned}$$

From this, on account of ($\star$), one has

$$\left\langle \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial y^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = \frac{1}{2}\theta_0(x)e_0(\alpha)\big(\delta_i^j - e_i(\alpha)\theta_j(x)\big).$$

Now, from ($\star\star$) we deduce

$$\theta_0(x)e_0(\alpha) = \frac{1}{1+\langle x,y\rangle}, \qquad e_i(\alpha)\theta_j(x) = \frac{y^i x^j}{1+\langle x,y\rangle},$$

hence

$$\left\langle \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial y^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = \frac{1}{2(1+\langle x,y\rangle)}\left(\delta_{ij} - \frac{y^i x^j}{1+\langle x,y\rangle}\right).$$

Similarly,

$$\left\langle \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial x^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = 0, \qquad \left\langle \frac{\partial}{\partial y^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial y^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = 0.$$

Hence the metric on $P(V \oplus V^*)$ has on U_0^+ the expression given in the statement.

(v) We have

$$J\frac{\partial}{\partial x^i}\bigg|_{\pi(x+\alpha)} = \pi_* J_0\left(\frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}\right) = \pi_*\left(\frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}\right) = \frac{\partial}{\partial x^i}\bigg|_{\pi(x+\alpha)},$$

and similarly,

$$J\frac{\partial}{\partial y^i}\bigg|_{\pi(x+\alpha)} = -\frac{\partial}{\partial y^i}\bigg|_{\pi(x+\alpha)}.$$

Hence

$$J = \sum_i \left(\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i - \frac{\partial}{\partial y^i} \otimes \mathrm{d}y^i \right).$$

Problem 6.171 Let $\mathfrak{os}$ be the Lie algebra with generators P, X, Y, Q, and non-null brackets

$$[X, Y] = P, \qquad [Q, X] = Y, \qquad [Q, Y] = -X.$$

The corresponding simply connected Lie group Os is called the harmonic oscillator group, or simply the *oscillator group*, and it can be realised as $\mathbb{R}^4$ with group operation

$$\left(p'', x'', y'', q''\right) = (p, x, y, q) \cdot \left(p', x', y', q'\right),$$

given by

$$p'' = p + p' + \frac{1}{2}\left(x'(x \sin q - y \cos q) + y'(x \cos q + y \sin q)\right),$$
$$x'' = x + x' \cos q - y' \sin q, \qquad y'' = y + x' \sin q + y' \cos q, \qquad q'' = q + q'.$$

Consider the family of Lorentz inner products on $\mathfrak{os}$ given, with respect to the basis above, by

$$\langle \cdot , \cdot \rangle_\varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \varepsilon \end{pmatrix}, \quad -1 < \varepsilon < 1.$$

(i) Show that g_ε has Lorentzian signature.
(ii) Find the explicit expression of the family of corresponding left-invariant Lorentz metrics g_ε on Os.
(iii) Compute the Levi-Civita connection, the curvature tensor field, the Ricci tensor, the scalar curvature, and the Einstein tensor

$$\mathbf{r} - \frac{1}{2} \mathbf{s} \, g_\varepsilon$$

of this space.

Remark The group Os was introduced by Streater [34], who named it *harmonic oscillator group* because $\mathfrak{os}$ has the same brackets that the operators in the harmonic oscillator problem,

$$P = 1, \qquad X = \frac{\partial}{\partial x}, \qquad Y = x, \qquad Q = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right),$$

acting on functions of x.

The group Os may be considered (see Problem 4.69) as a semi-direct product $H \rtimes \mathbb{R}$, where $H \equiv \mathbb{R} \times \mathbb{C}$ is the three-dimensional Heisenberg group, with group operation

$$(p, z, q) \cdot (p', z', q') = \left(p + p' + \frac{1}{2} \operatorname{Im}\big(\bar{z}\, e^{iq} z'\big),\ z + e^{iq} z',\ q + q' \right).$$

Moreover, if $\varepsilon = 0$, the corresponding Lorentzian metric is also right-invariant and hence (Os, g_0) is a symmetric space. In the other cases, g_ε is not bi-invariant.

The relevant theory is developed, for instance, in Medina [22], Medina and Revoy [23] and [9].

Solution

(i) It suffices to give a suitable g_ε-orthonormal basis. If we put

$$E = \frac{P - Q}{\sqrt{2 - 2\varepsilon}}, \qquad F = \frac{P + Q}{\sqrt{2 + 2\varepsilon}},$$

then it is immediate that $\{E, X, Y, F\}$ is an orthonormal basis of $(\mathfrak{os}, g_\varepsilon)$, that is, the matrix of g_ε with respect to this basis is

$$g_\varepsilon \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) In general, if g denote a left-invariant metric on a Lie group G, e the identity element of G and s an arbitrary element of G, one has

$$\begin{aligned} g_e\left(\left.\frac{\partial}{\partial x^i}\right|_e, \left.\frac{\partial}{\partial x^j}\right|_e \right) &= g_s\left(L_{s*}\left(\left.\frac{\partial}{\partial x^i}\right|_e \right), L_{s*}\left(\left.\frac{\partial}{\partial x^j}\right|_e \right) \right) \\ &= \left({}^t L_{s*} g_s L_{s*} \right)\left(\left.\frac{\partial}{\partial x^i}\right|_e, \left.\frac{\partial}{\partial x^j}\right|_e \right), \end{aligned}$$

that is,

$$g_s = {}^t L_{s*}^{-1} g_e\, L_{s*}^{-1}.$$

In the present case we have

$$L_{(p,x,y,q)*} = \begin{pmatrix} \frac{\partial p''}{\partial p'} & \frac{\partial p''}{\partial x'} & \frac{\partial p''}{\partial y'} & \frac{\partial p''}{\partial q'} \\ \frac{\partial x''}{\partial p'} & \frac{\partial x''}{\partial x'} & \frac{\partial x''}{\partial y'} & \frac{\partial x''}{\partial q'} \\ \frac{\partial y''}{\partial p'} & \frac{\partial y''}{\partial x'} & \frac{\partial y''}{\partial y'} & \frac{\partial y''}{\partial q'} \\ \frac{\partial q''}{\partial p'} & \frac{\partial q''}{\partial x'} & \frac{\partial q''}{\partial y'} & \frac{\partial q''}{\partial q'} \end{pmatrix}_{(p',x',y',q')=0}$$

$$= \begin{pmatrix} 1 & \frac{1}{2}(x\ \sin q - y\ \cos q) & \frac{1}{2}(x - \cos q + y \sin q) & 0 \\ 0 & \cos q & -\sin q & 0 \\ 0 & \sin q & \cos q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so after computation we deduce that

$$(g_\varepsilon)_{(p,x,y,q)} = {}^t L^{-1}_{(p,x,y,q)*} (g_\varepsilon)_{(0,0,0,0)}\, L^{-1}_{(p,x,y,q)*}$$
$$= \begin{pmatrix} \varepsilon & \frac{\varepsilon}{2}y & -\frac{\varepsilon}{2}x & 1 \\ \frac{\varepsilon}{2}y & \frac{\varepsilon}{4}y^2+1 & -\frac{\varepsilon}{4}xy & \frac{1}{2}y \\ -\frac{\varepsilon}{2}x & -\frac{\varepsilon}{4}xy & \frac{\varepsilon}{4}x^2+1 & -\frac{1}{2}x \\ 1 & \frac{1}{2}y & -\frac{1}{2}x & \varepsilon \end{pmatrix}.$$

(iii) The Koszul formula for the Levi-Civita connection ∇ gives, for each fixed ε,

$$2\, g_\varepsilon(\nabla_U V, W) = g_\varepsilon\big([U,V], W\big) - g_\varepsilon\big([V,W], U\big) + g_\varepsilon\big([W,U], V\big),$$

for all $U, V, W \in \mathfrak{os}$. So, the covariant derivatives between generators are given by

$$\nabla_P X = \nabla_X P = -\frac{\varepsilon}{2}Y, \qquad \nabla_X Q = -\nabla_Q X = -\frac{1}{2}Y, \qquad \nabla_P Y = \nabla_Y P = \frac{\varepsilon}{2}X,$$
$$\nabla_Y Q = -\nabla_Q Y = \frac{1}{2}X, \qquad \nabla_X Y = -\nabla_Y X = \frac{1}{2}P.$$

The non-trivial components of the curvature tensor field, with

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W,$$

are thus given by

$$R(P,X)P = \frac{\varepsilon^2}{4}X, \qquad R(P,Y)P = \frac{\varepsilon^2}{4}Y, \qquad R(P,X)X = -\frac{\varepsilon}{4}P,$$
$$R(P,Y)Y = -\frac{\varepsilon}{4}P, \qquad R(P,X)Q = \frac{\varepsilon}{4}X, \qquad R(P,Y)Q = \frac{\varepsilon}{4}Y,$$
$$R(X,Y)X = -\frac{3\varepsilon}{4}Y, \qquad R(X,Y)Y = \frac{3\varepsilon}{4}X, \qquad R(X,Q)P = -\frac{\varepsilon}{4}X,$$
$$R(Y,Q)P = -\frac{\varepsilon}{4}Y, \qquad R(X,Q)X = \frac{1}{4}P, \qquad R(Y,Q)Y = \frac{1}{4}P,$$
$$R(X,Q)Q = -\frac{1}{4}X, \qquad R(Y,Q)Q = -\frac{1}{4}Y.$$

The Ricci tensor is defined by $\mathbf{r}(U, V) = \mathrm{tr}(W \mapsto R(U, W)V)$ so, with respect to the basis $\{P, X, Y, Q\}$, it is given by

$$\mathbf{r} = \begin{pmatrix} \frac{\varepsilon^2}{2} & 0 & 0 & \frac{\varepsilon}{2} \\ 0 & -\frac{\varepsilon}{2} & 0 & 0 \\ 0 & 0 & -\frac{\varepsilon}{2} & 0 \\ \frac{\varepsilon}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

and the scalar curvature (which is defined by $\mathrm{tr}(\mathbf{r})$) is given in terms of the orthonormal basis in (i) above by

$$\mathbf{s} = -\mathbf{r}(E, E) + \mathbf{r}(X, X) + \mathbf{r}(Y, Y) + \mathbf{r}(F, F) = -\frac{\varepsilon}{2}.$$

The Einstein tensor is thus given by

$$\mathbf{r} - \frac{1}{2}\mathbf{s}\, g_\varepsilon = \frac{1}{4}\begin{pmatrix} 3\,\varepsilon^2 & 0 & 0 & 3\,\varepsilon \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ 3\,\varepsilon & 0 & 0 & 2+\varepsilon^2 \end{pmatrix}.$$

Problem 6.172 Let

$$G = \left\{ \begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{R}) : a > 0 \right\}.$$

Prove:

(i) G is a closed subgroup of $\mathrm{GL}(3, \mathbb{R})$.
(ii) G does not admit a pseudo-Riemannian bi-invariant metric.

Hint (to (ii)) Show that $g = \lambda\omega_1 \otimes \omega_1$, where $\omega_1 = \mathrm{d}a/a$, is the general expression of a bi-invariant metric; but such a metric g is singular.

Solution

(i) We have

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/a' & 0 & 0 \\ 0 & a' & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/aa' & 0 & 0 \\ 0 & aa' & ab'+b \\ 0 & 0 & 1 \end{pmatrix} \in G \qquad (\star)$$

and

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1/a & -b/a \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Therefore, G is an abstract subgroup of $\mathrm{GL}(3,\mathbb{R})$.

If a sequence in G of matrices

$$\begin{pmatrix} 1/a_n & 0 & 0 \\ 0 & a_n & b_n \\ 0 & 0 & 1 \end{pmatrix}$$

goes as $n \to \infty$ to the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R}),$$

computing the limit we have

$$a_{12} = a_{13} = a_{21} = a_{31} = a_{32} = 0, \qquad a_{33} = 1,$$
$$\lim_{n\to\infty} \frac{1}{a_n} = a_{11}, \qquad \lim_{n\to\infty} a_n = a_{22},$$

and thus $a_{11} \geqslant 0$, $a_{22} \geqslant 0$, $a_{11}a_{22} = 1$, so one has $a_{11} > 0$, $a_{22} > 0$, then $A \in G$. Hence, G is a closed subgroup of $\mathrm{GL}(3,\mathbb{R})$.

(ii) Suppose

$$X = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The equations of translations are

$$L_A \equiv \begin{cases} \bar{x} = ax, \\ \bar{y} = ay + b, \end{cases} \qquad R_A \equiv \begin{cases} \bar{x} = ax, \\ \bar{y} = bx + y. \end{cases}$$

A basis of left-invariant 1-forms is $\{\omega_1 = \mathrm{d}x/x,\ \omega_2 = \mathrm{d}y/x\}$. In fact,

$$L_A^*\omega_1 = \frac{\mathrm{d}\bar{x}}{\bar{x}} = \frac{\mathrm{d}x}{x} = \omega_1, \qquad L_A^*\omega_2 = \frac{\mathrm{d}\bar{y}}{\bar{x}} = \frac{\mathrm{d}y}{x} = \omega_2.$$

A basis of right-invariant differential 1-forms is

$$\{\bar{\omega}_1 = \omega_1,\ \bar{\omega}_2 = x\omega_2 - y\omega_1\}.$$

In fact,

$$R_A^*\bar{\omega}_1 = \frac{\mathrm{d}\bar{x}}{\bar{x}} = \frac{\mathrm{d}x}{x} = \bar{\omega}_1, \qquad R_A^*\bar{\omega}_2 = \bar{x}\frac{\mathrm{d}\bar{y}}{\bar{x}} - \bar{y}\frac{\mathrm{d}\bar{x}}{\bar{x}} = \mathrm{d}y - \frac{y}{x}\,\mathrm{d}x = \bar{\omega}_2.$$

Hence, the most general form of a left-invariant symmetric bilinear $(0,2)$ tensor is

$$g = \lambda\,\omega_1 \otimes \omega_1 + \mu(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + \nu\,\omega_2 \otimes \omega_2, \quad \lambda, \mu, \nu \in \mathbb{R}.$$

Suppose g is also right-invariant. Since

$$R_A^*\omega_2 = \frac{\mathrm{d}\bar{y}}{\bar{x}} = \frac{b\,\mathrm{d}x + \mathrm{d}y}{ax} = \frac{b}{a}\,\omega_1 + \frac{1}{a}\,\omega_2,$$

we would have

$$R_A^*\,g = \lambda\,\omega_1\otimes\omega_1 + \mu\left\{\omega_1\otimes\left(\frac{b}{a}\,\omega_1 + \frac{1}{a}\,\omega_2\right) + \left(\frac{b}{a}\,\omega_1 + \frac{1}{a}\,\omega_2\right)\otimes\omega_1\right\}$$
$$+\,\nu\left(\frac{b}{a}\,\omega_1 + \frac{1}{a}\,\omega_2\right)\otimes\left(\frac{b}{a}\,\omega_1 + \frac{1}{a}\,\omega_2\right).$$

If $R_A^*g = g$ for all $A\in G$, we necessarily have $\mu=\nu=0$, thus $g=\lambda\omega_1\otimes\omega_1$ is the most general expression of the bi-invariant metric. But it is singular.

Problem 6.173 Let M be the pseudo-Euclidean space with metric $g=\sum_{i=1}^n \varepsilon_i\,\mathrm{d}x^i \otimes \mathrm{d}x^i$, $\varepsilon_i=\pm 1$, and let

$$\Delta = -\sum_{i=1}^n \varepsilon_i\,\frac{\partial^2}{\partial (x^i)^2}$$

be the Laplacian on M.

Prove that the Laplace equation $\Delta f = 0$, $f\in C^\infty M$, has solution $f=\psi(\Omega)$, where

$$\psi(\Omega) = \begin{cases} A\log|\Omega| + B & \text{if } n=2,\\ \frac{A}{|\Omega|^{\frac{1}{2}(n-2)}} + B & \text{if } n>2,\end{cases}$$

and $\Omega = \frac{1}{2}\sum_{i=1}^n \varepsilon_i (x^i - x_0^i)^2$, in any neighbourhood in which Ω does not vanish and has constant sign.

The reader can find much relevant geometric theory in Ruse, Walker and Willmore [30].

Solution If $n=2$, one has

$$\Delta\,\psi(\Omega) = -\sum_{i=1}^2 \varepsilon_i\,\frac{\partial^2}{\partial (x^i)^2}\bigl(A\log|\Omega| + B\bigr) = -\sum_{i=1}^2 \varepsilon_i\,A\,\frac{\partial}{\partial x^i}\,\frac{\varepsilon_i (x^i - x_0^i)}{|\Omega|}$$
$$= -A\sum_{i=1}^2 \frac{\Omega - (x^i - x_0^i)\varepsilon_i (x^i - x_0^i)}{\Omega^2} = -A\,\frac{2\Omega - 2\Omega}{\Omega^2} = 0.$$

For $n\geqslant 3$, we have

$$\Delta\,\psi(\Omega) = -\sum_{i=1}^n \varepsilon_i\,\frac{\partial^2}{\partial (x^i)^2}\left(\frac{A}{|\Omega|^{\frac{1}{2}(n-2)}} + B\right)$$

$$= -A\sum_{i=1}^{n}\varepsilon_i\frac{\partial^2}{\partial(x^i)^2}\left|\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\left(x^j-x_0^j\right)^2\right|^{-\frac{n}{2}+1}$$

$$= -A\left(-\frac{n}{2}+1\right)\sum_{i=1}^{n}\varepsilon_i\frac{\partial}{\partial x^i}\left[\left(\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\left(x^j-x_0^j\right)^2\right)^{-\frac{n}{2}}\right.$$
$$\left.\times\left(\pm\varepsilon_i\left(x^i-x_0^i\right)\right)\right]$$

$$= -A\left(-\frac{n}{2}+1\right)\sum_{i=1}^{n}\left[-\frac{n}{2}\varepsilon_i\left\{\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\left(x^j-x_0^j\right)^2\right\}^{-\frac{n}{2}-1}\right.$$
$$\left.\times\left\{\pm\varepsilon_i\left(x^i-x_0^i\right)\right\}\left\{\pm\varepsilon_i\left(x^i-x_0^i\right)\right\}+\left(\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j\left(x^j-x_0^j\right)^2\right)^{-\frac{n}{2}}(\pm 1)\right]$$

$$= \begin{cases} -A(-\frac{n}{2}+1)(-n\Omega^{-\frac{n}{2}-1}\Omega+n\Omega^{-\frac{n}{2}})=0 & \text{if } \Omega>0, \\ -A(-\frac{n}{2}+1)(n\Omega^{-\frac{n}{2}-1}\Omega-n\Omega^{-\frac{n}{2}})=0 & \text{if } \Omega<0. \end{cases}$$

Problem 6.174 Let $(\mathbb{R}^2, g)$ be the pseudo-Riemannian manifold with

$$g=\frac{4}{c}\left(\cosh^2 y\,\mathrm{d}x^2-\mathrm{d}y^2\right).$$

Calculate the Laplacian Δ on functions $f\in C^\infty\mathbb{R}^2$.

Solution The non-vanishing Christoffel symbols are

$$\Gamma_{12}^1=\tanh y,\qquad \Gamma_{11}^2=\frac{1}{2}\sinh 2y.$$

Now applying the usual formula (valid in every local coordinate system)

$$\Delta f=-\sum_{i,j}g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j}-\sum_k\Gamma_{ij}^k\frac{\partial f}{\partial x^k}\right),$$

we deduce

$$\Delta f=-\frac{c}{4}\left(\frac{1}{\cosh^2 y}\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}-\tanh y\frac{\partial f}{\partial y}\right).$$

Problem 6.175 Let $x\colon [a,b]\times(-\delta,\delta)\to M$ be a variation of a segment $\gamma(u)=x(u,0)$ of a geodesic on a Riemannian manifold (M,g). For each $v\in(-\delta,\delta)$, let $L_x(v)$ be the length of the longitudinal curve $u\to x(u,v)$. L_x is a real-valued function, where $L_x(0)$ denotes the length of the given segment γ of the geodesic.

Compute the *second variation of arc length*, $L''_x(0)$, by the usual formula and also directly from L_x, in the following cases (equipped with the respective usual metrics):

(i) In S^2, $x(u,v) = (\cos v \cos u, \cos v \sin u, \sin v)$, $0 \leqslant u \leqslant \pi$.
(ii) In $\mathbb{R}^2$, $x(u,v) = (u \cosh v,\ v)$, $-1 \leqslant u \leqslant 1$.
(iii) In $\mathbb{R}^2$, $x(u,v) = \begin{cases} (u,\ vu) & \text{if } u \in [0,1], \\ (u,\ v(2-u)) & \text{if } u \in [1,2]. \end{cases}$

The relevant theory is developed, for instance, in O'Neill [26, Chap. 10].

Solution

(i) The ends of $x(u,v)$ are $x(0,v) = (\cos v, 0, \sin v)$ and $x(\pi, v) = (-\cos v, 0, \sin v)$, for $v \in (-\delta, \delta)$. For $v = 0$ we have the curve

$$\gamma : x(u,0) = (\cos u, \sin u, 0), \quad 0 \leqslant u \leqslant \pi,$$

with origin $(1,0,0)$ and end $(-1,0,0)$. For $v = -\delta$ we have the curve

$$x(u,-\delta) = (\cos\delta \cos u, \cos\delta \sin u, -\sin\delta),$$

with origin $(\cos\delta, 0, -\sin\delta)$ and end $(-\cos\delta, 0, -\sin\delta)$; for $v = \delta$,

$$x(u,\delta) = (\cos\delta \cos u, \cos\delta \sin u, \sin\delta).$$

The curve $x(u,0)$ is a segment of a geodesic. The length of $x(u,v)$, for a given v, is

$$L_x(v) = \int_0^\pi \sqrt{\big(x_u(u,v)\big)^2 \, \mathrm{d}u^2} = \int_0^\pi \cos v \, \mathrm{d}u = \pi \cos v.$$

The length of γ is $L_x(0) = \pi$. The second variation of the arc on x is

$$L''(0) = \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} = (-\pi \cos v)_{v=0} = -\pi,$$

where $L = L_x$.

Since γ is a geodesic, it must be $L'(0) = 0$. In fact, we have

$$L'(0) = \left.\frac{\mathrm{d}L}{\mathrm{d}v}\right|_{v=0} = (-\pi \sin v)_{v=0} = 0.$$

As for Synge's formula (see p. 598), since S^2 is a space of constant curvature 1, one has

$$g\big(R(V,\gamma')V,\gamma'\big) = g\big(V,\gamma'\big)g\big(V,\gamma'\big) - g(V,V)g\big(\gamma',\gamma'\big),$$

where V denotes the variation vector field $V(u) = (\partial x/\partial v)_{v=0}$, given by

$$V(u) = (-\sin v \cos u, -\sin v \sin u, \cos v)_{v=0} = (0,0,1),$$

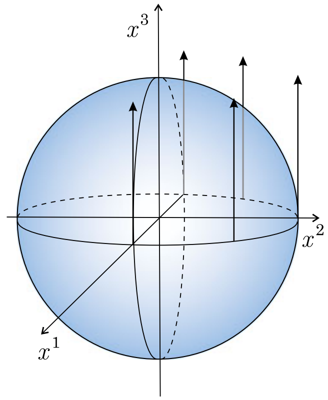


Fig. 6.19 The variation vector field on γ

(see Fig. 6.19) and $\gamma'(u) = (-\sin u, \cos u, 0)$, thus $c = |\gamma'| = 1$. Therefore,

$$g(V, V) = 1, \qquad g(\gamma', \gamma') = 1,$$
$$g(V, \gamma') = 0, \qquad g\big(R(V, \gamma')V, \gamma'\big) = -1.$$

We have $V' = (0, 0, 0)$, thus $g(V'^{\perp}, V'^{\perp}) = 0$. On the other hand, the *transverse acceleration vector field* $A(u)$ on γ is given by

$$\begin{aligned} A(u) &= \frac{\partial^2}{\partial v^2}\bigg|_{v=0} (\cos v \cos u, -\cos v \sin u, \sin v) \\ &= (-\cos v \cos u, -\cos v \sin u, -\sin v)_{v=0} = (-\cos u, -\sin u, 0), \end{aligned}$$

from which $g(\gamma', A) = 0$ and $L''(0) = -\int_0^{\pi} \mathrm{d}u = -\pi$; that is, the same result as before.

(ii) The ends of $x(u, v)$ are

$$x(-1, v) = (-\cosh v, v), \qquad x(1, v) = (\cosh v, v).$$

One has $v \in (-\delta, \delta)$. For $v = 0$ we have the curve in $\mathbb{R}^2$

$$\gamma : x(u, 0) = (u, 0), \quad -1 \leqslant u \leqslant 1,$$

which is obviously a segment of a geodesic. For $v = -\delta$, we have the curve

$$x(u, -\delta) = (u \cosh \delta, -\delta),$$

with origin $(-\cosh \delta, -\delta)$ and end $(\cosh \delta, -\delta)$. For $v = \delta$, we have the curve

$$x(u, \delta) = (u \cosh \delta, \delta),$$

with origin $(-\cosh \delta, \delta)$ and end $(\cosh \delta, \delta)$. The length of $x(u, v)$ is

$$L_x(v) = \int_{-1}^{1} \big(\cosh^2 v\big)^{\frac{1}{2}} \, \mathrm{d}u = 2 \cosh v.$$

The length of γ is $L_x(0) = 2$. The second variation of the arc on x is

$$L''(0) = \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} = (2\cosh v)_{v=0} = 2.$$

As γ is a geodesic, it must be $L'(0) = 0$. In fact, we have

$$L'(0) = (2\sinh v)_{v=0} = 0.$$

As for Synge's formula, we have $c = |\gamma'| = 1$ and $R = 0$ as $M = \mathbb{R}^2$ with its usual metric. Moreover,

$$V(u) = \left.\frac{\partial}{\partial v}\right|_{v=0} (u\cosh v, v) = (u\sinh v, 1)_{v=0} = (0, 1),$$

and thus $V' = (0,0)$, so that $V'^{\perp} = (0,0)$. We have

$$A(u) = \left.\frac{\partial^2}{\partial v^2}\right|_{v=0} x(u, v) = (u\cosh v, 0)_{v=0} = (u, 0),$$

so that one has $g(\gamma', A) = u$. Hence $L''(0) = [g(\gamma', A)]_{-1}^{1} = 2$, as before.

(iii) For $v = 0$, we have the curve

$$\gamma : x(u, 0) = (u, 0), \quad u \in [0, 2],$$

which is a segment of a geodesic. For $v = -\delta$, we have the curve

$$x(u, -\delta) = \begin{cases} (u, -\delta u) & \text{if } u \in [0, 1], \\ (u, -\delta(2-u)) & \text{if } u \in [1, 2], \end{cases}$$

and for $v = \delta$ the symmetric one with respect to the u-axis. The length of $x(u, v)$ is

$$L_x(v) = \int_0^1 \left(1 + v^2\right)^{\frac{1}{2}} \mathrm{d}u + \int_1^2 \left(1 + v^2\right)^{\frac{1}{2}} \mathrm{d}u = 2\sqrt{1 + v^2}.$$

The length of γ is $L_x(0) = 2$. The second variation of the arc on x is

$$L''(0) = \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} = \left.\frac{\mathrm{d}}{\mathrm{d}v}\right|_{v=0} \left(2\frac{v}{\sqrt{1 + v^2}}\right) = 2.$$

As γ is a geodesic, it must be $L'(0) = 0$. In fact,

$$L'(0) = \left(\frac{2v}{\sqrt{1 + v^2}}\right)_{v=0} = 0.$$

As for Synge's formula, we have $c = |\gamma'| = 1$. Furthermore, one has $R = 0$, and

$$V = \frac{\partial}{\partial v}\bigg|_{v=0} x(u,v) = \begin{cases} (0,u) & \text{if } u \in [0,1], \\ (0,2-u) & \text{if } u \in [1,2]. \end{cases}$$

$$V' = \begin{cases} (0,1) & \text{if } u \in [0,1], \\ (0,-1) & \text{if } u \in [1,2]. \end{cases}$$

$$g(V',\gamma') = \begin{cases} (0,1)\cdot(1,0) = 0 & \text{if } u \in [0,1], \\ (0,-1)\cdot(1,0) = 0 & \text{if } u \in [1,2]. \end{cases}$$

Thus $V'^{\perp} = V'$;

$$g(V'^{\perp}, V'^{\perp}) = 1, \qquad A = \frac{\partial^2}{\partial v^2}\bigg|_{v=0} x(u,v) = \begin{cases} (0,0) & \text{if } u \in [0,1], \\ (0,0) & \text{if } u \in [1,2]. \end{cases}$$

Hence, $L''(0) = \int_0^2 \mathrm{d}u = 2$, as before.

Problem 6.176 Let M be an embedded submanifold of the *paracomplex projective space* $P(V \oplus V^*)$ (see Problem 6.170), such that the metric inherited on M from g is non-degenerate, and denote by $\mathscr{N}$ the normal bundle $\mathscr{N} = \bigcup_{p\in M} \mathscr{N}_p$, where $\mathscr{N}_p = (T_pM)^{\perp}$, which exists by the non-degeneracy of the induced metric. Such a submanifold is said to be totally umbilical if there exists $\xi \in \Gamma\mathscr{N}$ such that

$$\alpha(X,Y) = g(X,Y)\xi, \qquad X, Y \in \mathfrak{X}(M),$$

where $\alpha(X,Y)$ is the second fundamental form and ξ is called the *normal curvature vector field*. Then, for such a submanifold:

(i) Find the expression of the Codazzi equation.
(ii) Find the expression of the Ricci equation.
(iii) Prove, applying Gauss equation, that if $J(TM) \subset \mathscr{N}$, then

$$R(X,Y,Z,W) = \big(1 + g(\xi,\xi)\big)\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big).$$

The relevant theory is developed, for instance, in [12].

Solution

(i) If ∇ denotes the Levi-Civita connection of any pseudo-Riemannian submanifold M, we have

$$\nabla_X Y = \tau\widetilde{\nabla}_X Y, \qquad \alpha(X,Y) = \nu\widetilde{\nabla}_X Y, \qquad A_\eta = -\tau\widetilde{\nabla}_X \eta, \qquad \nabla^{\perp}_X \eta = \nu\widetilde{\nabla}_X \eta,$$

where $X, Y \in \mathfrak{X}(M)$; $\eta \in \Gamma\mathfrak{N}$; τ and ν denote the "tangential part" and the "normal part", respectively; $\widetilde{\nabla}$ is the Levi-Civita connection of $P(V \oplus V^*)$;

$\nabla^{\perp}$ denotes the connection induced in $\mathcal{N}$; and

$$g(A_\eta X, Y) = g\big(\alpha(X, Y), \eta\big).$$

Codazzi's equation is written in general as

$$-\nu\widetilde{R}(X, Y)Z = (\widehat{\nabla}_X\alpha)(Y, Z) - (\widehat{\nabla}_Y\alpha)(X, Z),$$

where $\widehat{\nabla}_X\alpha$ is defined by

$$(\widehat{\nabla}_X\alpha)(Y, Z) = \nabla^{\perp}_X\big(\alpha(Y, Z)\big) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

If the pseudo-Riemannian manifold M is moreover totally umbilical, then the previous equation reduces to

$$\begin{aligned}(\widehat{\nabla}_X\alpha)(Y, Z) &= \nabla^{\perp}_X\big(g(Y, Z)\xi\big) - g(\nabla_X Y, Z)\xi - g(Y, \nabla_X Z)\xi \\ &= X\big(g(Y, Z)\big)\xi + g(Y, Z)\nabla^{\perp}_X\xi - g(\nabla_X Y, Z)\xi - \big(g(Y, \nabla_X Z)\big)\xi \\ &= g(Y, Z)\nabla^{\perp}_X\xi + (\nabla_X g)(Y, Z) = g(Y, Z)\nabla^{\perp}_X\xi.\end{aligned}$$

Hence, on account of the expression for the curvature of $P(V \oplus V^*)$ in the remark in Problem 6.170, we have for Codazzi's equation

$$\begin{aligned}-\nu\widetilde{R}(X, Y)Z &= -\nu\big(g(X, Z)Y - g(Y, Z)X \\ &\quad + g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\big) \\ &= -g(X, JZ)\nu JY + g(Y, JZ)\nu JX + 2g(X, JY)\nu JZ \\ &= g(Y, Z)\nabla^{\perp}_X\xi - g(X, Z)\nabla^{\perp}_Y\xi.\end{aligned}$$

That is,

$$\begin{aligned}&g(X, \tau JZ)\nu JY - g(Y, \tau JZ)\nu JX + 2g(X, \tau JY)\nu JZ \\ &\quad = g(Y, Z)\nabla^{\perp}_X\xi - g(X, Z)\nabla^{\perp}_Y\xi.\end{aligned}$$

(ii) Let $R_{\nabla^{\perp}}$ be the curvature tensor field of the connection $\nabla^{\perp}$ in $\mathscr{N}$. Then, Ricci's equation is

$$\begin{aligned}&\nu\widetilde{R}(X, Y)\eta = R_{\nabla^{\perp}}(X, Y)\eta - \alpha(A_\eta X, Y) + \alpha(A_\eta Y, X), \\ &\quad X, Y \in \mathfrak{X}(M),\ \eta \in \mathcal{N}.\end{aligned}$$

As

$$g(A_\eta X, Y) = g\big(\alpha(X, Y), \eta\big) = g(X, Y)g(\xi, \eta),$$

we have $A_\eta X = g(\xi, \eta)X$ and $\alpha(A_\eta X, Y) = g(\xi, \eta)g(X, Y)\xi$. Hence, Ricci's equation reduces to

$$\nu\widetilde{R}(X, Y)\eta = R_{\nabla^{\perp}}(X, Y)\eta.$$

(iv) If $J(TM) \subset \mathcal{N}$, direct application of Gauss' equation gives us

$$\begin{aligned}
R(X, Y, Z, W) &= \widetilde{R}(X, Y, Z, W) + g\big(\alpha(X, Z), \alpha(Y, W)\big) \\
&\quad - g\big(\alpha(Y, Z), \alpha(X, W)\big) \\
&= g(X, Z)g(Y, W) - g(X, W)g(Y, Z) - g(X, JZ)g(Y, JW) \\
&\quad + g(X, JW)g(Y, JZ) - 2g(X, JY)g(Z, JW) \\
&\quad + g(X, Z)g(Y, W)g(\xi, \xi) - g(Y, Z)g(X, W)g(\xi, \xi) \\
&= \big(1 + g(\xi, \xi)\big)\big(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\big),
\end{aligned}$$

as wanted.

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Chapter 7
Some Formulas and Tables

Abstract This chapter contains a 56-page-long list of formulae from the calculus on manifolds, and tables concerning different topics: Lie groups, Lie algebras and symmetric spaces, a list of Poincaré polynomials of compact simple Lie groups, an overview of real forms of classical complex simple Lie algebras and their corresponding simple Lie groups, a table of irreducible Riemannian symmetric spaces of type I and III, a table of Riemannian symmetric spaces of classical type with noncompact isotropy group, etc. One can find the formulae for Christoffel symbols, the curvature tensor, Bianchi identities, Ricci tensor, the basic differential operators, the expression for conformal changes of Riemannian metrics, Cartan structure equations for pseudo-Riemannian manifolds, and many more. Several of these formulae are used throughout the book; others are not, but they have been included since such a collection might prove useful as an aide-mémoire, also to lecturers and researchers.

Chapter 1

- Stereographic projection σ (from either the north pole or the south pole) of the sphere $S^n((0,\dots,0),1)$ with centre $(0,\dots,0)\in\mathbb{R}^{n+1}$ and radius 1 onto the equatorial hyperplane:

$$\begin{array}{ccc} U_N & \xrightarrow{\sigma_N} & \mathbb{R}^n \\ (x^1,\dots,x^{n+1}) & \longmapsto & \left(\dfrac{x^1}{1-x^{n+1}},\dots,\dfrac{x^n}{1-x^{n+1}}\right) \\ U_S & \xrightarrow{\sigma_S} & \mathbb{R}^n \\ (x^1,\dots,x^{n+1}) & \longmapsto & \left(\dfrac{x^1}{1+x^{n+1}},\dots,\dfrac{x^n}{1+x^{n+1}}\right) \end{array}$$

where

$$S^n=\left\{(x^1,\dots,x^{n+1})\in\mathbb{R}^{n+1}:\sum_{i=1}^{n+1}(x^i)^2=1\right\},$$

$$U_N=\{(x^1,\dots,x^{n+1})\in S^{n+1}:x^{n+1}\neq 1\},$$

$$U_S=\{(x^1,\dots,x^{n+1})\in S^{n+1}:x^{n+1}\neq -1\}.$$

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*, Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7_7,

- Inverse map σ_N^{-1} of the stereographic projection from the north pole of the sphere $S^n((0,\dots,0),1)$ onto the equatorial hyperplane:

$$\sigma_N^{-1}\big(y^1,\dots,y^n\big)=\left(\frac{2y^1}{|y|^2+1},\dots,\frac{2y^n}{|y|^2+1},\frac{|y|^2-1}{|y|^2+1}\right),\quad |y|^2=\sum_{i=1}^n\big(y^i\big)^2.$$

- Stereographic projection σ_N from the north pole of $S^n((0,\dots,0,r),r)\in\mathbb{R}^{n+1}$ with centre $(0,\dots,0,r)\in\mathbb{R}^{n+1}$ and radius r onto the hyperplane $x^{n+1}=0$ tangent to the south pole:

$$\sigma_N\big(x^1,\dots,x^{n+1}\big)=\left(\frac{2rx^1}{2r-x^{n+1}},\dots,\frac{2rx^n}{2r-x^{n+1}}\right).$$

- Inverse map σ_N^{-1} of the stereographic projection from the north pole of the sphere $S^n((0,\dots,0,r),r)$ onto the hyperplane $x^{n+1}=0$:

$$\sigma_N^{-1}\big(y^1,\dots,y^n\big)=\left(\frac{4r^2y^1}{4r^2+|y|^2},\dots,\frac{4r^2y^n}{4r^2+|y|^2},\frac{2r|y|^2}{4r^2+|y|^2}\right).$$

- Differential of a map $\Phi: M\to N$ between differentiable manifolds at $p\in M$, in terms of coordinate systems $(U,x^1,\dots,x^m)$ and $(V,y^1,\dots,y^n)$ around p and $\Phi(p)$:

$$\Phi_{*p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)=\sum_{j=1}^n\frac{\partial(y^j\circ\Phi)}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{\Phi(p)},\qquad i=1,\dots,m.$$

- A diffeomorphism between $\mathbb{R}^n$ and the open cube $(-1,1)^n\subset\mathbb{R}^n$:

$$\varphi:\mathbb{R}^n\to(-1,1)^n,\qquad \big(x^1,\dots,x^n\big)\mapsto\big(\tanh x^1,\dots,\tanh x^n\big).$$

- Standard local coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$ of the tangent bundle (TM,π,M) on a coordinate neighbourhood $\pi^{-1}(U)$ of TM over a coordinate neighbourhood U for a coordinate system $(U,x^1,\dots,x^n)$ around $p\in M$:

$$\begin{aligned}&\big(x^1,\dots,x^n,y^1,\dots,y^n\big)(v)\\&\quad=\big(\big(x^1\circ\pi\big)(v),\dots,\big(x^n\circ\pi\big)(v),\mathrm{d}y^1(v),\dots,\mathrm{d}y^n(v)\big),\quad v\in T_pM.\end{aligned}$$

- A property of the bracket of vector fields ($f,g\in C^\infty M$; $X,Y\in\mathfrak{X}(M)$):

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X.$$

- Jacobi identity for vector fields:

$$\big[[X,Y],Z\big]+\big[[Y,Z],X\big]+\big[[Z,X],Y\big]=0.$$

- A parallelisation of S^3 by unit vectors fields:

$$X_p = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}\right)_p,$$

$$Y_p = \left(-z\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial t}\right)_p,$$

$$Z_p = \left(-t\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}\right)_p$$

  $(p \in S^3 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\})$.
- Image vector field $\Phi \cdot X \in \mathfrak{X}(N)$ of $X \in \mathfrak{X}(M)$ by the diffeomorphism $\Phi : M \to N$:

$$(\Phi \cdot X)_p = \Phi_*(X_{\Phi^{-1}(p)}), \quad p \in N.$$

- A non-vanishing vector field on the sphere S^{2n+1}:

$$X_p = -x^2 \frac{\partial}{\partial x^1}\bigg|_p + x^1\frac{\partial}{\partial x^2}\bigg|_p + \cdots - x^{2n+2}\frac{\partial}{\partial x^{2n+1}}\bigg|_p + x^{2n+1}\frac{\partial}{\partial x^{2n+2}}\bigg|_p.$$

Chapter 2

- Nijenhuis torsion of two $(1, 1)$ tensor fields A, B:

$$\begin{aligned} S(X, Y) &= [AX, BY] + [BX, AY] + AB[X, Y] + BA[X, Y] \\ &\quad - A[X, BY] - A[BX, Y] - B[X, AY] - B[AX, Y]. \end{aligned}$$

- Nijenhuis tensor of a $(1, 1)$ tensor field J:

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y],$$

$$N^i_{jk} = \sum_l \left(J^l_j \frac{\partial J^i_k}{\partial x^l} - J^l_k \frac{\partial J^i_j}{\partial x^l} + J^i_l \frac{\partial J^l_j}{\partial x^k} - J^i_l \frac{\partial J^l_k}{\partial x^j}\right).$$

- Kulkarni–Nomizu product of two symmetric $(0, 2)$ tensors h, k:

$$\begin{aligned} (h \owedge k)(X, Y, Z, W) &= h(X, Z)k(Y, W) + h(Y, W)k(X, Z) \\ &\quad - h(X, W)k(Y, Z) - h(Y, Z)k(X, W). \end{aligned}$$

- Exterior or “wedge” or “Grassmann” product of differential forms:

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p, \quad p \in M,\ \alpha \in \Lambda^r M,\ \beta \in \Lambda^s M;$$

$$\begin{aligned} &(\alpha_p \wedge \beta_p)(X_1, \dots, X_{r+s}) \\ &= \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} (\operatorname{sgn} \sigma)\alpha_p(X_{\sigma(1)}, \dots, X_{\sigma(r)})\beta_p(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)}) \end{aligned}$$

$$= \sum_{\substack{\sigma\in\mathfrak{S}_{r+s}\\ \sigma(1)<\cdots<\sigma(r)\\ \sigma(r+1)<\cdots<\sigma(r+s)}} (\operatorname{sgn}\sigma)\alpha_p(X_{\sigma(1)},\dots,X_{\sigma(r)})\beta_p(X_{\sigma(r+1)},\dots,X_{\sigma(r+s)}),$$

$$X_i \in T_pM, \quad i = 1,\dots,r+s;$$

$$\alpha\wedge\beta = (-1)^{rs}\beta\wedge\alpha, \quad \alpha\in\Lambda^rM,\ \beta\in\Lambda^sM.$$

- Exterior differential $\mathrm{d}\colon \Lambda^*M \to \Lambda^*M$:
  (i) If $f \in C^\infty M$, then $\mathrm{d}f \in \Lambda^1 M$ is the usual differential of f.
  (ii) d is a linear map such that $\mathrm{d}(\Lambda^r M) \subset \Lambda^{r+1}M$.
  (iii) $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge\mathrm{d}\beta$ (α homogeneous).
  (iv) $\mathrm{d}^2 = 0$.

- Relation between the bracket product of vector fields and the exterior differential of a differential 1-form:

$$(\mathrm{d}\omega)(X,Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big). \tag{7.1}$$

- Relation between the bracket product of vector fields and the exterior differential of a differential r-form:

$$\begin{aligned}(\mathrm{d}\omega)&(X_0,\dots,X_r)\\ &= \sum_{i=0}^{r}(-1)^iX_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big)\\ &\quad + \sum_{0\leqslant i<j\leqslant r}(-1)^{i+j}\omega\big([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r\big).\end{aligned} \tag{7.2}$$

Induced (or pull-back of a) differential form $\Phi^*\theta$ of $\theta = f_i\,\mathrm{d}y^i$ for $\Phi\colon M\to N$ (in terms of local coordinates $(x^1,\dots,x^m)$, $(y^1,\dots,y^n)$ on M, N, respectively):

$$\Phi^*\theta \equiv \begin{pmatrix} \frac{\partial(y^1\circ\Phi)}{\partial x^1} & \cdots & \frac{\partial(y^n\circ\Phi)}{\partial x^1}\\ \vdots & & \vdots\\ \frac{\partial(y^1\circ\Phi)}{\partial x^m} & \cdots & \frac{\partial(y^n\circ\Phi)}{\partial x^m}\end{pmatrix}\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix} \equiv \frac{\partial(y^i\circ\Phi)}{\partial x^j} f_i\,\mathrm{d}x^j.$$

- Basis of differential 1-forms $\{\mu^k = \mu^k_l\,\mathrm{d}x^l\}$ dual to the basis of vector fields $\{e_i = \lambda_i^j\,\partial/\partial x^j\}$:

$$\big(\mu^i_j\big) = {}^t\big(\lambda^i_j\big)^{-1}.$$

- Some formulas for the Lie derivative:

$$L_X f = Xf, \quad f\in C^\infty M;$$

$$(L_XY)_p = \lim_{t\to 0}\frac{1}{t}\big(Y_p - \varphi_{t*}Y_{\varphi_t^{-1}(p)}\big), \quad \varphi_t = \text{local flow of } X;$$

$$L_X Y = [X, Y];$$

$$L_Y\big(\omega(X_1, \dots, X_r)\big)$$

$$= (L_Y\omega)(X_1, \dots, X_r) + \sum_{i=1}^{r} \omega\big(X_1, \dots, X_{i-1}, [Y, X_i], X_{i+1}, \dots, X_r\big);$$

$$(L_X T)\big(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s\big)$$

$$= X\big(T\big(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s\big)\big) - \sum_{i=1}^{r} T\big(\omega^1, \dots, L_X\omega^i, \dots, \omega^r, Y_1, \dots, Y_s\big)$$

$$- \sum_{i=1}^{s} T\big(\omega^1, \dots, \omega^r, Y_1, \dots, L_X Y_i, \dots, Y_s\big);$$

$$L_X(T_1 \otimes T_2) = (L_X T_1) \otimes T_2 + T_1 \otimes (L_X T_2),$$

$$L_{[X,Y]} = [L_X, L_Y];$$

$$L_X \mathrm{d} = \mathrm{d} L_X$$

$(X, Y, X_i, Y_i \in \mathfrak{X}(M);\ \omega, \omega^i \in \Lambda^* M;\ T, T_i \in \mathcal{T}_s^r M)$.

- Interior product:

$$(i_X\omega)(X_1, \dots, X_{r-1}) = \omega(X, X_1, \dots, X_{r-1}), \quad \omega \in \Lambda^r M;$$

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^r \alpha \wedge i_X\beta, \quad \alpha \in \Lambda^r M,\ \beta \in \Lambda^s M;$$

$$L_X\omega = i_X\, \mathrm{d}\omega + \mathrm{d} i_X\omega;$$

$$[L_X, i_Y] = i_{[X,Y]}. \tag{7.3}$$

- Canonical 1-form ϑ and canonical symplectic form Ω on the cotangent bundle (T^*M, π, M):

$$\vartheta_\omega(X) = \omega(\pi_* X), \quad \omega \in T^*M,\ X \in T_\omega T^*M;$$

$$\vartheta = \sum_i p_i \mathrm{d}q^i; \qquad \Omega = \mathrm{d}\vartheta = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$$

$((q^1, \dots, q^n, p_1, \dots, p_n) =$ local coordinates on$T^*M)$.

- Hamilton equations:

$$i_{\sigma'}(\Omega \circ \sigma) + \mathrm{d}H \circ \sigma = 0$$

$(H \in C^\infty(T^*M)$ and $\sigma : (a, b) \to T^*M$ a C^∞ curve with tangent vector $\sigma')$.

Chapter 3

- Divergence of a vector field X on an oriented manifold M with fixed volume element v:

$$(\operatorname{div} X)v = L_X v.$$

- Stokes' Theorem I:

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega$$

(see Theorem 3.3).
- Stokes' Theorem II:

$$\int_{\partial D} \omega = \int_D \mathrm{d}\omega$$

(see Theorem 3.6).
- Green's theorem: for any vector field X on an oriented compact manifold M with a fixed volume element v,

$$\int_M (\operatorname{div} X) v = 0.$$

Chapter 4

Some Usual Lie Groups

- General linear group:

$$\mathrm{GL}(n, \mathbb{C}) = \left\{ A \in M(n, \mathbb{C}) : \det A \neq 0 \right\}.$$

- Special linear group:

$$\mathrm{SL}(n, \mathbb{C}) = \left\{ A \in \mathrm{GL}(n, \mathbb{C}) : \det A = 1 \right\}.$$

- Unitary group:

$$\mathrm{U}(n) = \left\{ A \in M(n, \mathbb{C}) : {}^t\bar{A} A = I \right\}$$

($t =$ transpose; bar $=$ complex conjugation; $I =$ identity matrix).
- Special unitary group:

$$\mathrm{SU}(n) = \left\{ A \in \mathrm{U}(n) : \det A = 1 \right\}.$$

- Complex orthogonal group:

$$\mathrm{O}(n, \mathbb{C}) = \left\{ A \in M(n, \mathbb{C}) : {}^t A A = I \right\}.$$

- Complex special orthogonal group:

$$\mathrm{SO}(n, \mathbb{C}) = \left\{ A \in \mathrm{O}(n, \mathbb{C}) : \det A = 1 \right\}.$$

- Symplectic group over $\mathbb{C}$:

$$\mathrm{Sp}(n, \mathbb{C}) = \left\{ A \in \mathrm{GL}(2n, \mathbb{C}) : {}^t A \Omega A = \Omega \right\}$$

($\Omega \equiv \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} \right) =$ symplectic 2-form on $\mathbb{C}^{2n}$).

- Real general linear group:

$$\mathrm{GL}(n,\mathbb{R}) = \left\{A \in M(n,\mathbb{R}) : \det A \neq 0\right\}.$$

- Real special linear group:

$$\mathrm{SL}(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{GL}(n,\mathbb{R}) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : \det A = 1\right\}.$$

- Orthogonal group:

$$\mathrm{O}(n) = \mathrm{U}(n) \cap \mathrm{GL}(n,\mathbb{R}) = \mathrm{O}(n,\mathbb{C}) \cap \mathrm{GL}(n,\mathbb{R}) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : {}^t\!AA = I\right\}.$$

- Special orthogonal group:

$$\mathrm{SO}(n) = \left\{A \in \mathrm{O}(n) : \det A = 1\right\}.$$

- Lorentz group:

$$\mathrm{O}(k, n-k) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : {}^t\!A \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} A = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}\right\}.$$

- Symplectic group over $\mathbb{R}$:

$$\mathrm{Sp}(n,\mathbb{R}) = \left\{A \in \mathrm{GL}(2n,\mathbb{R}) : {}^t\!A\Omega A = \Omega\right\}$$

($\Omega = \sum_{k=1}^n \mathrm{d}x^k \wedge \mathrm{d}x^{n+k} \equiv \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) =$ symplectic 2-form on $\mathbb{R}^{2n}$).

- Symplectic group:

$$\begin{aligned}
\mathrm{Sp}(n) = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n) &= \left\{C \in \mathrm{GL}(2n,\mathbb{C}) : {}^t\bar{C}C = I, {}^tC\Omega C = \Omega\right\} \\
&= \left\{\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \in M(2n,\mathbb{C}) : {}^t\!\bar{A}A + {}^t\!\bar{B}B = I, {}^t\!AB - {}^t\!BA = 0\right\} \\
&= \left\{A + \mathbf{j}B \in M(n,\mathbb{H}) : {}^t(\overline{A + \mathbf{j}B})(A + \mathbf{j}B) = I\right\} \\
&= \left\{A + \mathbf{j}B \text{ preserving the quaternionic Hermitian product } \langle\,,\rangle \text{ on } \mathbb{H}^n\right\}
\end{aligned}$$

($\mathbb{H} \equiv \mathbb{C} + \mathbf{j}\mathbb{C}$; $\mathbf{j}^2 = -1$, $\overline{a + \mathbf{j}b} = \bar{a} - \mathbf{j}\bar{b}$, $\mathbf{j}b = \bar{b}\mathbf{j}\ \forall a, b \in \mathbb{C}$, $\langle u, v\rangle = \sum_r \bar{p}^r q^r$, $u = (p^1, \dots, p^n)$, $v = (q^1, \dots, q^n) \in \mathbb{H}^n$).

- $\mathrm{Sp}(n)\mathrm{Sp}(1)$: Let $\iota\colon \mathrm{Sp}(1) \equiv \{q \in \mathbb{H} : |q| = 1\} \to \mathrm{SO}(4n)$ be the inclusion given by $\iota(q) = \mathrm{diag}(B_q, \overset{(n)}{\dots}, B_q)$ with

$$B_q = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \in \mathrm{SO}(4).$$

Then $\iota(\pm 1) = \pm I_{4n} \subset \mathrm{SO}(4n)$. The group $\mathrm{Sp}(n)$ is a subgroup of $\mathrm{SO}(4n)$ and $\mathrm{Sp}(n) \cap \iota(\mathrm{Sp}(1)) = \{\pm I_{4n}\}$. Then, omitting the ι, one defines

$$\mathrm{Sp}(n)\mathrm{Sp}(1) = \big(\mathrm{Sp}(n) \times \mathrm{Sp}(1)\big)/\{\pm I_{4n}\} \subset \mathrm{SO}(4n).$$

The action of this group on $\mathbb{R}^{4n} \equiv \mathbb{H}^n$ is given by $(B,q)v = Bv\bar{q}$, $B \in \mathrm{Sp}(n)$, $q \in \mathrm{Sp}(1)$, $v \in \mathbb{H}^n$, and $\bar{q}$ the quaternionic conjugate of q.

On the other hand, let V be a $4n$-dimensional real vector space. A quaternionic structure on V is a three-dimensional space of $\operatorname{End} V$ given by

$$Q = \mathbb{R}J_1 + \mathbb{R}J_2 + \mathbb{R}J_3, \qquad J_k^2 = -I, \qquad J_3 = J_1 J_2,$$
$$J_k J_l = -J_l J_k, \quad k,l = 1,2,3.$$

An orthogonal automorphism $A \in \mathrm{SO}(4n)$ belongs to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ if and only if $A \circ J_a = \sum_{b=1}^3 m_a^b J_b \circ A$, $a = 1,2,3$, for a certain matrix $(m_b^a) \in \mathrm{SO}(3)$, which is obtained from the projection homomorphism $\mathrm{Sp}(n)\mathrm{Sp}(1) \to \mathrm{Sp}(1)/\{\pm\mathrm{Id}\} = \mathrm{SO}(3)$. Let

$$S(Q) = \left\{ J = a_1 J_1 + a_2 J_2 + a_3 J_3 \in Q : a_1^2 + a_2^2 + a_3^2 = 1 \right\}.$$

A Euclidean metric g on V is called Hermitian with respect to Q if $g(JX, JY) = g(X,Y)$ for $J \in S(Q)$, $X, Y \in V$. The pair (Q, g) is called a quaternionic Hermitian structure. Then, for V as a right module over $\mathbb{H}$,

$$\mathrm{Sp}(n)\mathrm{Sp}(1) = \mathrm{Aut}(Q,g) = \left\{ \varphi \in \mathrm{GL}(V) : \varphi \text{ preserves } (Q,g) \right\}.$$

Some Topological Properties of Some Usual Lie Groups

Group	$\dim_{\mathbb{R}}$	Type	Group	$\dim_{\mathbb{R}}$	Type
$\mathrm{GL}(n,\mathbb{C})$	$2n^2$	cn	$\mathrm{O}(n,\mathbb{C})$	$n(n-1)$	2 cc
$\mathrm{SL}(n,\mathbb{C})$	$2(n^2-1)$	cn, sc	$\mathrm{SO}(n,\mathbb{C})$	$n(n-1)$	cn
$\mathrm{GL}(n,\mathbb{R})$	n^2	2 cc	$\mathrm{O}(n)$	$n(n-1)/2$	2 cc, cp
$\mathrm{SL}(n,\mathbb{R})$	n^2-1	cn	$\mathrm{SO}(n)$	$n(n-1)/2$	cn, cp
$\mathrm{U}(n)$	n^2	cn, cp	$\mathrm{SO}(p,q)$	$(p+q)(p+q-1)/2$	2 cc (∗)
$\mathrm{SU}(n)$	n^2-1	cn, sc, cp	$\mathrm{Sp}(n,\mathbb{C})$	$2(2n^2+n)$	cn
$\mathrm{SU}(p,q)$	$(p+q)^2-1$	cn	$\mathrm{Sp}(n)$	$n(2n+1)$	cn, sc, cp
$\mathrm{SU}^*(2n)$	$2(2n^2-1)$	cn	$\mathrm{Sp}(p,q)$	$(p+q)(2(p+q)+1)$	cn
$\mathrm{SO}^*(2n)$	$2n(n-1)$	cn	$\mathrm{Sp}(n,\mathbb{R})$	$n(2n+1)$	cn

cn = connected; sc = simply connected; 2 cc = 2 connected components; cp = compact; (∗) $0 < p < p+q$

Isomorphisms of Spin(n) *with Some Classical Groups*

Spin(2)	U(1)
Spin(3)	SU(2)
Spin(4)	SU(2) × SU(2)
Spin(5)	Sp(2)
Spin(6)	SU(4)

(See, for instance, Harvey [16].)

Miscellaneous

- Euler angles (of rotations around the x, y, z-axes):

$$\mathrm{SO}(3) = \left\{ g(\varphi, \theta, \psi) = R_z(\varphi) R_x(\theta) R_z(\psi), 0 \leqslant \varphi, \psi \leqslant 2\pi, 0 \leqslant \theta \leqslant \pi \right\},$$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Compact Connected Lie Groups G Acting Effectively and Transitively on Some Sphere

Sphere	G	Isotropy
S^{n-1}	$\mathrm{SO}(n)$	$\mathrm{SO}(n-1)$
S^{2n-1}	$\mathrm{U}(n)$	$\mathrm{U}(n-1)$
	$\mathrm{SU}(n)$	$\mathrm{SU}(n-1)$
S^{4n-1}	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{Sp}(n-1)\mathrm{Sp}(1)$
	$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathrm{Sp}(n-1)\mathrm{U}(1)$
	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n-1)$
S^6	G_2	$\mathrm{SU}(3)$
S^7	$\mathrm{Spin}(7)$	G_2
S^{15}	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$

Real Lie Algebras of Some Lie Groups (Non-vanishing Brackets)

- Two-dimensional solvable non-Abelian Lie algebra with basis $\{X, Y\}$:

$$[X, Y] = X.$$

- Special orthogonal $\mathfrak{so}(3)$ with basis $\{X, Y, Z\}$:

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y.$$

- Lie algebra $\mathfrak{h}$ with basis $\{X, Y, Z\}$ of the Heisenberg group:

$$[X, Y] = Z.$$

- Lie algebra with basis $\{A, X_1, \dots, X_{n-1}\}$ of a solvable Lie group that acts simply transitively on the real hyperbolic space $\mathbb{R}\mathrm{H}^n$:

$$[A, X_i] = X_i.$$

- Lie algebra with basis $\{A, X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, U\}$ of a solvable Lie group that acts simply transitively on the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$:

$$[A, X_i] = X_i, \qquad [A, Y_i] = Y_i, \qquad [A, U] = 2U, \qquad [X_i, Y_i] = -2U.$$

- Lie algebra with basis $\{A, X_i, Y_i, Z_i, W_i, U_j\}$, $i = 1, \ldots, n-1$; $j = 1, 2, 3$, of a solvable Lie group that acts simply transitively on the quaternionic hyperbolic space $\mathbb{H}\mathrm{H}^n$:

$$[A, X_i] = X_i, \qquad [A, Y_i] = Y_i, \qquad [A, Z_i] = Z_i, \qquad [A, W_i] = W_i,$$
$$[A, U_j] = 2U_j, \qquad [X_i, Y_i] = -[Z_i, W_i] = -2U_1,$$
$$[X_i, Z_i] = [Y_i, W_i] = -2U_2, \qquad [X_i, W_i] = -[Y_i, Z_i] = -2U_3.$$

For some of these formulas, see [10, 11].

Some Isomorphisms of Classical Lie Algebras

$$\begin{aligned}
&\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1), && \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1) \cong \mathfrak{sp}(1,\mathbb{R}),\\
&\mathfrak{so}(5) \cong \mathfrak{sp}(2), && \mathfrak{so}(3,2) \cong \mathfrak{sp}(2,\mathbb{R}),\\
&\mathfrak{so}(4) \cong \mathfrak{sp}(1) \times \mathfrak{sp}(1), && \mathfrak{so}(4,1) \cong \mathfrak{sp}(1,1),\\
&\mathfrak{su}(4) \cong \mathfrak{so}(6), && \mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3),\\
&\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,3), && \mathfrak{su}^*(4) \cong \mathfrak{so}(5,1),\\
&\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2), && \mathfrak{su}(3,1) \cong \mathfrak{so}^*(6),\\
&\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2), && \mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}),\\
&\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}), && \mathfrak{so}^*(4) \cong \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R}).
\end{aligned}$$

Unimodular Three-Dimensional Real Lie Algebras and Their Corresponding Lie Groups

$$[e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \qquad [e_1, e_2] = \lambda_3 e_3, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group	Description
$+, +, +$	SU(2) or SO(3)	Compact, simple
$+, +, -$	SL(2, $\mathbb{R}$) or O(1, 2)	Noncompact, simple
$+, +, 0$	$E(2)$ $(*)$	Solvable
$+, -, 0$	$E(1, 1)$ $(**)$	Solvable
$+, 0, 0$	Heisenberg group	Nilpotent
$0, 0, 0$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	Abelian

$(*)$ group of rigid motions of Euclidean 2-space

$(**)$ group of rigid motions of Minkowski 2-space, which is a semi-direct product of subgroups isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{R}$, where each $t \in \mathbb{R}$ acts on $\mathbb{R} \oplus \mathbb{R}$ by the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

Killing Form B for Some Lie Algebras $\mathfrak{g}$

$$B(X,Y)=\operatorname{tr}(\operatorname{ad}_X\circ\operatorname{ad}_Y),\qquad X,Y\in\mathfrak{g}$$

$\mathfrak{g}$	$B(X,Y)$
$\mathfrak{gl}(n,\mathbb{R})$	$2n\operatorname{tr}(XY)-2\operatorname{tr}(X)\operatorname{tr}(Y)$
$\mathfrak{sl}(n,\mathbb{R})$	$2n\operatorname{tr}(XY)$
$\mathfrak{su}(n)$	$2n\operatorname{tr}(XY)$
$\mathfrak{so}(n,\mathbb{C})$	$(n-2)\operatorname{tr}XY$
$\mathfrak{so}(n)$	$(n-2)\operatorname{tr}XY$
$\mathfrak{sp}(n,\mathbb{F})$	$(2n+2)\operatorname{tr}(XY)$ ($\mathbb{F}=\mathbb{R},\mathbb{C}$)

Maurer–Cartan Equations

$$\begin{aligned} \mathrm{d}\omega(X,Y)&=-\omega\big([X,Y]\big),\qquad \omega\in\mathfrak{g}^*,\ X,Y\in\mathfrak{g},\\ \mathrm{d}\theta^i&=-\sum_{j<k}c^i_{jk}\theta^j\wedge\theta^k \end{aligned}\tag{7.4}$$

($\{\theta^i\}$ = a basis of left-invariant differential 1-forms on the Lie group G with Lie algebra $\mathfrak{g}$; c^i_{jk} = structure constants with respect to that basis of differential forms).

The Exponential Map

- Product of exponentials (five first summands of the Campbell–Baker–Hausdorff formula, cf. [18, p. 670], [24, Sem. V, Lect. 4], [28, Sect. 2.15], [32]):

$$\begin{aligned} \exp tX\cdot\exp tY = 1 &+ t(X+Y)+\frac{t^2}{2}[X,Y]+\frac{t^3}{12}\big([X,[X,Y]]+[Y,[Y,X]]\big)\\ &-\frac{t^4}{24}[X,[Y,[X,Y]]]-\frac{t^5}{720}\{[[[[X,Y],Y],Y],Y]\\ &+[[[[Y,X],X],X],X]\\ &-2\big([[[[X,Y],Y],Y],X]+[[[[Y,X],X],X],Y]\big)\\ &+6\big([[[[X,Y],Y],X],Y]+[[[[Y,X],X],Y],X]\big)\}+\cdots. \end{aligned}\tag{7.5}$$

Simply Connected Compact Simple Lie Groups

G	$\dim G$	$\operatorname{rank} G$	$(*)$
$\mathrm{SU}(n)$	n^2-1	$n-1$	$n \geqslant 2$
$\mathrm{Spin}(2n+1)$	$2n^2+n$	n	$n \geqslant 2$
$\mathrm{Sp}(n)$	$2n^2+n$	n	$n \geqslant 3$
$\mathrm{Spin}(2n)$	$2n^2-n$	n	$n \geqslant 4$
G_2	14	2	
F_4	52	4	
E_6	78	6	
E_7	133	7	
E_8	248	8	

$(*)$ To avoid repetitions

Centre of Some Usual Lie Groups

G	$Z(G)$
$\mathrm{SO}(2,\mathbb{R})$	$\mathrm{SO}(2,\mathbb{R}) \cong S^1$
$\mathrm{SO}(2n+1,\mathbb{R})$	$\{I\}$
$\mathrm{SO}(2n,\mathbb{R}), n > 1$	$\{\pm I\} \cong \mathbb{Z}_2$
$\mathrm{U}(n)$	$\{\mathrm{e}^{2\pi\mathrm{i}\theta} I : \theta \in \mathbb{R}/\mathbb{Z}\} \cong S^1$
$\mathrm{SU}(n)$	$\{\omega I : \omega^n = 1\} \cong \mathbb{Z}_n$
$\mathrm{Sp}(n)$	$\{\pm I\} \cong \mathbb{Z}_2$
$\mathrm{Spin}(2n+1,\mathbb{C})$	$\{\pm I\} \cong \mathbb{Z}_2$
$\mathrm{Spin}(2n,\mathbb{C})$	$\{\pm I, \pm\tau\} \cong \mathbb{Z}_2 + \mathbb{Z}_2$ (if n even) $(*)$
$\mathrm{Spin}(2n,\mathbb{C})$	$\{\pm I, \pm\tau\} \cong \mathbb{Z}_4$ (if n odd)

$(*)$ The centre of $\mathrm{Spin}(2n,\mathbb{C})$ is the group $\{\pm 1, \pm\tau\}$ with three elements of order 2

Poincaré Polynomials of the Compact Simple Lie Groups

$$p_{A_n}(t) = p_{\mathrm{SU}(n+1)}(t) = \left(1+t^3\right)\left(1+t^5\right)\cdots\left(1+t^{2n+1}\right),$$

$$p_{B_n}(t) = p_{\mathrm{SO}(2n+1)}(t) = \left(1+t^3\right)\left(1+t^7\right)\cdots\left(1+t^{4n-1}\right),$$

$$p_{C_n}(t) = p_{\mathrm{Sp}(n)}(t) = \left(1+t^3\right)\left(1+t^7\right)\cdots\left(1+t^{4n-1}\right),$$

$$p_{D_n}(t) = p_{\mathrm{SO}(2n)}(t) = \left(1+t^3\right)\left(1+t^7\right)\cdots\left(1+t^{2n-1}\right)\left(1+t^{4n-5}\right),$$

$$p_{\mathrm{G}_2}(t) = \left(1+t^3\right)\left(1+t^{11}\right),$$

$$p_{\mathrm{F}_4}(t) = \left(1+t^3\right)\left(1+t^{11}\right)\left(1+t^{15}\right)\left(1+t^{23}\right),$$

$$p_{\mathrm{E}_6}(t) = \left(1+t^3\right)\left(1+t^9\right)\left(1+t^{11}\right)\left(1+t^{15}\right)\left(1+t^{17}\right)\left(1+t^{23}\right),$$

$$p_{\mathrm{E}_7}(t) = (1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35}),$$

$$p_{\mathrm{E}_8}(t) = (1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59}).$$

Lie Groups G for the Simple Lie Algebras $\mathfrak{g}$ over $\mathbb{C}$, Their Compact Real Forms U and Centre of $\widetilde{U}$ (Universal Covering Group of U)

$\mathfrak{g}$	G	U	$Z(\widetilde{U})$	$\dim U$
$\mathfrak{a}_n$ $(n \geqslant 1)$	$\mathrm{SL}(n+1,\mathbb{C})$	$\mathrm{SU}(n+1)$	$\mathbb{Z}_{n+1}$	n^2+2n
$\mathfrak{b}_n$ $(n \geqslant 2)$	$\mathrm{SO}(2n+1,\mathbb{C})$	$\mathrm{SO}(2n+1)$	$\mathbb{Z}_2$	$2n^2+n$
$\mathfrak{c}_n$ $(n \geqslant 3)$	$\mathrm{Sp}(n,\mathbb{C})$	$\mathrm{Sp}(n)$	$\mathbb{Z}_2$	$2n^2+n$
$\mathfrak{d}_n$ $(n \geqslant 4)$	$\mathrm{SO}(2n,\mathbb{C})$	$\mathrm{SO}(2n)$	$\mathbb{Z}_4$ (n odd) $\mathbb{Z}_2+\mathbb{Z}_2$ (n even)	$2n^2-n$
$\mathfrak{g}_2$	$\mathrm{G}_2^{\mathbb{C}}$	G_2	$\mathbb{Z}_1$	14
$\mathfrak{f}_4$	$\mathrm{F}_4^{\mathbb{C}}$	F_4	$\mathbb{Z}_1$	52
$\mathfrak{e}_6$	$\mathrm{E}_6^{\mathbb{C}}$	E_6	$\mathbb{Z}_3$	78
$\mathfrak{e}_7$	$\mathrm{E}_7^{\mathbb{C}}$	E_7	$\mathbb{Z}_2$	133
$\mathfrak{e}_8$	$\mathrm{E}_8^{\mathbb{C}}$	E_8	$\mathbb{Z}_1$	248

Connected Complex Lie Groups G for a Given Complex Simple Lie Algebra $\mathfrak{g}$

$\mathfrak{g}$	G
$\mathfrak{sl}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$
	$\mathrm{SL}(n,\mathbb{C})/\{\mathrm{e}^{2\pi \ell \mathrm{i}/m} I, \ell = 0,\dots,m-1\}$ $(*)$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathrm{Spin}(2n+1,\mathbb{C})$
	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$
	$\mathrm{PSp}(2n,\mathbb{C})$
$\mathfrak{so}(2n,\mathbb{C})$ (n odd)	$\mathrm{Spin}(2n,\mathbb{C})$
	$\mathrm{SO}(2n,\mathbb{C})$
	$\mathrm{PSO}(2n,\mathbb{C})$
$\mathfrak{so}(2n,\mathbb{C})$ (n even)	$\mathrm{Spin}(2n,\mathbb{C})$
	$\mathrm{SO}(2n,\mathbb{C})$
	$\mathrm{PSO}(2n,\mathbb{C})$
	$\mathrm{Spin}(2n,\mathbb{C})/\{1,\tau\}$ $(**)$
	$\mathrm{Spin}(2n,\mathbb{C})/\{1,-\tau\}$

$(*)$ For m prime, only $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PSL}(n,\mathbb{C})$ remain

$(**)$ For the meaning of τ, see the table on p. 558 (see [12, p. 369])

Real Forms of the Classical Simple Lie Algebras over $\mathbb{C}$ and Their Corresponding Simple Lie Groups

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$\mathfrak{sl}(n,\mathbb{C}) \qquad (\sim \mathfrak{a}_{n-1}, n > 1)$

$$\begin{aligned}
\mathfrak{su}(n) &= \left\{A \in \mathfrak{gl}(n,\mathbb{C}) : A + {}^t\bar{A} = 0, \operatorname{tr} A = 0\right\}\\
\mathrm{SU}(n) &= \left\{A \in \mathrm{GL}(n,\mathbb{C}) : {}^t\bar{A}A = I, \det A = 1\right\}\\
\mathfrak{sl}(n,\mathbb{R}) &= \left\{A \in \mathfrak{gl}(n,\mathbb{R}) : \operatorname{tr} A = 0\right\}\\
\mathrm{SL}(n,\mathbb{R}) &= \left\{A \in \mathrm{GL}(n,\mathbb{R}) : \det A = 1\right\}\\
\mathfrak{su}(p,q) &= \left\{\begin{pmatrix} A_1 & A_2 \\ {}^t\bar{A}_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{C}) : A_1 \in \mathfrak{gl}(p,\mathbb{C}),\, A_1 + {}^t\bar{A}_1 = 0,\right.\\
&\qquad \left. A_3 \in \mathfrak{gl}(q,\mathbb{C}),\, A_3 + {}^t\bar{A}_3 = 0, \operatorname{tr} A_1 + \operatorname{tr} A_3 = 0,\, A_2 \text{ arbitrary}\right\},\\
&\qquad p+q = n,\, p \geqslant q\\
\mathrm{SU}(p,q) &= \left\{A \in \mathrm{SL}(p+q,\mathbb{C}) : Q(Az) = Q(z)\right.\\
&= \left. -z^1\bar{z}^1 - \cdots - z^p\bar{z}^p + z^{p+1}\bar{z}^{p+1} + \cdots + \cdots z^{p+q}\bar{z}^{p+q}\right\},\\
&\qquad (p+q = n,\, p \geqslant q)\\
&= \left\{A \in \mathrm{SL}(p+q,\mathbb{C}) : {}^tAI_{p,q}\bar{A} = I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\right\}\\
&\qquad \text{(pseudo-unitary groups if } q \neq 0\text{; } \mathrm{SU}(n) \text{ if } q = 0)\\
\mathfrak{su}^*(2n) &= \left\{\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) : A, B \in \mathfrak{gl}(n,\mathbb{C}), \operatorname{tr} A + \operatorname{tr} \bar{A} = 0\right\}\\
\mathrm{SU}^*(2n) &= \left\{A \in \mathrm{SL}\big(2n,\mathbb{C}\big) : A\tau = \tau A,\, \tau : \mathbb{C}^{2n} \to \mathbb{C}^{2n},\right.\\
&\qquad \left. \tau : \big(z^1, \ldots, z^{2n}\big) \to (\bar{z}^{n+1}, \ldots, \bar{z}^{2n}, -\bar{z}^1, \ldots, -\bar{z}^n)\right\}
\end{aligned}$$

$\mathfrak{so}(2n+1,\mathbb{C}) \qquad (\sim \mathfrak{b}_n, n \geqslant 1)$

$$\begin{aligned}
\mathfrak{so}(2n+1) &= \left\{A \in \mathfrak{gl}(2n+1,\mathbb{R}) : A + {}^tA = 0\right\}\\
\mathrm{SO}(2n+1) &= \left\{A \in \mathrm{GL}(2n+1,\mathbb{R}) : {}^tAA = I\right\}\\
\mathfrak{so}(p,q) &= \left\{\begin{pmatrix} A_1 & A_2 \\ {}^tA_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{R}) : A_1 \in \mathfrak{gl}(p,\mathbb{R}),\, A_3 \in \mathfrak{gl}(q,\mathbb{R}),\right.\\
&\qquad \left. A_1 + {}^tA_1 = 0,\, A_3 + {}^tA_3 = 0,\, A_2 \text{ arbitrary}\right\},\\
&\qquad p+q = 2n+1,\, p \geqslant q\\
\mathrm{SO}(p,q) &= \left\{A \in \mathrm{SL}(p+q,\mathbb{R}) : Q(Ax) = Q(x)\right.\\
&= \left. -\big(x^1\big)^2 + \cdots - \big(x^p\big)^2 + \big(x^{p+1}\big)^2 + \cdots + \big(x^{2n+1}\big)^2\right\},\\
&\qquad (p+q = 2n+1,\, p \geqslant q)\\
&= \left\{A \in \mathrm{SL}(p+q,\mathbb{R}) : {}^tAI_{p,q}A = I_{p,q}\right\}
\end{aligned}$$

$\mathfrak{sp}(n,\mathbb{C}) \qquad (\sim \mathfrak{c}_n, n \geqslant 1)$

$$\mathfrak{sp}(n) = \left\{A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) : A + {}^t\bar{A} = 0, \operatorname{tr} A = 0, \right.$$
$$\left. A_i \in \mathfrak{gl}(n,\mathbb{C}),\ A_2 = {}^tA_2,\ A_3 = {}^tA_3 \right\}, \quad \big(\mathfrak{sp}(n) = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{su}(2n)\big)$$

$$\mathrm{Sp}(n) = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(2n)$$

$$\mathfrak{sp}(n,\mathbb{R}) = \left\{\begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{R}) : A_i \in \mathfrak{gl}(n,\mathbb{R}),\ A_2 = {}^tA_2,\ A_3 = {}^tA_3\right\}$$

$$\mathrm{Sp}(n,\mathbb{R}) = \left\{A \in \mathrm{GL}(2n,\mathbb{R}) : {}^tA\,\Omega\,A = \Omega\right\}$$
$$\big(\Omega = x^1 \wedge x^{n+1} + x^2 \wedge x^{n+2} + \cdots + x^n \wedge x^{2n},$$
symplectic form on $\mathbb{R}^{2n}\big)$

$$\mathfrak{sp}(p,q) = \left\{\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ {}^t\bar{A}_{12} & A_{22} & {}^tA_{14} & A_{24} \\ -\bar{A}_{13} & \bar{A}_{14} & \bar{A}_{11} & -\bar{A}_{12} \\ {}^t\bar{A}_{14} & -\bar{A}_{24} & -{}^tA_{12} & \bar{A}_{22} \end{pmatrix} \in \mathfrak{gl}\big(2(p+q),\mathbb{C}\big) : \right.$$
$$A_{11}, A_{13} \in \mathfrak{gl}(p,\mathbb{C}),\ A_{12}, A_{14} \in M(p \times q,\mathbb{C}),\ A_{11} + {}^t\bar{A}_{11} = 0,$$
$$\left. A_{22} + {}^t\bar{A}_{22} = 0,\ A_{13} = {}^tA_{13},\ A_{24} = {}^tA_{24} \right\}$$

$$\mathrm{Sp}(p,q) = \left\{A \in \mathrm{Sp}(p+q,\mathbb{C}) : {}^tA I_{p,q,p,q} \bar{A} = I_{p,q,p,q} = \operatorname{diag}(-I_p, I_q, -I_p, I_q)\right\}$$
$$\big(\mathrm{Sp}(p) \text{ if } q = 0,\ \mathrm{Sp}(n) = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n),$$
$$\mathrm{Sp}(p,q) = \mathrm{Sp}(p+q,\mathbb{C}) \cap \mathrm{U}(2p,2q)\big)$$

$\mathfrak{so}(2n,\mathbb{C}) \qquad (\sim \mathfrak{d}_n, n \geqslant 1)$

$$\mathfrak{so}(2n) = \left\{A \in \mathfrak{gl}(2n,\mathbb{R}) : A + {}^tA = 0\right\}$$

$$\mathfrak{so}(p,q) = \left\{\begin{pmatrix} A_1 & A_2 \\ {}^tA_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{R}) : A_1 \in \mathfrak{gl}(p,\mathbb{R}),\ A_1 + {}^tA_1 = 0, \right.$$
$$\left. A_2 \text{ arbitrary},\ A_3 \in \mathfrak{gl}(q,\mathbb{R}),\ A_3 + {}^tA_3 = 0 \right\}, \quad p + q = 2n,\ p \geqslant q$$

$$\mathrm{SO}(p,q) \quad p + q = 2n,\ p \geqslant q \quad \text{as} \quad \mathrm{SO}(p,q), \quad p + q = 2n+1,\ p \geqslant q$$

$$\mathfrak{so}^*(2n) = \left\{\begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) : A_1, A_2 \in \mathfrak{gl}(n,\mathbb{C}), \right.$$
$$\left. A_1 + {}^tA_1 = 0,\ A_2 = \bar{A}_2 \right\}$$

$$\mathrm{SO}^*(2n) = \left\{A \in \mathrm{SO}(2n,\mathbb{C}) : Q(Az) = Q(z)\right.$$
$$\left. = -z^1\bar{z}^{n+1} + z^{n+1}\bar{z}^1 - z^2\bar{z}^{n+2} + z^{n+2}\bar{z}^2 + \cdots - z^n\bar{z}^{2n} + z^{2n}\bar{z}^n\right\}$$
$$= \left\{A \in \mathrm{SO}(2n,\mathbb{C}) : {}^tA \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \bar{A} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\right\}$$

(See Barut and Rączka [3].)

System Δ of Roots, Set of Positive Roots Δ^+ and Set of Simple Roots Π for the Simple Lie Algebras over $\mathbb{C}$

$(\varepsilon_1, \ldots, \varepsilon_n)$ usual basis of $\mathbb{R}^n$	
$\mathfrak{a}_n$ $(n \geqslant 1)$	$V = \{v \in \mathbb{R}^{n+1} : \langle v, \varepsilon_1 + \cdots + \varepsilon_{n+1} \rangle = 0\}$ $\Delta = \{\varepsilon_i - \varepsilon_j, i \neq j; i, j = 1, \ldots, n\}$ $\Delta^+ = \{\varepsilon_i - \varepsilon_j, i < j\}$ $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n\}$
$\mathfrak{b}_n$ $(n \geqslant 2)$	$V = \mathbb{R}^n$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\pm\varepsilon_i\}, i, j = 1, \ldots, n$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\varepsilon_i\}$ $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n, i = 1, \ldots, n-1\}$
$\mathfrak{c}_n$ $(n \geqslant 3)$	$V = \mathbb{R}^n$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\pm 2\varepsilon_i\}, i, j = 1, \ldots, n$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{2\varepsilon_i\}$ $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, 2\varepsilon_n, i = 1, \ldots, n-1\}$
$\mathfrak{d}_n$ $(n \geqslant 4)$	$V = \mathbb{R}^n$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j; i, j = 1, \ldots, n\}$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i < j\}$ $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n, i = 1, \ldots, n-2\}$
$\mathfrak{g}_2$	$V = \{v \in \mathbb{R}^3 : \langle v, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \rangle = 0\}$ $\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$ $\quad \cup\{\pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}$ $\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\}$ $\quad \cup\{-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, -2\varepsilon_2 + \varepsilon_1 + \varepsilon_3, -2\varepsilon_3 + \varepsilon_1 + \varepsilon_2\}$ $\Pi = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$
$\mathfrak{f}_4$	$V = \mathbb{R}^8$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\pm\varepsilon_i\} \cup \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\varepsilon_i\} \cup \{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ $\Pi = \{\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \varepsilon_4, \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3\}$
$\mathfrak{e}_6$	$V = \mathbb{R}^8$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j \leqslant 5\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}\varepsilon_i, \sum_{i=1}^{8} n(i)\,\text{even}, n(i) = 0, 1\}$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i > j\}$ $\quad \cup\{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5}(-1)^{n(i)}\varepsilon_i, \sum_{i=1}^{5} n(i)\,\text{odd}\}$ $\Pi = \{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1),$ $\quad \varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4\}$

$(\varepsilon_1, \dots, \varepsilon_n)$ usual basis of $\mathbb{R}^n$	
$\mathfrak{e}_7$	$V = \{v \in \mathbb{R}^8 : \langle v, \varepsilon_7 + \varepsilon_8 \rangle = 0\}$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j \leqslant 6\} \cup \{\pm(\varepsilon_7 - \varepsilon_8)\}$ $\cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}\varepsilon_i, \sum_{i=1}^{8} n(i) \text{ even}\}$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i > j\} \cup \{\varepsilon_8 - \varepsilon_7\}$ $\cup \{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^{6}(-1)^{n(i)}\varepsilon_i), \sum_{i=1}^{6} n(i) \text{ even}\}$ $\Pi = \{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1),$ $\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_5\}$
$\mathfrak{e}_8$	$V = \{v \in \mathbb{R}^8 : \langle v, \varepsilon_6 - \varepsilon_7 \rangle = \langle v, \varepsilon_7 + \varepsilon_8 \rangle = 0\}$ $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, i < j\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}\varepsilon_i, \sum_{i=1}^{8} n(i) \text{ even}\}$ $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, i > j\} \cup \{\frac{1}{2}(\varepsilon_8 + \sum_{i=1}^{7}(-1)^{n(i)}\varepsilon_i), \sum_{i=1}^{7} n(i) \text{ even}\}$ $\Pi = \{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1),$ $\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_6\}$

For more details and a wealth of related information, see Knapp [18].

Some Usual Homogeneous Spaces

- Sphere:

$$S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n) \cong \mathrm{SO}(n+1)/\mathrm{SO}(n), \quad n \geqslant 1,$$
$$S^{2n+1} \cong \mathrm{U}(n+1)/\mathrm{U}(n) \cong \mathrm{SU}(n+1)/\mathrm{SU}(n), \quad n \geqslant 1.$$

- Real Grassmann manifold of k-planes in $\mathbb{R}^n$:

$$G_k(\mathbb{R}^n) \cong \mathrm{O}(n)\big/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big).$$

- Real projective space:

$$\mathbb{R}\mathrm{P}^n \cong G_1(\mathbb{R}^{n+1}) \cong \mathrm{O}(n+1)\big/\big(\mathrm{O}(1) \times \mathrm{O}(n)\big) \cong \mathrm{SO}(n+1)/\mathrm{O}(n).$$

- Real Stiefel manifold of k-tuples of orthonormal vectors in $\mathbb{R}^n$:

$$V_k(\mathbb{R}^n) \cong \mathrm{O}(n)/\mathrm{O}(n-k).$$

- Complex projective n-space:

$$\mathbb{C}\mathrm{P}^n \cong \mathrm{U}(n+1)\big/\big(\mathrm{U}(1) \times \mathrm{U}(n)\big) \cong \mathrm{SU}(n+1)\big/\mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(n)\big)$$
$$\cong \big(\mathrm{SU}(n+1)\big/\mathbb{Z}_{n+1}\big)/\big(\mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(n)\big)/\mathbb{Z}_{n+1}\big)$$

  ($\mathbb{Z}_{n+1} = \mathrm{centre}(\mathrm{SU}(n+1))$).
- Quaternionic projective space:

$$\mathbb{H}\mathrm{P}^n \cong \mathrm{Sp}(n+1)\big/\big(\mathrm{Sp}(1) \times \mathrm{Sp}(n)\big).$$

Some Inclusions of Lie Groups and Their Homogeneous Spaces

$$\mathrm{F}_4 \supset \mathrm{Spin}(9) \supset \mathrm{Spin}(8) \supset \mathrm{Spin}(7) \supset \mathrm{G}_2 \supset \mathrm{SU}(3) \supset S^3 \supset 1.$$

(Respective dimensions: 52, 36, 28, 21, 14, 8, 3, 1.)

M	$\cong G/H$
S^5	$\mathrm{SU}(3)/\mathrm{SU}(2)$
S^6	$\mathrm{G}_2/\mathrm{SU}(3)$
S^7	$\mathrm{Spin}(7)/\mathrm{G}_2$
	$\mathrm{Spin}(8)/\mathrm{Spin}(7)$
S^8	$\mathrm{Spin}(9)/\mathrm{Spin}(8)$
$V_2(\mathbb{R}^7)$	$\mathrm{G}_2/\mathrm{SU}(2)$
$S^7 \times S^7$	$\mathrm{Spin}(8)/\mathrm{G}_2$
S^{15}	$\mathrm{Spin}(9)/\mathrm{Spin}(7)$
$\mathbb{O}\mathrm{P}^2$	$\mathrm{F}_4/\mathrm{Spin}(9)$

Algebra $\mathbb{H}$ of Quaternions Basis: $\{e_0, e_1, e_2, e_3\}$ satisfying

$$e_0^2 = e_0, \qquad e_i^2 = -e_0, \qquad e_0 e_i = e_i e_0 = e_i, \qquad e_i e_j = -e_j e_i = e_k$$

($(i, j, k) =$ even permutation of $(1, 2, 3)$).

Conjugate quaternion of $q = \sum_{i=0}^{3} a_i e_i \in \mathbb{H}$, and relation with the product:

$$\bar{q} = a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3, \qquad \overline{q_1 q_2} = \bar{q}_2 \bar{q}_1.$$

Norm $n(q)$ and inverse q^{-1} of q:

$$n(q) = \sqrt{q\bar{q}} = \sqrt{\sum_{i=0}^{3} a_i^2} \in \mathbb{R},$$

$$q^{-1} = \frac{1}{|q|^2}(a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3) = \frac{\bar{q}}{|q|^2}.$$

Algebra $\mathbb{O}$ of (the Usual) Octonions, Multiplication Table, and Some Associated Spaces

$$\begin{aligned} \mathbb{O} = \big\{ & x = z + u \in \mathbb{C} \oplus \mathbb{C}^3 : \\ & (z+u)\big(z' + v\big) = \big(zz' - \langle u, v\rangle\big) + zv + \bar{z}'u + u * v, \\ & \langle\, ,\rangle : \mathbb{C}^3 \to \mathbb{C}^3 \text{ the usual Hermitian product}, \\ & \langle u, v * w\rangle = \det(u, v, w), \alpha, \beta \in \mathbb{C}, u, v, w \in \mathbb{C}^3 \big\}. \end{aligned}$$

Conjugate, trace and norm of $x = z + u \in \mathbb{O}$:

$$\bar{x} = \bar{z} - u, \qquad t(x) = x + \bar{x} \in \mathbb{R}, \qquad n(x) = \sqrt{x\bar{x}} = \sqrt{|z|^2 + |u|^2} \in \mathbb{R}.$$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	e_7	e_6
e_2	e_2	e_3	e_0	e_1	$-e_6$	$-e_7$	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	e_0	$-e_7$	$-e_6$	e_5	e_4
e_4	e_4	e_5	e_6	e_7	e_0	e_1	e_2	e_3
e_5	e_5	e_4	e_7	e_6	$-e_1$	e_0	$-e_3$	$-e_2$
e_6	e_6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	e_1
e_7	e_7	$-e_6$	e_5	$-e_4$	$-e_3$	e_2	$-e_1$	e_0

$$\begin{aligned}
S^7 &= \big\{x \in \mathbb{O} : n^2(x) = 1\big\},\\
S^6 &= \big\{x \in \mathbb{O} : n^2(x) = 1, t(x) = 0\big\},\\
S^{15} &= \big\{x \in \mathbb{O} \times \mathbb{O} : n^2(x) + n^2(y) = 1\big\},\\
\mathrm{Spin}(7) &= \mathrm{Aut}\big(\mathbb{O}, \{\,,\,,\,\}\big), \qquad \{x, y, z\} = (x\bar{y})z,\\
\mathrm{G}_2 &= \big\{\varphi \in \mathrm{Spin}(7) : \varphi(1) = 1\big\}, \qquad 1 \in S^7.
\end{aligned}$$

Chapter 5

- Hopf bundles $\pi_{\mathbb{C}}\colon S^3 \subset \mathbb{C}^2 \to S^2$ and $\pi_{\mathbb{H}}\colon S^7 \subset \mathbb{H}^2 \to S^4$:
$$\pi(x, y) = \big(2y\bar{x}, |x|^2 - |y|^2\big).$$

- Fundamental vector fields on the bundle of linear frames FM over M:
$$A_z^* = \sum_{i,j,k} x_k^i(z) a_j^k \partial/\partial x_j^i$$
($A = (a_j^i) \in M(n, \mathbb{R})$; $z = (X_1, \dots, X_n) \in FM$; $x^i(z) = x^i(\pi(z))$; $x_j^i(z) = \mathrm{d}x^i(X_j)$).
- Connection form ω on a principal bundle $P(M, G)$ in terms of forms $\omega_i = \sigma_i^* \omega$, with local sections σ_i, defined on open subsets U_i of M:
$$\omega_j = \mathrm{Ad}_{\psi_{ij}^{-1}} \omega_i + \theta_{ij} \quad \text{on } U_i \cap U_j$$
($\{U_i\}$ = open covering of M; $\psi_{ij}(U_i \cap U_j) \to G$ = transition functions; $\theta_{ij} = \mathfrak{g}$-valued 1-form $\psi_{ij}^*\theta$; θ = canonical 1-form on G: $\theta(X) = X$).
- Exterior covariant derivative $D\varphi$ of a tensorial 1-form of type $\mathrm{Ad}\, G$ with respect to a connection in the principal bundle P with connection form ω ($X, Y \in T_u P$, $u \in P$):
$$D\varphi(X, Y) = \mathrm{d}\varphi(X, Y) + \big[\varphi(X), \omega(Y)\big] + \big[\omega(X), \varphi(Y)\big].$$

- Cartan's structure equation (principal bundle).

ω the connection form of a connection in a principal G-bundle P, with curvature form Ω; $\omega = \sum_i \omega^i \otimes e_i$, $\Omega = \sum_i \Omega^i \otimes e_i$, $\{e_i\}$ basis of $\mathfrak{g}$; c^i_{jk} structure constants with respect to $\{e_i\}$:

$$\mathrm{d}\omega(X, Y) = -\big[\omega(X), \omega(Y)\big] + \Omega(X, Y), \quad X, Y \in T_u P,\ u \in P;$$

$$\mathrm{d}\omega = -[\omega, \omega] + \Omega \quad \text{(simplified expression);}$$

$$\mathrm{d}\omega^i = -\sum_{j<k} c^i_{jk} \omega^j \wedge \omega^k + \Omega^i.$$

- Structure constants of $\mathrm{GL}(n, \mathbb{R})$ with respect to the standard basis $\{E^i_j\}$ of $\mathfrak{gl}(n, \mathbb{R})$ (also for $\mathbb{C}$):

$$c^{rs}_{ij,kl} = \delta^r_i \delta^j_k \delta^s_l - \delta^r_k \delta^l_i \delta^s_j.$$

- Cartan's structure equation (vector bundle).
  $E(M, \mathbb{F}^n, \mathrm{GL}(n, \mathbb{F}), P)$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, a vector bundle associated to the principal fibre bundle P; $\{E^i_j\}$, $i, j = 1, \ldots, n$, is a basis of $\mathfrak{gl}(n, \mathbb{F})$; $\omega = \sum_{i<j} \omega^j_i \otimes E^i_j$ and $\Omega = \sum_{i<j} \Omega^j_i \otimes E^i_j$ the connection form and the curvature form of a connection in P:

$$\mathrm{d}\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j, \quad i, j = 1, \ldots, n.$$

Linear Connections

- Canonical 1-form θ on the frame bundle (FM, π, M):

$$\theta(X) = z^{-1}(\pi_* X), \qquad \theta^i = \sum_j Y^i_j \,\mathrm{d}x^j$$

  ($X \in T_z(FM)$, $z \in FM$; $\{x^i, x^i_j\}$, $i = 1, \ldots, n = \dim M$, local coordinates on FM; $Y = (x^i_j)^{-1}$).
- Components (or Christoffel symbols) Γ^i_{jk} of a linear connection ∇ on M with connection form $\omega = \sum_{i<j} \omega^i_j \otimes E^j_i$; $\sigma = (X_1, \ldots, X_n)$ a section of FM over an open subset U of M; $\omega_U = \sigma^* \omega$ (which is a $\mathfrak{gl}(n, \mathbb{R})$-valued 1-form on U). Then:

$$\omega_U = \sum_{i,j,k} \Gamma^i_{jk} \,\mathrm{d}x^j \otimes E^k_i.$$

  Also, for local coordinate functions (x^i) on M,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

- Connection form ω of a linear connection with Christoffel symbols Γ^i_{jk} in terms of the local coordinates (x^i, x^j_k) on FM:

$$\omega^i_j = \sum_k Y^i_k \big(\mathrm{d}x^k_j + \sum_{m,l} \Gamma^k_{ml} x^l_j \mathrm{d}x^m\big), \quad i, j, k, l, m = 1, \ldots, n = \dim M.$$

- Structure equations (frame bundle).

  ∇ a linear connection on M, with connection form ω, torsion form $\Theta = (\Theta^i)$, and curvature form $\Omega = (\Omega^i_j)$; θ the canonical 1-form on FM; $X, Y \in T_z(TM)$; $i, j, k = 1, \dots, n = \dim M$:

$$\begin{aligned}
\mathrm{d}\theta(X,Y) &= -\big(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)\big) + \Theta(X,Y),\\
\mathrm{d}\omega(X,Y) &= -\big[\omega(X),\omega(Y)\big] + \Omega(X,Y),\\
\mathrm{d}\theta^i &= -\sum_j \omega^i_j \wedge \theta^j + \Theta^i,\\
\mathrm{d}\omega^i_j &= -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j.
\end{aligned}$$

- Covariant differentiation ∇_X ($\mathcal{T}(M)$ = algebra of tensor fields on M):

  (i) $\nabla_X : \mathcal{T}(M) \to \mathcal{T}(M)$ is a type-preserving derivation.
  (ii) ∇_X commutes with every contraction.
  (iii) $\nabla_X f = Xf$, $f \in C^\infty M$.
  (iv) $\nabla_{X+Y} = \nabla_X + \nabla_Y$, $X, Y \in \mathfrak{X}(M)$.
  (v) $\nabla_X(fK) = (Xf)K + f\nabla_X K$, $K \in \mathcal{T}(M)$.

- Covariant derivative of a $(0,r)$ tensor field Ψ:

$$(\nabla_Y\Psi)(X_1,\dots,X_r) = Y\big(\Psi(X_1,\dots,X_r)\big) - \sum_{i=1}^{r} \Psi(X_1,\dots,\nabla_Y X_i,\dots,X_r).$$

- Relation between exterior differential and covariant derivative for a differential r-form α:

$$\mathrm{d}\alpha(X_0,\dots,X_r) = \sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\alpha)(X_0,\dots,\widehat{X}_i,\dots,X_r). \tag{7.6}$$

- Second covariant derivative:

$$\big(\nabla^2 s\big)_{X,Y} = \nabla_X\nabla_Y s - \nabla_{\nabla_X Y} s.$$

- Torsion tensor and curvature tensor field of a linear connection in terms of covariant differentiation:

$$\begin{aligned}
T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y],\\
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.
\end{aligned}$$

- Torsion tensor and curvature tensor field:

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \qquad R(e_i,e_j)e_k = \sum_l R^l_{kij} e_l,$$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \sum_r \big(\Gamma^r_{lj}\Gamma^i_{kr} - \Gamma^r_{kj}\Gamma^i_{lr}\big),$$

$$R^k_{lij} = -R^k_{lji}, \qquad R^k_{lij} = -R^l_{kij}, \qquad R^l_{kij} + R^l_{ijk} + R^l_{jki} = 0,$$
$$R(X,Y)Z = -R(Y,X)Z, \qquad R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

- Bianchi identities:

$$\text{(1st)} \qquad \underset{XYZ}{\mathfrak{S}} R(X,Y)Z = \underset{XYZ}{\mathfrak{S}} \big\{T\big(T(X,Y),Z\big) + (\nabla_X T)(Y,Z)\big\},$$
$$(\text{1st}, T=0) \qquad \underset{XYZ}{\mathfrak{S}} R(X,Y)Z = 0,$$
$$\text{(2nd)} \qquad \underset{XYZ}{\mathfrak{S}} \big\{(\nabla_X R)(Y,Z) + R\big(T(X,Y),Z\big)\big\} = 0,$$
$$(\text{2nd}, T=0) \qquad \underset{XYZ}{\mathfrak{S}} (\nabla_X R)(Y,Z) = 0.$$

- Covariant and double covariant derivative of tensor fields for a linear connection Γ^i_{jk} with torsion tensor T^i_{jk} and curvature tensor field R^i_{jkl}:

  Vector field with components X^i:

$$X^i_{;j} = \partial_j X^i + \sum_r \Gamma^i_{jr} X^r.$$

  Differential 1-form with components ω_i:

$$\omega_{j;i} = \partial_i \omega_j - \sum_r \Gamma^r_{ij} \omega_r, \qquad \omega_{i;jk} - \omega_{i;kj} = \sum_r \big(R^r_{ijk}\omega_r + 2T^r_{jk}\omega_{i;r}\big).$$

  $(1,1)$ tensor field with components J^i_j:

$$J^i_{j;k} = \partial_k J^i_j + \sum_r J^r_j \Gamma^i_{kr} - \sum_r \Gamma^r_{kj} J^i_r.$$

  $(0,2)$ tensor field with components τ_{ij}:

$$\tau_{ij;k} = \partial_i \tau_{jk} - \sum_r \Gamma^r_{ij} \tau_{rk} - \sum_r \Gamma^r_{ik} \tau_{jr}.$$

  (r,s) tensor field with components $K^{i_1\dots i_r}_{j_1\dots j_s}$:

$$K^{i_1\dots i_r}_{j_1\dots j_s;k} = \frac{\partial K^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^k} + \sum_{\alpha=1}^{r} \big(\Gamma^{i_\alpha}_{kl} K^{i_1\dots l\dots i_r}_{j_1\dots j_s}\big) - \sum_{\beta=1}^{s} \big(\Gamma^m_{kj_\beta} K^{i_1\dots i_r}_{j_1\dots m\dots j_s}\big).$$

  The Ricci identity:

$$\nabla_l \nabla_k K^{i_1\dots i_r}_{j_1\dots j_s} - \nabla_k \nabla_l K^{i_1\dots i_r}_{j_1\dots j_s} = \sum_{\rho=1}^{r} K^{i_1\dots i_{\rho-1} i i_{\rho+1}\dots i_r}_{j_1\dots j_s} R^{i_\rho}_{ikl}.$$
$$- \sum_{\sigma=1}^{r} K^{i_1\dots i_r}_{j_1\dots j_{\sigma-1} j j_{\sigma+1}\dots j_s} R^j_{j_\sigma kl} - \nabla_i \delta^{i_1\dots i_r}_{j_1\dots j_s} T^i_{kl}.$$

- Components of the torsion and curvature forms: $\sigma = (X_1, \ldots, X_n)$ a moving frame on an open subset U of M; $T(X_j, X_k) = \sum_i T^i_{jk} X_i$, $R(X_k, X_l)X_j = \sum_i R^i_{jkl} X_i$, the torsion and curvature tensors of a linear connection on M. Define $\widetilde{T}^i_{jk}, \widetilde{R}^i_{jkl} \in C^\infty(FM)$ by

$$\Theta^i = \sum_{j<k} \widetilde{T}^i_{jk} \theta^j \wedge \theta^k = \frac{1}{2} \widetilde{T}^i_{jk} \theta^j \wedge \theta^k, \quad \widetilde{T}^i_{jk} = -\widetilde{T}^i_{kj},$$

$$\Omega^i_j = \sum_{k<l} \widetilde{R}^i_{jkl} \theta^k \wedge \theta^l = \frac{1}{2} \sum_{k,l} \widetilde{R}^i_{jkl} \theta^k \wedge \theta^l, \quad \widetilde{R}^i_{jkl} = -\widetilde{R}^i_{jlk}.$$

Then

$$\sigma^* \widetilde{T}^i_{jk} = T^i_{jk}, \qquad \sigma^* \widetilde{R}^i_{jkl} = R^i_{jkl}.$$

- Cartan's structure equations (moving frame).
  $\sigma = (X_1, \ldots, X_n)$ a moving frame defined on an open subset U of M; $\widetilde{\theta}^i = \sigma^* \theta^i$, $\widetilde{\omega}^i_j = \sigma^* \omega^i_j$. Then:

$$\mathrm{d}\widetilde{\theta}^i = -\sum_j \widetilde{\omega}^i_j \wedge \widetilde{\theta}^j + \frac{1}{2} \sum_{j,k} T^i_{jk} \widetilde{\theta}^j \wedge \widetilde{\theta}^k,$$

$$\mathrm{d}\widetilde{\omega}^i_j = -\sum_k \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j + \frac{1}{2} \sum_{k,l} R^i_{jkl} \widetilde{\theta}^k \wedge \widetilde{\theta}^l.$$

- Structure equations (geodesic polar coordinates).
  $\{e_1, \ldots, e_n\}$ a basis of T_pM, $p \in M$; x^i normal coordinates defined by $\{e_i\}$ on a normal coordinate neighbourhood U of p; $(X_1, \ldots, X_n)$ the moving frame defined on U by parallel transport of $\{e_1, \ldots, e_n\}$ along geodesic rays from p. Let F map an open set of $\mathbb{R}^{n+1}$ into U by $x^i(F(t; a^1, \ldots, a^n)) = ta^i$. Define f^i, β^i, β^i_j by

$$F^* \widetilde{\theta}^i = f^i \mathrm{d}t + \beta^i, \qquad F^* \widetilde{\omega}^i_j = \beta^i_j,$$

where $\widetilde{\theta}^i = \sigma^* \theta^i$, $\widetilde{\omega}^i_j = \sigma^* \omega^i_j$, and β^i does not depend on $\mathrm{d}t$. Then $f^i(t; a^1, \ldots, a^n) = a^i$; β^i_j does not depend on $\mathrm{d}t$, and we have (structure equations):

$$\frac{\partial \beta^i}{\partial t} = \mathrm{d}a^i + \sum_j a^j \beta^i_j + \sum_{j,k} T^i_{jk} a^j \beta^k, \qquad \frac{\partial \beta^i_j}{\partial t} = \sum_{k,l} R^i_{jkl} a^k \beta^l,$$

with initial data $\beta^i(t; a^k, \mathrm{d}a^l)_{t=0} = 0 = \beta^j_i(t; a^k, \mathrm{d}a^l)_{t=0}$.

- Differential equations of geodesics ($i, j, k = 1, \ldots, n = \dim M$):

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \sum_{j,k} \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0.$$

- Covariant derivative on a vector bundle (E, π, M) over M ($\Gamma E = (C^\infty M)$-module of C^∞ sections of E):

$$\begin{aligned}
&\nabla\colon \mathfrak{X}(M) \times \Gamma E \to \Gamma E, \qquad (X, s) \mapsto \nabla_X s,\\
&\nabla_{fX+hY} s = f\nabla_X s + h\nabla_Y s, \quad f, h \in C^\infty M,\ X, Y \in \mathfrak{X}(M),\\
&\nabla_X(s+t) = \nabla_X s + \nabla_X t, \quad s, t \in \Gamma E,\\
&\nabla_X(fs) = (Xf)s + f\nabla_X s.
\end{aligned}$$

Reductive Homogeneous Spaces

- G-invariant connections: $M = G/H$ a reductive homogeneous space, G (with Lie algebra $\mathfrak{g}$ and identity element e) acting transitively and effectively on M. Reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad \operatorname{Ad}(H)\mathfrak{m} \subset \mathfrak{m}.$$

  Linear isotropy representation $\lambda\colon H \to \operatorname{Aut}(T_o M)$, $\lambda(h) = (L_h)_{*o}$, and $o = eH \in G/H$ the origin. P a G-invariant K-structure over M. There is a one-to-one correspondence between the set of G-invariant connections in P and the set of linear maps $\Lambda_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{k}$ such that

$$\Lambda_{\mathfrak{m}}(\operatorname{Ad}_h X) = \operatorname{Ad}_{\lambda(h)}\big(\Lambda_{\mathfrak{m}}(X)\big), \quad X \in \mathfrak{m},\ h \in H,$$

  with $\Lambda_{\mathfrak{m}}$ corresponding to $\Lambda\colon \mathfrak{g} \to \mathfrak{k}$ with

$$\begin{cases} \Lambda(X) = \lambda_*(X), & X \in \mathfrak{h},\\ \Lambda(\operatorname{Ad}_h X) = \operatorname{Ad}_{\lambda(h)}(\Lambda(X)), & X \in \mathfrak{m}, h \in H, \end{cases}$$

  where λ_* is the Lie algebra homomorphism induced by λ.
- Torsion tensor and curvature operator at $o \in G/H$ for the invariant connection corresponding to $\Lambda_{\mathfrak{m}}$ ($X, Y \in \mathfrak{m}$):

$$\begin{aligned}
T(X,Y)_o &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}},\\
R(X,Y)_o &= \big[\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)\big] - \Lambda_{\mathfrak{m}}\big([X,Y]_{\mathfrak{m}}\big) - \lambda\big([X,Y]_{\mathfrak{h}}\big).
\end{aligned}$$

- Curvature form Ω of the canonical invariant connection ω on G/H:

$$\Omega(X,Y) = -\frac{1}{2}[X,Y]_{\mathfrak{h}}, \quad X, Y \in \mathfrak{m}.$$

- Torsion tensor and curvature tensor field at $o \in G/H$ of the canonical connection ∇ ($\Lambda_{\mathfrak{m}} = 0$), ($X, Y, Z \in \mathfrak{m}$):

$$\begin{aligned}
T(X,Y)_o &= -[X,Y]_{\mathfrak{m}},\\
\big(R(X,Y)Z\big)_o &= -\big[[X,Y]_{\mathfrak{h}}, Z\big], \quad \nabla T = 0,\ \nabla R = 0.
\end{aligned}$$

- Unique torsionless G-invariant connection on G/H with the same geodesics as the canonical connection:

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}.$$

For these and many other formulas in this chapter, see Kobayashi and Nomizu [19].

Almost Complex Manifolds

- Canonical complex structure of $\mathbb{R}^{2n}$ induced from that of $\mathbb{C}^n$:

$$\begin{gathered}\mathbb{R}^{2n} \to \mathbb{R}^{2n} \\ \left(x^1, \ldots, x^n, y^1, \ldots, y^n\right) \mapsto \left(y^1, \ldots, y^n, -x^1, \ldots, -x^n\right).\end{gathered}$$

Matrix with respect to the natural basis of $\mathbb{R}^{2n}$:

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

- Real representation of the general linear group:

$$\begin{gathered}\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R}) \\ A + \mathrm{i}B \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.\end{gathered}$$

- V vector space with complex structure J; $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ complexified space of V; again J the extension of J to V^c. Eigenspaces of J in V^c:

$$V^{1,0} = \left\{Z \in V^c : JZ = \mathrm{i}Z\right\} = \{X - \mathrm{i}JX : X \in V\},$$

$$V^{0,1} = \left\{Z \in V^c : JZ = -\mathrm{i}Z\right\} = \{X + \mathrm{i}JX : X \in V\}.$$

- Standard almost complex structure J on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ defined by

$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}.$$

- Cauchy–Riemann equations:

$f : U \subset \mathbb{C}^n = \left\{z^l = x^l + \mathrm{i}y^l\right\} \to \mathbb{C}^m = \left\{w^k = u^k + \mathrm{i}v^k\right\}$ holomorphic:

$$\frac{\partial u^k}{\partial x^l} = \frac{\partial v^k}{\partial y^l}, \qquad \frac{\partial u^k}{\partial y^l} = -\frac{\partial v^k}{\partial x^l}.$$

- Torsion of an almost complex structure J:

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

- Basis of $T_p^{1,0}M$ and $T_p^{0,1}M$ for a complex manifold M:

$$\left\{\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right), \ \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + \mathrm{i}\frac{\partial}{\partial y^k}\right)\right\}$$

$(z^1, \dots, z^n, z^k = x^k + \mathrm{i}y^k =$ complex local coordinate functions; and $\mathrm{d}z^k = \mathrm{d}x^k + \mathrm{i}\,\mathrm{d}y^k$, $\mathrm{d}\bar{z}^k = \mathrm{d}x^k - \mathrm{i}\,\mathrm{d}y^k$).

- Holomorphic vector field on a complex manifold of complex dimension n:

$$Z = f^k \frac{\partial}{\partial z^k}, \qquad f^k \text{ a holomorphic function } (\bar{\partial} f = 0),\ k = 1, \dots, n.$$

- Cartan's structure equations (almost complex linear connection).

$C(M)$ the bundle of complex linear frames on an almost complex manifold M of real dimension $2n$; θ the canonical form on $C(M)$ = restriction of θ on FM to $C(M)$; $\omega =$ connection form of an almost complex linear connection with torsion form Θ and curvature form Ω; ω and Ω are valued on the subalgebra $\mathfrak{gl}(n, \mathbb{C})$ of $\mathfrak{gl}(2n, \mathbb{R})$. Set

$$\begin{aligned} &\varphi^\alpha = \theta^\alpha + \mathrm{i}\theta^{n+\alpha}, && \Phi^\alpha = \Theta^\alpha + \mathrm{i}\Theta^{n+\alpha}, && \alpha = 1, \dots, n; \\ &\psi^\alpha_\beta = \omega^\alpha_\beta + \mathrm{i}\omega^\alpha_{n+\beta}, && \Psi^\alpha_\beta = \Omega^\alpha_\beta + \mathrm{i}\Omega^\alpha_{n+\beta}, && \alpha, \beta = 1, \dots, n \end{aligned}$$

($\varphi = (\varphi^\alpha)$ and $\Phi = (\Phi^\alpha)$ are $\mathbb{C}^n$-valued; $\psi = (\psi^\alpha_\beta)$ and $\Psi = (\Psi^\alpha_\beta)$ with values in $\mathfrak{gl}(n, \mathbb{C})$, as the Lie algebra of $n \times n$ complex matrices). Then, besides the real structure equations we can write:

$$\mathrm{d}\varphi^\alpha = -\sum_\beta \psi^\alpha_\beta \wedge \varphi^\beta + \Phi^\alpha, \quad \alpha = 1, \dots, n;$$

$$\mathrm{d}\psi^\alpha_\beta = -\sum_\gamma \psi^\alpha_\gamma \wedge \psi^\gamma_\beta + \Psi^\alpha_\beta, \quad \alpha, \beta = 1, \dots, n.$$

Some Properties of Spheres

	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^n (*)
Lie group	**Y**	n	**Y**	n	n	n	n	n
Parallelisable	**Y**	n	**Y**	n	n	n	**Y**	n
Almostcomplex	n	**Y**	n	n	n	**Y**	n	n

Y = yes, n = no, (*) $n > 7$

Chapter 6

- Musical isomorphisms associated to a metric g on M:

$$\flat \colon T_pM \to T^*_pM, \qquad X^\flat\big(= \flat(X)\big) = g(X, \cdot);$$

$$\sharp \colon T^*_pM \to T_pM, \qquad \alpha^\sharp\big(= \sharp(\alpha)\big) = g^{-1}(\alpha, \cdot).$$

- Arc length $L(\sigma)$ of a differentiable curve $\sigma = x_t$, $a \leqslant t \leqslant b$, in a Riemannian n-manifold (M, g) ($(x^1, \dots, x^n)$ local coordinates):

$$L(\sigma) = \int_a^b \sqrt{g\big(x'_t, x'_t\big)}\,\mathrm{d}t, \qquad L(\sigma) = \int_a^b \sqrt{\sum_{i,j} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t.$$

- Energy of a curve $\sigma : [a, b] \to M$ in (M, g):

$$E(\sigma) = \frac{1}{2} \int_a^b \left|\sigma'(t)\right|^2 \mathrm{d}t.$$

- Poincaré upper half-plane:

$$M = \{(x, y) \in \mathbb{R}^2 : y > 0\}, \qquad \mathrm{d}s^2 = \frac{1}{y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).$$

- Koszul formula for the Levi-Civita connection:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad + g\big([X, Y], Z\big) - g\big([Y, Z], X\big) + g\big([Z, X], Y\big). \end{aligned} \tag{7.7}$$

- Christoffel symbols:

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

- Geodesic through $p \in S^n$ (with the round metric) with initial velocity vector $v \in T_p S^n$:

$$\gamma(t) = \big(\cos |v|t\big)p + \big(\sin |v|t\big)\frac{v}{|v|}.$$

Riemann–Christoffel Curvature Tensor

- Riemann–Christoffel curvature tensor ($g_{ij} = g(e_i, e_j)$; (e_i) a local frame):

$$\begin{aligned} R(X, Y, Z, W) &= g\big(R(Z, W)Y, X\big), \\ g\big(R(X, Y)Z, W\big) &= -g\big(R(X, Y)W, Z\big) = -g\big(R(Y, X)Z, W\big), \\ g\big(R(X, Y)Z, W\big) &= g\big(R(Z, W)X, Y\big), \\ R_{ijkl} &= R(e_i, e_j, e_k, e_l) = g\big(R(e_k, e_l)e_j, e_i\big) = \sum_h g_{ih} R^h_{jkl}, \\ R_{klij} &= -R_{lkij} = -R_{klij}, \\ R_{klij} &= R_{ijkl}, \\ R_{lkij} + R_{lijk} + R_{ljki} &= 0. \end{aligned}$$

- Metric and Riemann–Christoffel curvature tensor near the origin p, $x^i(p) = 0$, of normal coordinates (x^i) (letting $R_{ikjl,r} = \partial R_{ikjl}/\partial x^r$):

$$\begin{aligned} g_{ij} &= \delta_{ij} - \frac{1}{3} \sum_{k,l} R_{ikjl} x^k x^l - \frac{1}{3!} \sum_{k,l,r} R_{ikjl,r} x^k x^l x^r \\ &\quad + \frac{1}{5!} \sum_{k,l,r,s} \left(-6 R_{ikjl,rs} + \frac{4}{3} \sum_t R_{iktl} R^t_{rjs} \right) x^k x^l x^r x^s + \cdots , \end{aligned}$$

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right).$$

- Sectional curvature for a plane $P \subset T_pM$:

$$K(P) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \quad X, Y \text{ basis of } P;$$

$$K(P) = R(X,Y,X,Y), \quad X, Y \text{ orthonormal basis.}$$

- Ricci tensor ($(e_i) =$ a local orthonormal frame):

$$\mathbf{r}(X,Y) = \text{trace of the map } Z \mapsto R(Z,X)Y \text{ of } T_pM,$$

$$\mathbf{r}(X,Y) = \sum_i g\big(R(e_i,X)Y, e_i\big) = \sum_i R(e_i,X,e_i,Y),$$

$$\mathbf{r}_{ij} = \sum_k R_{ikjk} = \sum_k R^k_{jki}.$$

- Ricci tensor ($(e_i) =$ any local frame):

$$\mathbf{r}_{ij} = \sum_k R^k_{jki} = \sum_{k,l} g^{kl} R_{ikjl}.$$

- Scalar curvature:

$$\mathbf{s} = \sum_{i,j} g^{ij}\mathbf{r}_{ij} = \sum_{i,j} g^{ij}\mathbf{r}(e_i,e_j) \quad ((e_i) \text{ a local frame}),$$

$$\mathbf{s} = \sum \mathbf{r}(e_i,e_i) \quad ((e_i) \text{ a local orthonormal frame}).$$

- Weyl conformal curvature tensor for a Riemannian n-manifold (M,g):

$$W(X,Y)Z = R(X,Y)Z + L(Y,Z)X - L(X,Z)Y + g(Y,Z)L^*X - g(X,Z)L^*Y$$

$$\left(L(X,Y) = -\tfrac{1}{n-2}\mathbf{r}(X,Y) + \tfrac{\mathbf{s}}{2(n-1)(n-2)}g(X,Y);\ g(L^*X,Y) = L(X,Y)\right),$$

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}\big(\mathbf{r}_{jk}\delta^i_l - \mathbf{r}_{jl}\delta^i_k + g_{jk}\mathbf{r}^i_l - g_{jl}\mathbf{r}^i_k\big) + \frac{\mathbf{s}}{(n-1)(n-2)}\big(g_{jk}\delta^i_l - g_{jl}\delta^i_k\big).$$

- Cotton tensor for a Riemannian n-manifold (M,g):

$$C_{ijk} = \nabla_k\mathbf{r}_{ij} - \nabla_j\mathbf{r}_{ik} - \frac{1}{2(n-1)}(\nabla_k\mathbf{s}g_{ij} - \nabla_j\mathbf{s}g_{ik}).$$

- Weyl projective curvature tensor ($n > 1$):

$$P^i_{jkl} = R^i_{jkl} - \frac{1}{n-1}\big(\mathbf{r}_{jk}\delta^i_l - \mathbf{r}_{jl}\delta^i_k\big).$$

Kähler Manifolds

- Hermitian metric on an almost complex manifold (M, J):

$$g(JX, JY) = g(X, Y),$$

$$g = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\,\mathrm{d}z^\alpha\,\mathrm{d}\bar{z}^\beta.$$

- (M, g, J) an almost Hermitian manifold. Holomorphic sectional curvature:

$$H(X) = g\big(R(X, JX)X, JX\big), \quad X \in T_pM,\ |X| = 1,\ p \in M.$$

- Fundamental 2-form of a Hermitian metric:

$$F(X, Y) = g(X, JY), \qquad F = -2\mathrm{i}\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\,\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta.$$

- Kähler metric:

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha} \quad \text{or} \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\bar{\gamma}}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial z^{\bar{\beta}}}.$$

- Curvature components:

$$R^\alpha_{\beta\gamma\bar{\delta}} = -\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial \bar{z}^\delta},$$

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z^\gamma \partial \bar{z}^\delta} - \sum_{\varepsilon,\tau} g^{\bar{\varepsilon}\tau}\frac{\partial g_{\alpha\bar{\varepsilon}}}{\partial z^\gamma}\frac{\partial g_{\bar{\beta}\tau}}{\partial \bar{z}^\delta}.$$

- Ricci form:

$$\rho(X, Y) = \mathbf{r}(X, JY),$$

$$\rho = -2\mathrm{i}\sum_{\alpha,\beta} \mathbf{r}_{\alpha\bar{\beta}}\,\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta.$$

- Structure equations (bundle of unitary frames).

  $U(M)$ = bundle of unitary frames; θ = canonical form on $U(M)$; $\omega = (\omega^i_j)$, $i, j = 1, \dots, 2n$ = connection form on $U(M)$ defining the Levi-Civita connection of the Kähler manifold M; Ω = curvature form (ω and Ω with values in the real representation of $\mathfrak{u}(n)$). Setting (for $\alpha, \beta = 1, \dots, n$)

$$\varphi^\alpha = \theta^\alpha + \mathrm{i}\theta^{n+\alpha}, \qquad \psi^\alpha_\beta = \omega^\alpha_\beta + \mathrm{i}\omega^\alpha_{n+\beta}, \qquad \Psi^\alpha_\beta = \Omega^\alpha_\beta + \mathrm{i}\Omega^\alpha_{n+\beta},$$

  we have

$$\omega^\alpha_\beta = \omega^{n+\alpha}_{n+\beta}, \qquad \omega^\alpha_{n+\beta} = -\omega^{n+\alpha}_\beta, \qquad \omega^\alpha_\beta = -\omega^\beta_\alpha, \qquad \omega^\alpha_{n+\beta} = \omega^\beta_{n+\alpha};$$

$$\Omega^\alpha_\beta = \Omega^{n+\alpha}_{n+\beta}, \qquad \Omega^\alpha_{n+\beta} = -\Omega^{n+\alpha}_\beta, \qquad \Omega^\alpha_\beta = -\Omega^\beta_\alpha, \qquad \Omega^\alpha_{n+\beta} = \Omega^\beta_{n+\alpha}.$$

  Hence,

$$\psi^\alpha_\beta = -\bar{\psi}^\beta_\alpha, \qquad \Psi^\alpha_\beta = -\bar{\Psi}^\beta_\alpha.$$

- Riemann–Christoffel curvature tensor and curvature form on the bundle of unitary frames $U(M)$, of a Kähler manifold (M,g) of constant holomorphic sectional curvature c:

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{c}{2}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\bar{\beta}\gamma}),$$
$$\Psi^{\alpha}_{\beta} = \frac{c}{2}\Big(\varphi^{\alpha}\wedge\bar{\varphi}^{\beta} + \delta_{\alpha\beta}\sum_{\gamma}\varphi^{\gamma}\wedge\bar{\varphi}^{\gamma}\Big).$$

- Bochner curvature tensor for a Kähler manifold (M, g, J) of real dimension n:

$$\begin{aligned} B(X,Y,Z,W) = {} & R(X,Y,Z,W) - L(X,W)g(Y,Z) - L(X,Z)g(Y,W) \\ & + L(Y,Z)g(X,W) - L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) \\ & - L(JX,Z)g(JY,W) + L(JY,Z)g(JX,W) \\ & - L(JY,W)g(JX,Z) \\ & - 2L(JX,Y)g(JZ,W) - 2L(JZ,W)g(JX,Y) \end{aligned}$$

$(L(X,Y) = -\frac{1}{n+4}r(X,Y) + \frac{s}{2(n+2)(n+4)}g(X,Y))$.

Characteristic Forms

- rth Chern class $c_r(E)$ of a complex vector bundle E over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle

$$\big(P, p, M, \mathrm{GL}(n,\mathbb{C})\big).$$

Represented by the Chern form $\alpha_r \in \Lambda^{2r}M$:

$$p^*(\alpha_r) = \frac{(-1)^r}{(2\pi \mathrm{i})^r r!}\sum \delta^{j_1\dots j_r}_{i_1\dots i_r}\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_r}_{j_r},$$

where one sums over all ordered subsets $(i_1,\dots,i_r)$ of $(1,\dots,n)$ and all permutations $(j_1,\dots,j_r)$ of $(i_1,\dots,i_r)$, and where $\delta^{j_1\cdots j_r}_{i_1\dots i_r}$ denotes the sign of the permutation.

- rth Pontrjagin class $p_r(E)$ of a real vector bundle E over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle

$$\big(P, p, M, \mathrm{GL}(n,\mathbb{R})\big).$$

Represented by the Pontrjagin form $\beta_r \in \Lambda^{4r}M$:

$$p^*(\beta_r) = \frac{1}{(2\pi)^{2r}(2r)!}\sum \delta^{j_1\cdots j_{2r}}_{i_1\dots i_{2r}}\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_{2r}}_{j_{2r}},$$

where one sums over all the ordered subsets $(i_1, \dots, i_{2r})$ of $2r$ elements of $(1, \dots, n)$ and all permutations $(j_1, \dots, j_{2r})$ of $(i_1, \dots, i_{2r})$.

- Euler class $e(E)$ of an oriented real vector bundle E of rank $2r$ (with a fibre metric) over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle

$$\big(P, p, M, \mathrm{SO}(2r)\big).$$

Represented by $\gamma \in \Lambda^{2r} M$:

$$p^*(\gamma) = \frac{(-1)^r}{2^{2r}\pi^r \cdot r!} \sum_{i_1,\dots,i_{2r}} \varepsilon_{i_1 \dots i_{2r}} \Omega^{i_1}_{i_2} \wedge \cdots \wedge \Omega^{i_{2r-1}}_{i_{2r}}.$$

Homogeneous Riemannian Manifolds

- $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ a Riemannian submersion; X, Y orthonormal vector fields on M with horizontal lifts $\widetilde{X}, \widetilde{Y}$; $Z^v =$ vertical lift of $Z \in \mathfrak{X}(M)$. Sectional curvature:

$$K_M(X, Y) = K_{\widetilde{M}}(\widetilde{X}, \widetilde{Y}) + \frac{3}{4}\big|[X, Y]^v\big|^2.$$

- Levi-Civita connection of $(M = G/H, g)$ reductive homogeneous; $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ reductive decomposition; $\langle\,,\rangle$ an $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$ ($\langle X, Y\rangle = g_o(X, Y)$, $X, Y \in \mathfrak{m}$, $\mathfrak{m} \equiv T_o M$):

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y),$$

$U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defined by

$$2\big\langle U(X, Y), Z\big\rangle = \big\langle [Z, X]_{\mathfrak{m}}, Y\big\rangle + \big\langle X, [Z, Y]_{\mathfrak{m}}\big\rangle, \quad X, Y, Z \in \mathfrak{m}.$$

- $M = (G/H, g)$ a naturally reductive homogeneous Riemannian manifold ($U \equiv 0$). Levi-Civita connection and Riemann–Christoffel curvature tensor at o ($X, Y \in \mathfrak{m}$):

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}},$$

$$g_o\big(R(X, Y)Y, X\big) = \frac{1}{4}\big\langle [X, Y]_{\mathfrak{m}}, [X, Y]_{\mathfrak{m}}\big\rangle - \big\langle \big[[X, Y]_{\mathfrak{h}}, Y\big], X\big\rangle.$$

- $M = (G/H, g)$ a normal homogeneous Riemannian manifold (there exists an $\mathrm{Ad}(G)$-invariant scalar product $\langle\,,\rangle$ on $\mathfrak{g}$ such that $\langle\,,\rangle_{\mathfrak{h}}$ is non-degenerate); $\mathfrak{m}$ the orthogonal complement to $\mathfrak{h}$ for $\langle\,,\rangle$; $X, Y \in \mathfrak{m}$. Sectional curvature:

$$g\big(R(X, Y)Y, X\big)_{\mathfrak{h}} = \frac{1}{4}\big\langle [X, Y]_{\mathfrak{m}}, [X, Y]_{\mathfrak{m}}\big\rangle_{\mathfrak{m}} + \big\langle [X, Y]_{\mathfrak{h}}, [X, Y]_{\mathfrak{h}}\big\rangle_{\mathfrak{h}}.$$

Curvature and Killing Vector Fields

- (M, g) Riemannian manifold; X, Y, Z, Killing vector fields; Levi-Civita connection:

$$2g(\nabla_X Y, Z) = g\big([X, Y], Z\big) + g\big([Y, Z], X\big) - g\big([Z, X], Y\big).$$

- $(M = G/H, g)$ Riemannian homogeneous; $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$; X, Y, Killing vector fields in $\mathfrak{m}$; $\mathfrak{m} \equiv T_o M$; Levi-Civita connection, curvature tensor and Ricci curvature at o, and scalar curvature at any point:

$$\begin{aligned}
(\nabla_X Y)_o &= -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \\
g_o\big(R(X, Y)Y, X\big) &= -\frac{3}{4}\big|[X, Y]_{\mathfrak{m}}\big|^2 - \frac{1}{2}\big\langle\big[X, [X, Y]_{\mathfrak{g}}\big]_{\mathfrak{m}}, Y\big\rangle \\
&\quad - \frac{1}{2}\big\langle\big[Y, [Y, X]_{\mathfrak{g}}\big]_{\mathfrak{m}}, X\big\rangle + \big|U(X, Y)\big|^2 - \big\langle U(X, X), U(Y, Y)\big\rangle, \\
\mathbf{r}_o(X, X) &= -\frac{1}{2}\sum_j \big|[X, e_i]_{\mathfrak{m}}\big|^2 - \frac{1}{2}\sum_i \big\langle\big[X, [X, e_i]_{\mathfrak{m}}\big]_{\mathfrak{m}}, e_i\big\rangle \\
&\quad - \sum_i \big\langle\big[X, [X, e_i]_{\mathfrak{h}}\big]_{\mathfrak{m}}, e_i\big\rangle + \frac{1}{4}\sum_{i,j}\big\langle[e_i, e_j]_{\mathfrak{m}}, X\big\rangle^2 \\
&\quad - \big\langle[Z, X]_{\mathfrak{m}}, X\big\rangle, \\
\mathbf{s} &= -\frac{1}{4}\sum_{i,j}\big|[e_i, e_j]_{\mathfrak{m}}\big|^2 - \frac{1}{2}\sum_i B(e_i, e_i) - |Z|^2
\end{aligned}$$

($\{e_i\}$ a $\langle\, ,\, \rangle$-orthonormal basis of $\mathfrak{m}$; $Z = \sum_i U(e_i, e_i)$) (see [5]).

Riemannian Symmetric Spaces

$$M = (G/H, g, \sigma), \qquad \mathfrak{h} = (\sigma_{*o})_+, \qquad \mathfrak{m} = (\sigma_{*o})_-, \qquad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$
$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

- Curvature tensor field ($X, Y, Z \in \mathfrak{m}$):

$$R(X, Y)Z = -\big[[X, Y], Z\big]. \tag{7.8}$$

- Ricci tensor for G/H Hermitian symmetric with G semi-simple and effective on G/H, and H compact, in terms of the Killing form B of $\mathfrak{g}$:

$$\mathbf{r}(X, Y) = -\frac{1}{2}B(X, Y), \qquad X, Y \in \mathfrak{m}.$$

- Invariant connections on Riemannian symmetric spaces (G, H, σ) of type:

 I. (G simple), and
 II. (G product of two copies of a simple Lie group interchanged by the involutive automorphism σ).

For this table and the next one, see Helgason [20] and Laquer [21, 22].

Type I		Set of invariant connections
AI	$\mathrm{SU}(n)/\mathrm{SO}(n),\ n \geqslant 3\ (*)$	1-dimensional family
AII	$\mathrm{SU}(n)/\mathrm{Sp}(n),\ n \geqslant 3$	1-dimensional family
EIV	$\mathrm{E}_6/\mathrm{F}_4$	1-dimensional family
	Other cases	Only the canonical connection
Type II		**Set of bi-invariant connections**
	$\mathrm{SU}(n), n \geqslant 3\ (*)$	2-dimensional family
	Other cases	1-dimensional family

$(*)$ $\mathrm{SO}(6)/(\mathrm{SO}(3) \times \mathrm{SO}(3))$ behaves as $\mathrm{SU}(4)/\mathrm{SO}(4)$ and $\mathrm{SO}(6)$ as $\mathrm{SU}(4)$

Irreducible Riemannian Symmetric Spaces of Type I and III

	Compact	Noncompact	rank	dim
A I	$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$	$n-1$	$\frac{(n-1)(n+2)}{2}$
A II	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$	$n-1$	$2n^2-n-1$
A III $(*)$	$\frac{\mathrm{SU}(p+q)}{S(\mathrm{U}(p)\times\mathrm{U}(q))}$	$\frac{\mathrm{SU}(p,q)}{S(\mathrm{U}(p)\times\mathrm{U}(q))}$	$\min(p,q)$	$2pq\ (**)$
BD I $(*)$	$\frac{\mathrm{SO}(p+q)}{\mathrm{SO}(p)\times\mathrm{SO}(q)}$	$\frac{\mathrm{SO}_0(p,q)}{\mathrm{SO}(p)\times\mathrm{SO}(q)}$	$\min(p,q)$	pq
D III $(*)$	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$[\frac{1}{2}n]$	$n(n-1)$
C I $(*)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	n	$n(n+1)$
C II	$\frac{\mathrm{Sp}(p+q)}{\mathrm{Sp}(p)\times\mathrm{Sp}(q)}$	$\frac{\mathrm{Sp}(p,q)}{\mathrm{Sp}(p)\times\mathrm{Sp}(q)}$	$\min(p,q)$	$4pq\ (**)$
E I	$(\mathfrak{e}_6^{-78}, \mathfrak{sp}(4))$	$(\mathfrak{e}_6^{6}, \mathfrak{sp}(4))$	6	42
E II	$(\mathfrak{e}_6^{-78}, \mathfrak{su}(6)+\mathfrak{su}(2))$	$(\mathfrak{e}_6^{2}, \mathfrak{su}(6)+\mathfrak{su}(2))$	4	40
E III $(*)$	$(\mathfrak{e}_6^{-78}, \mathfrak{so}(10)+\mathbb{R})$	$(\mathfrak{e}_6^{-14}, \mathfrak{so}(10)+\mathbb{R})$	2	32
E IV	$(\mathfrak{e}_6^{-78}, \mathfrak{f}_4)$	$(\mathfrak{e}_6^{-26}, \mathfrak{f}_4)$	2	26
E V	$(\mathfrak{e}_7^{-133}, \mathfrak{su}(8))$	$(\mathfrak{e}_7^{7}, \mathfrak{su}(8))$	7	70
E VI	$(\mathfrak{e}_7^{-133}, \mathfrak{so}(12)+\mathfrak{su}(2))$	$(\mathfrak{e}_7^{-5}, \mathfrak{so}(12)+\mathfrak{su}(2))$	4	64
E VII $(*)$	$(\mathfrak{e}_7^{-133}, \mathfrak{e}_6+\mathbb{R})$	$(\mathfrak{e}_7^{-25}, \mathfrak{e}_6+\mathbb{R})$	3	54
E VIII	$(\mathfrak{e}_8^{-248}, \mathfrak{so}(16))$	$(\mathfrak{e}_8^{8}, \mathfrak{so}(16))$	8	128
E IX	$(\mathfrak{e}_8^{-248}, \mathfrak{e}_7+\mathfrak{su}(2))$	$(\mathfrak{e}_8^{-24}, \mathfrak{e}_7+\mathfrak{su}(2))$	4	112
F I	$(\mathfrak{f}_4^{-52}, \mathfrak{sp}(3)+\mathfrak{su}(2))$	$(\mathfrak{f}_4^{4}, \mathfrak{sp}(3)+\mathfrak{su}(2))$	4	28
F II	$(\mathfrak{f}_4^{-52}, \mathfrak{so}(9))$	$(\mathfrak{f}_4^{-20}, \mathfrak{so}(9))$	1	16
G	$(\mathfrak{g}_2^{-14}, \mathfrak{su}(2)+\mathfrak{su}(2))$	$(\mathfrak{g}_2^{2}, \mathfrak{su}(2)+\mathfrak{su}(2))$	2	8

$(*)$ Hermitian symmetric (for BD I, only if $q=2$)

$(**)$ $(p \leqslant q)$

Remark The superindices for the exceptional simple Lie algebras denote the signature of the corresponding Killing form B, where the signature is defined here as the number of positive values minus the number of negative values when B is expressed in diagonal form (see [20]): $\mathfrak{e}_6$: $-78, -26, -14, 2, 6$; $\mathfrak{e}_7$: $-133, -25, -5, 7$; $\mathfrak{e}_8$: $-248, -24, 8$; $\mathfrak{f}_4$: $-52, -20, 4$; $\mathfrak{g}_2$: $-14, 2$.

Symmetric Spaces G/H of Classical Type with Noncompact Isotropy Group[1]

$G = SL(n, \mathbb{C})$	$G = SL(n, \mathbb{R})$	$G = \mathrm{SU}(p, q)$
$SL(p, \mathbb{C}) \times SL(q, \mathbb{C}) \times \mathbb{C}$	$SL(p, \mathbb{R}) \times SL(q, R) \times \mathbb{R}$	$\mathrm{SU}(k, k+h) \times \mathrm{SU}(p-k, n-k-h) \times \mathrm{U}(1)$
$\mathrm{SL}(n, \mathbb{R})$	$\mathrm{SO}(p, q)$	$\mathrm{SO}(p, q)$
$\mathrm{SO}(n, \mathbb{C})$	$\mathrm{Sp}(n/2, \mathbb{R})$	$\mathrm{Sp}(p/2, q/2)$
$\mathrm{SU}(p, q)$	$\mathrm{Sp}(n/2, \mathbb{C}) \times \mathbb{R}$	$\mathrm{SO}^*(n)$ $(*)$
$\mathrm{Sp}(n/2, \mathbb{C})$		$\mathrm{Sp}(n, \mathbb{R})$ $(*)$
$\mathrm{SU}^*(n)$		$\mathrm{SL}(n, \mathbb{C}) \times \mathbb{R}$ $(*)$

$G = \mathrm{SU}^*(n)$	$G = \mathrm{SO}(n, \mathbb{C})$	$G = \mathrm{SO}(p, q)$
$\mathrm{SU}^*(p) \times \mathrm{SU}^*(q) \times \mathbb{R}$	$\mathrm{SO}(p, \mathbb{C}) \times \mathrm{SO}(q, \mathbb{C})$ $(\dagger)$	$\mathrm{SO}(k, k+h) \times \mathrm{SO}(p-k, n-k-h)$ $(**)$
$\mathrm{Sp}(p/2, q/2)$	$\mathrm{SO}(n-2) \times \mathbb{C}$	$\mathrm{SO}(p-2, q) \times \mathrm{U}(1)$
$\mathrm{SO}^*(n)$	$\mathrm{SO}(p, q)$	$\mathrm{SO}(p-1, q-1) \times \mathbb{R}$
$\mathrm{SL}(n, \mathbb{C}) \times \mathrm{U}(1)$	$\mathrm{SL}(n/2, \mathbb{C}) \times \mathbb{C}$	$\mathrm{SU}(p/2, q/2) \times \mathrm{U}(1)$
	$\mathrm{SO}^*(n)$	$\mathrm{SL}(n/2, \mathbb{R}) \times \mathbb{R}$ $(*)$
	$\mathrm{SO}^*(n/2, \mathbb{C})$	$\mathrm{SO}(n/2, \mathbb{C})$ $(*)$
	$\mathrm{SU}(p, q) \times \mathrm{U}(1)$	$\mathrm{SO}(n-2) \times \mathrm{U}(1)$ $(\ddagger)$

$G = \mathrm{Sp}(n, \mathbb{C})$	$G = \mathrm{Sp}(n, \mathbb{R})$	$G = \mathrm{Sp}(p, q)$
$\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}$	$\mathrm{Sp}(p, \mathbb{R}) \times \mathrm{Sp}(q, \mathbb{R})$	$\mathrm{Sp}(k, k+h) \times \mathrm{Sp}(p-k, n-k-h)$
$\mathrm{Sp}(n, \mathbb{R})$	$\mathrm{SU}(p, q) \times \mathrm{U}(1)$	$\mathrm{SU}(p, q) \times \mathrm{U}(1)$
$\mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C})$	$\mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}$	
$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(n/2, \mathbb{C})$	
	$\mathrm{SU}^*(n) \times \mathbb{R}$ $(*)$	
	$\mathrm{Sp}(n/2, \mathbb{C})$	

$(*)$ $p = q = n/2$, $(\dagger)$ $p = 1$ or $p, q > 2$, $(**)$ $k + h > 2$, $n - k - h > 2$, $(\ddagger)$ $p = 2$, $q = n - 2$

[1] The isotropy groups for each group G are listed under it. In all cases but for $\mathrm{Sp}(n, \mathbb{C})$ and $\mathrm{Sp}(n, \mathbb{R})$, the expression of the first listed isotropy group has been broken in two lines.

Simply Connected Normal Homogeneous Spaces of Positive Curvature

Type	$\dim_{\mathbb{R}}$	δ	(i)
S^n	n	1	(ii)
$\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2$	$2n, 4n, 16$	$\frac{1}{4}$	(ii)
$\mathbb{C}P^n = \mathrm{Sp}(m+1)/(\mathrm{Sp}(m) \times \mathrm{U}(1))$	$2m+1$	$\frac{1}{16}$	(iii)
$S^{2m+1} = (\mathrm{SU}(m+1)/\mathrm{SU}(m), g_s)$	$4m+3$	$\frac{s(m+1)}{8m-3s(m+1)}$	(iv)
$(S^{4m+3} = \mathrm{Sp}(m+1)/\mathrm{Sp}(m), g_s)$	$4m+3$	$\begin{cases} \frac{s}{8-3s} & \text{if } s \geq \frac{2}{3} \\ \frac{s^2}{4} & \text{if } s < \frac{2}{3} \end{cases}$	
$B^7 = \mathrm{Sp}(2)/\mathrm{SU}(2)$	7	$\frac{1}{37}$	(vi)
$B^{13} = \mathrm{SU}(5)/H$	13	$\frac{16}{29 \cdot 37}$	(vii)
$(W^7 = (\mathrm{SO}(3) \times \mathrm{SU}(3))/U^{\bullet}(2), g_s)$	7	$\begin{cases} \frac{t^2}{4} & \text{if } t \leq \frac{8-2\sqrt{13}}{3} \\ \frac{t}{16-3t} & \text{if } \frac{8-2\sqrt{13}}{3} \leq t \leq \frac{2}{5} \\ \frac{16(1-t)^3}{(16-3t)(4+16t-11t^2)} & \text{if } \frac{2}{5} \leq t < 1 \end{cases}$	(viii)

(i) $\delta =$ Pinching constant

(ii) Compact rank-one symmetric spaces with their standard metrics

(iii) Equipped with a standard $\mathrm{Sp}(m+1)$-homogeneous metric

(iv) Berger spheres, $0 < s \leq 1$

(v) $0 < s \leq 1$

(vi) Berger space B^7 ([4]) equipped with a standard $\mathrm{Sp}(2)$-homogeneous metric

(vii) Berger space B^{13} ([4]) equipped with a standard $\mathrm{SU}(5)$-homogeneous metric

(viii) Wilking space [30] equipped with a one-parameter family g_s, $s > 0$, of $\mathrm{SO}(3) \times \mathrm{SU}(3)$-homogeneous metrics. Here $t = \frac{2s}{2s+3}$

Compact Rank-One Riemannian Symmetric Spaces

Type	$\dim_{\mathbb{R}}$	$\chi(M)$	$\tau(M)$	Volume
$S^n(\lambda)$	n	$1+(-1)^n$	$n(n-1)\lambda$	$\frac{\pi^{\frac{n+1}{2}}(n+1)}{(\frac{n+1}{2})!\lambda^{\frac{n}{2}}}$
$\mathbb{R}P^n(\lambda)$	n	$\frac{1}{2}(1+(-1)^n)$	$n(n-1)\lambda$	$\frac{\pi^{\frac{n+1}{2}}}{(\frac{n-1}{2})!\lambda^{\frac{n}{2}}}$
$\mathbb{C}P^n(\lambda)$	$2n$	$n+1$	$4n(n+1)\lambda$	$\frac{1}{n!}(\frac{\pi}{\lambda})^n$
$\mathbb{H}P^n(\lambda)$	$4n$	$n+1$	$16n(n+1)\lambda$	$\frac{\pi^{2n}}{(2n+1)!\lambda^{2n}}$
$\mathbb{O}P^2(\lambda)$	16	3	576λ	$\frac{6\pi^8}{11!\lambda^8}$

$\chi(M)$: Euler–Poincaré characteristic of M

$\tau(M)$: signature of M

λ: constant curvature λ for $S^n(\lambda)$, $\mathbb{R}P^n(\lambda)$; constant holomorphic (resp. quaternionic, Cayley) sectional curvature 4λ for $\mathbb{C}P^n$ (resp. $\mathbb{H}P^n(\lambda)$, $\mathbb{O}P^2(\lambda)$). See [6]

Simply Irreducible Pseudo-Hermitian Symmetric Spaces G/H

(A)

$$\frac{\mathrm{SL}(2n,\mathbb{R})}{\mathrm{SL}(n,\mathbb{C})\times\mathrm{U}(1)},\qquad \frac{\mathrm{SU}^*(2n)}{\mathrm{SL}(n,\mathbb{C})\times\mathrm{U}(1)},\qquad \frac{\mathrm{SU}(n-i,i)}{\mathrm{S}(\mathrm{U}(h,k)\times\mathrm{U}(n-i-h,i-k))},$$

(BD)

$$\frac{\mathrm{SO}^*(2n)}{\mathrm{SO}^*(2n-2)\times\mathrm{U}(1)},\qquad \frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n-k,k)},$$
$$\frac{\mathrm{SO}(n-k,k)}{\mathrm{SO}(n-k,k-2)\times\mathrm{U}(1)},\qquad \frac{\mathrm{SO}(2(n-k),2k)}{\mathrm{U}(n-k,k)},$$

(C)

$$\frac{\mathrm{Sp}(n-i,i)}{\mathrm{U}(n-i,i)},$$

(E_6)

$$\frac{\mathrm{E}_{6(2)}}{\mathrm{SO}^*(10)\times\mathrm{U}(1)},\qquad \frac{\mathrm{E}_{6(2)}}{\mathrm{SO}(6,4)\times\mathrm{U}(1)},$$
$$\frac{\mathrm{E}_{6(-14)}}{\mathrm{SO}(8,2)\times\mathrm{U}(1)},\qquad \frac{\mathrm{E}_{6(-14)}}{\mathrm{SO}^*(10)\times\mathrm{U}(1)},$$

(E_7)

$$\frac{\mathrm{E}_{7(7)}}{\mathrm{E}_{6(2)}\times\mathrm{U}(1)},\qquad \frac{\mathrm{E}_{7(-5)}}{\mathrm{E}_{6(-14)}\times\mathrm{U}(1)},\qquad \frac{\mathrm{E}_{7(-5)}}{\mathrm{E}_{6(2)}\times\mathrm{U}(1)},\qquad \frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_{6(-14)}\times\mathrm{U}(1)}$$

(see [26]).

Simply Reducible Pseudo-Hermitian Symmetric Spaces G/H

(A)

$$\frac{\mathrm{SL}(2n,\mathbb{C})}{\mathrm{S}(\mathrm{GL}(n-k,\mathbb{C})\times\mathrm{GL}(k,\mathbb{C}))},$$

(BD)

$$\frac{\mathrm{SO}(n,\mathbb{C})}{\mathrm{SO}(n-2,\mathbb{C})\times\mathrm{SO}(2,\mathbb{C})},\qquad \frac{\mathrm{SO}(2n,\mathbb{C})}{\mathrm{SL}(n,\mathbb{C})\times\mathbb{C}^*},$$

(C)

$$\frac{\mathrm{Sp}(n,\mathbb{C})}{\mathrm{SL}(n,\mathbb{C})\times\mathbb{C}^*},$$

(E_6)

$$\frac{\mathrm{E}_6^{\mathbb{C}}}{\mathrm{SO}(10,\mathbb{C})\times\mathbb{C}^*},$$

(E_7)

$$\frac{E_7^{\mathbb{C}}}{E_6^{\mathbb{C}} \times \mathbb{C}^*},$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. (See [26].)

Simply Connected Symmetric Pseudo-Quaternionic Kähler Spaces G/H

(A)

$$\frac{SU(p+2,q)}{S(U(2)\times U(p,q))}, \qquad \frac{SL(n+1,\mathbb{H})}{S(GL(1,\mathbb{H})\times GL(n,\mathbb{H}))},$$

(BD)

$$\frac{SO_0(p+4,q)}{SO(4)\times SO_0(p,q)}, \qquad \frac{SO^*(2l+4)}{SO^*(4)\times SO^*(2l)},$$

(C)

$$\frac{Sp(p+1,q)}{Sp(1)\times Sp(p,q)},$$

(G_2)

$$\frac{G_{2(-14)}}{SO(4)}, \qquad \frac{G_{2(2)}}{SO(4)},$$

(F_4)

$$\frac{F_{4(-52)}}{Sp(1)Sp(3)}, \qquad \frac{F_{4(4)}}{Sp(1)Sp(3)}, \qquad \frac{F_{4(4)}}{Sp(1)Sp(1,2)}, \qquad \frac{F_{4(-20)}}{Sp(2)Sp(1,2)},$$

(E_6)

$$\frac{E_{6(-78)}}{SU(2)SU(6)}, \qquad \frac{E_{6(2)}}{SU(2)SU(6)}, \qquad \frac{E_{6(2)}}{SU(2)SU(2,4)},$$

$$\frac{E_{6(-14)}}{SU(2)SU(2,4)}, \qquad \frac{E_{6(6)}}{Sp(1)SL(3,\mathbb{H})}, \qquad \frac{E_{6(-26)}}{Sp(1)SL(3,\mathbb{H})},$$

(E_7)

$$\frac{E_{7(-133)}}{SU(2)Spin(12)}, \qquad \frac{E_{7(-5)}}{SU(2)Spin(12)}, \qquad \frac{E_{7(-5)}}{SU(2)Spin_0(4,8)},$$

$$\frac{E_{7(7)}}{SU(2)SO^*(12)}, \qquad \frac{E_{7(-25)}}{SU(2)SO^*(12)},$$

(E_8)

$$\frac{E_{8(-248)}}{SU(2)E_{7(133)}}, \qquad \frac{E_{8(-24)}}{SU(2)E_{7(133)}}, \qquad \frac{E_{8(-24)}}{SU(2)E_{7(-5)}}, \qquad \frac{E_{8(8)}}{SU(2)E_{7(-5)}}.$$

(see [2]).

Berger List of Riemannian Holonomy Groups

$\dim M = n$	Group	Name	Einstein	Ricci flat
n	$\mathrm{SO}(n)$	Generic	–	–
$2m$	$\mathrm{U}(m)$	Kähler	–	–
$2m$	$\mathrm{SU}(m)$	Special Kähler	–	Yes
$4m$	$\mathrm{Sp}(m)$	Hyper-Kähler	–	Yes
$4m$	$\mathrm{Sp}(1)\mathrm{Sp}(m)$	Quaternion-Kähler	Yes	–
7	G_2		–	Yes
8	Spin(7)		–	Yes
16	Spin(9)		Yes	–

Berger List of Pseudo-Riemannian Holonomy Groups Let (M, g) be a simply connected pseudo-Riemannian manifold of dimension $n = r + s$ and signature (r, s) that is not locally symmetric. If the holonomy group of (M, g) acts irreducibly, then it is either $\mathrm{SO}_0(r, s)$ or one of the following (modulo conjugation in $\mathrm{O}(r, s)$):

$$\begin{aligned}
\mathrm{U}(p,q) \quad \text{or} \quad \mathrm{SU}(p,q) &\subset \mathrm{SO}(2p,2q), \quad n \geq 4;\\
\mathrm{Sp}(p,q) \quad \text{or} \quad \mathrm{Sp}(p,q)\mathrm{Sp}(1) &\subset \mathrm{SO}(4p,4q), \quad n \geq 8;\\
\mathrm{SO}(r,\mathbb{C}) &\subset \mathrm{SO}(r,r), \qquad n \geq 4;\\
\mathrm{Sp}(p,\mathbb{R})\mathrm{SL}(2,\mathbb{R}) &\subset \mathrm{SO}(2p,2q), \quad n \geq 8;\\
\mathrm{Sp}(p,\mathbb{C})\mathrm{SL}(2,\mathbb{C}) &\subset \mathrm{SO}(4p,4q), \quad n \geq 16;\\
\mathrm{G}_2 &\subset \mathrm{SO}(7);\\
G^*_{2(2)} &\subset \mathrm{SO}(4,3);\\
\mathrm{G}_2^{\mathbb{C}} &\subset \mathrm{SO}(7,7);\\
\mathrm{Spin}(7) &\subset \mathrm{SO}(8);\\
\mathrm{Spin}(4,3) &\subset \mathrm{SO}(4,4);\\
\mathrm{Spin}(7)^{\mathbb{C}} &\subset \mathrm{SO}(8,8).
\end{aligned}$$

As pointed out in [13], this list is due to Berger and also to the efforts of some other authors [1, 7, 8, 27].

Spaces of Constant (Ordinary, Holomorphic, Quaternionic) Curvature and Cayley Planes (In the next formulas, unless the tensor product sign is used, it is understood that the product of differentials is the symmetric product.)

- *Constant ordinary curvature*
- Riemann–Christoffel curvature tensor for constant curvature c:

$$R(X,Y,Z,W) = c\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big).$$

- Metric g of non-zero constant curvature $1/r$ on

$$M = \left\{\left(x^1, \ldots, x^{n+1}, t\right) \in \mathbb{R}^{n+1} : \left(x^1\right)^2 + \cdots + \left(x^n\right)^2 + rt^2 = r\right\} :$$

$$g = \frac{r\{(r + \sum(y^i)^2)(\sum(\mathrm{d}y^i)^2) - (\sum y^i \mathrm{d}y^i)^2\}}{(r + \sum(y^i)^2)^2}, \quad y^i = x^i/t.$$

- Poincaré half-space model (H^n, g) of the real hyperbolic space $\mathbb{R}\mathrm{H}(n)$, with constant curvature $c < 0$:

$$\left(H^n, g\right) = \left(\left\{\left(u^1, \ldots, u^n\right) \in \mathbb{R}^n : u^1 > 0\right\}, -\frac{1}{c(u^1)^2}\sum_{i=1}^{n} \mathrm{d}u^i \otimes \mathrm{d}u^i\right).$$

- Open unit ball model (B^n, g) of the real hyperbolic space $\mathbb{R}\mathrm{H}(n)$ with negative constant curvature c:

$$B^n = \left\{\left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : \sum_{i=1}^{n}\left(x^i\right)^2 < 1\right\},$$

$$g = -\frac{4}{c(1 - \sum_{i=1}^{n}(x^i)^2)^2}\sum_{i=1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

- *Constant holomorphic curvature*
- Riemann–Christoffel curvature tensor for constant holomorphic curvature c:

$$\begin{aligned} R(X, Y, Z, W) = \frac{c}{4}\big\{&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ &+ g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) \\ &+ 2g(X, JY)g(Z, JW)\big\}. \end{aligned}$$

- Fubini–Study metric of positive constant holomorphic sectional curvature c on the complex projective space $\mathbb{C}\mathrm{P}(n)$:

$$g = \frac{4}{c}\frac{(1 + \sum z^k\bar{z}^k)(\sum \mathrm{d}z^k\,\mathrm{d}\bar{z}^k) - (\sum \bar{z}^k\,\mathrm{d}z^k)(\sum z^k\,\mathrm{d}\bar{z}^k)}{(1 + \sum z^k\bar{z}^k)^2}.$$

- Hermitian metric on the open unit ball model (D^n, h, J) of the complex hyperbolic space $\mathbb{C}\mathrm{H}(n)$ for $c < 0$ (see Goldberg [14, p. 227], Goldman [15, p. 73]):

$$D^n = \left\{\left(z^1, \ldots, z^n\right) \in \mathbb{C}^n : \sum_{k=1}^{n}\left|z^k\right|^2 < 1\right\},$$

$$\begin{aligned} h = -\frac{4}{c(1 - \sum_{k=1}^{n}|z^k|^2)^2}\Bigg(&\left(1 - \sum_{k=1}^{n}\left|z^k\right|^2\right)\left(\sum_{k=1}^{n}\mathrm{d}z^k\,\mathrm{d}\bar{z}^k\right) \\ &+ \left(\sum_{k=1}^{n}\bar{z}^k\,\mathrm{d}z^k\right)\left(\sum_{k=1}^{n}z^k\,\mathrm{d}\bar{z}^k\right)\Bigg). \end{aligned}$$

- Matrix expression of h:

$$(h_{k\bar{l}}) = -\frac{4}{c(1-\sum_{j=1}^{n}|z^j|^2)^2}\left(\delta_{kl}\left(1-\sum_{j=1}^{n}\left|z^j\right|^2\right)+\bar{z}^k z^l\right).$$

- Riemannian metric g, Kähler form ω ($h = g + \mathrm{i}\omega$), and complex structure J ($\omega(X,Y) = g(X, JY)$) (with real coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$ on D^n defined by $z^k = x^k + \mathrm{i}y^k$):

$$\begin{aligned}
g = -\frac{4}{c(1-\sum_{j=1}^{n}|z^j|^2)^2}&\left(\sum_{k=1}^{n}\left[1-\sum_{j\neq k}\left|z^j\right|^2\right]\cdot\left(\left(\mathrm{d}x^k\right)^2+\left(\mathrm{d}y^k\right)^2\right)\right.\\
&+\sum_{k\neq l}\left[\left(x^k x^l+y^k y^l\right)\left(\mathrm{d}x^k\,\mathrm{d}x^l+\mathrm{d}y^k\,\mathrm{d}y^l\right)\right.\\
&\left.\left.+\left(x^k y^l-x^l y^k\right)\left(\mathrm{d}x^k\,\mathrm{d}y^l-\mathrm{d}y^k\,\mathrm{d}x^l\right)\right]\right),\\
\omega = -\frac{4}{c(1-\sum_{j=1}^{n}|z^j|^2)^2}&\cdot\left(\sum_{k=1}^{n}\left[\sum_{j\neq k}\left|z^j\right|^2-1\right]\mathrm{d}x^k\wedge\mathrm{d}y^k\right.\\
&+\sum_{k<l}\left[\left(x^k y^l-x^l y^k\right)\left(\mathrm{d}x^k\wedge\mathrm{d}x^l+\mathrm{d}y^k\wedge\mathrm{d}y^l\right)\right.\\
&\left.\left.-\left(x^k x^l+y^k y^l\right)\left(\mathrm{d}x^k\wedge\mathrm{d}y^l+\mathrm{d}x^l\wedge\mathrm{d}y^k\right)\right]\right),\\
J = \sum_{k=1}^{n}&\left(\frac{\partial}{\partial y^k}\otimes\mathrm{d}x^k-\frac{\partial}{\partial x^k}\otimes\mathrm{d}y^k\right).
\end{aligned}$$

- Hermitian metric on the Siegel domain model (D_+, h_+, J_+) of $\mathbb{C}\mathrm{H}(n)$:

$$\begin{aligned}
D_+ &= \left\{\left(u^1 = x+\mathrm{i}y, u^2,\dots,u^n\right)\in\mathbb{C}^n : y-\sum_{k=2}^{n}\left|u^k\right|^2>0\right\},\\
h_+ &= -\frac{1}{c(y-\sum_{k=2}^{n}|u^k|^2)}\left\{\mathrm{d}u^1\,\mathrm{d}\bar{u}^1+2\mathrm{i}\sum_{k=2}^{n}\left(u^k\,\mathrm{d}u^1\,\mathrm{d}\bar{u}^k-\bar{u}^k\,\mathrm{d}u^k\,\mathrm{d}\bar{u}^1\right)\right.\\
&\qquad\left.+4\sum_{k=2}^{n}\left(y-\sum_{\substack{j>1\\ j\neq k}}\left|u^j\right|^2\right)\mathrm{d}u^k\,\mathrm{d}\bar{u}^k+4\left(\sum_{\substack{k,l>1\\ k\neq l}}\bar{u}^k u^l\,\mathrm{d}u^k\,\mathrm{d}\bar{u}^l\right)\right\}.
\end{aligned}$$

- Matrix expression of h_+:

$$
\big((h_+)_{k\bar{l}}\big) = -\frac{1}{c(y-\sum_{j>1}|u^j|^2)^2}
\times \left(\begin{array}{c|ccc}
1 & 2\mathrm{i}u^2 & \cdots & 2\mathrm{i}u^n \\
\hline
-2\mathrm{i}\bar{u}^2 & & & \\
\vdots & & 4(\delta_{kl}(y-\sum_{j>1}|u^j|^2)+\bar{u}^k u^l) & \\
-2\mathrm{i}\bar{u}^n & & &
\end{array}\right).
$$

- Corresponding Riemannian metric g_+ and complex structure J_+ ($u^k = v^k + \mathrm{i}w^k$, $k = 2, \dots, n$):

$$
\begin{aligned}
g_+ = -\frac{y-\sum_{j>1}|u^j|^2}{c}\Bigg\{ & \mathrm{d}x^2 + \mathrm{d}y^2 + 4\sum_{k>1}\Bigg[y - \sum_{\substack{j>1\\ j\neq k}}|u^j|^2\Bigg]\Big(\big(\mathrm{d}v^k\big)^2 + \big(\mathrm{d}w^k\big)^2\Big) \\
& - 4\Bigg[\mathrm{d}x\sum_{k>1}\big(w^k\,\mathrm{d}v^k - v^k\,\mathrm{d}w^k\big) + \mathrm{d}y\sum_{k>1}\big(v^k\,\mathrm{d}v^k + w^k\,\mathrm{d}w^k\big)\Bigg] \\
& + 4\sum_{\substack{k,l>1\\ k\neq l}}\Big[\big(v^k v^l + w^k w^l\big)\big(\mathrm{d}v^k\,\mathrm{d}v^l + \mathrm{d}w^k\,\mathrm{d}w^l\big) \\
& + \big(v^k w^l - v^l w^k\big)\big(\mathrm{d}v^k\,\mathrm{d}w^l - \mathrm{d}w^k\,\mathrm{d}v^l\big)\Big]\Bigg\},
\end{aligned}
$$

$$
J_+ = \frac{\partial}{\partial y}\otimes \mathrm{d}x - \frac{\partial}{\partial x}\otimes \mathrm{d}y + \sum_{k=2}^{n}\left(\frac{\partial}{\partial w^k}\otimes \mathrm{d}v^k - \frac{\partial}{\partial v^k}\otimes \mathrm{d}w^k\right).
$$

- *Constant quaternionic curvature*
- Riemann–Christoffel curvature tensor for constant quaternionic curvature c:

$$
\begin{aligned}
R(X,Y,Z,W) = \frac{c}{4}\Bigg\{ & g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\
& + \sum_{a=1}^{3}\Big(g(X,JZ)g(Y,JW) - g(X,JW)g(Y,JZ) \\
& + 2g(X,JY)g(Z,JW)\Big)\Bigg\}.
\end{aligned}
$$

- Watanabe's metric on the quaternionic projective space $\mathbb{H}\mathrm{P}(n)$:

$$\hat{h} = \frac{4}{c(1+\sum_{k=1}^{n}|q^k|^2)^2}\left(\left(1+\sum_{k=1}^{n}|q^k|^2\right)\sum_{k=1}^{n}\mathrm{d}q^k\,\mathrm{d}\bar{q}^k\right.$$
$$\left.-\left(\sum_{k=1}^{n}\bar{q}^k\,\mathrm{d}q^k\right)\left(\sum_{k=1}^{n}\mathrm{d}\bar{q}^k q^k\right)\right).$$

- Watanabe's Riemannian metric $\hat{g}$ (with $q^k = x^k + \mathbf{i}y^k + \mathbf{j}z^k + \mathbf{k}w^k$, $k = 1, \dots, n$, see Watanabe [29]):

$$\hat{g} = \frac{4}{c(1+\sum_{k=1}^{n}|q^k|^2)^2}\left\{\left(1+\sum_{k=1}^{n}|q^k|^2\right)\right.$$
$$\cdot\sum_{k=1}^{n}\left(\mathrm{d}x^k\,\mathrm{d}x^k + \mathrm{d}y^k\,\mathrm{d}y^k + \mathrm{d}z^k\,\mathrm{d}z^k + \mathrm{d}w^k\,\mathrm{d}w^k\right)$$
$$-\sum_{k,l=1}^{n}\big[\big(A_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l + B_{kl}\,\mathrm{d}x^k\,\mathrm{d}y^l + C_{kl}\,\mathrm{d}x^k\,\mathrm{d}z^l + D_{kl}\,\mathrm{d}x^k\,\mathrm{d}w^l$$
$$- B_{kl}\,\mathrm{d}y^k\,\mathrm{d}x^l + A_{kl}\,\mathrm{d}y^k\,\mathrm{d}y^l + D_{kl}\,\mathrm{d}y^k\,\mathrm{d}z^l - C_{kl}\,\mathrm{d}y^k\,\mathrm{d}w^l$$
$$- C_{kl}\,\mathrm{d}z^k\,\mathrm{d}x^l - D_{kl}\,\mathrm{d}z^k\,\mathrm{d}y^l + A_{kl}\,\mathrm{d}z^k\,\mathrm{d}z^l + B_{kl}\,\mathrm{d}z^k\,\mathrm{d}w^l$$
$$\left.- D_{kl}\,\mathrm{d}w^k\,\mathrm{d}x^l + C_{kl}\,\mathrm{d}w^k\,\mathrm{d}y^l - B_{kl}\,\mathrm{d}w^k\,\mathrm{d}z^l + A_{kl}\,\mathrm{d}w^k\,\mathrm{d}w^l\big)\big]\right\},$$

where

$$\begin{aligned} A_{kl} &= x^k x^l + y^k y^l + z^k z^l + w^k w^l, \\ B_{kl} &= x^k y^l - y^k x^l + z^k w^l - w^k z^l, \\ C_{kl} &= x^k z^l - y^k w^l - z^k x^l + w^k y^l, \\ D_{kl} &= x^k w^l + y^k z^l - z^k y^l - w^k x^l. \end{aligned} \tag{$\star$}$$

- Open unit ball model $(E^n, \hat{h}, \upsilon^3)$ of the quaternionic hyperbolic space $\mathbb{H}\mathrm{H}(n)$ (with $c < 0$):

$$E^n = \left\{\left(q^1, \dots, q^n\right) \in \mathbb{H}^n : \sum_{k=1}^{n}|q^k|^2 < 1\right\},$$
$$\hat{h} = -\frac{4}{c(1-\sum_{k=1}^{n}|q^k|^2)^2}\left(\left(1-\sum_{k=1}^{n}|q^k|^2\right)\sum_{k=1}^{n}\mathrm{d}q^k\,\mathrm{d}\bar{q}^k\right.$$
$$\left.+\left(\sum_{k=1}^{n}\bar{q}^k\,\mathrm{d}q^k\right)\left(\sum_{k=1}^{n}\mathrm{d}\bar{q}^k q^k\right)\right).$$

- Corresponding Riemannian metric $\hat{g}$ (with $q^k = x^k + \mathbf{i}y^k + \mathbf{j}z^k + \mathbf{k}w^k$, $k = 1, \dots, n$):

$$\begin{aligned}
\hat{g} = -\frac{4}{c(1-\sum_{k=1}^n |q^k|^2)^2} \Bigg\{ & \Bigg(1 - \sum_{k=1}^n |q^k|^2\Bigg) \\
& \times \sum_{k=1}^n \big(\mathrm{d}x^k\,\mathrm{d}x^k + \mathrm{d}y^k\,\mathrm{d}y^k + \mathrm{d}z^k\,\mathrm{d}z^k + \mathrm{d}w^k\,\mathrm{d}w^k\big) \\
& + \sum_{k,l=1}^n \big(A_{kl}\big(\mathrm{d}x^k\,\mathrm{d}x^l + \mathrm{d}y^k\,\mathrm{d}y^l + \mathrm{d}z^k\,\mathrm{d}z^l + \mathrm{d}w^k\,\mathrm{d}w^l\big) \\
& + B_{kl}\big(\mathrm{d}x^k\,\mathrm{d}y^l - \mathrm{d}y^k\,\mathrm{d}x^l + \mathrm{d}z^k\,\mathrm{d}w^l - \mathrm{d}w^k\,\mathrm{d}z^l\big) \\
& - C_{kl}\big(\mathrm{d}z^k\,\mathrm{d}x^l - \mathrm{d}x^k\,\mathrm{d}z^l + \mathrm{d}y^k\,\mathrm{d}w^l - \mathrm{d}w^k\,\mathrm{d}y^l\big) \\
& - D_{kl}\big(\mathrm{d}w^k\,\mathrm{d}x^l - \mathrm{d}x^k\,\mathrm{d}w^l - \mathrm{d}y^k\,\mathrm{d}z^l + \mathrm{d}z^k\,\mathrm{d}y^l\big)\big) \Bigg\},
\end{aligned}$$

with $A_{kl}, B_{kl}, C_{kl}, D_{kl}$ as in ($\star$) above.

- A standard basis of the almost quaternionic structure υ^3 on E^n:

$$J_1 = \sum_{k=1}^n \left(-\frac{\partial}{\partial x^k} \otimes \mathrm{d}y^k + \frac{\partial}{\partial y^k} \otimes \mathrm{d}x^k + \frac{\partial}{\partial z^k} \otimes \mathrm{d}w^k - \frac{\partial}{\partial w^k} \otimes \mathrm{d}z^k\right),$$

$$J_2 = \sum_{k=1}^n \left(-\frac{\partial}{\partial x^k} \otimes \mathrm{d}z^k - \frac{\partial}{\partial y^k} \otimes \mathrm{d}w^k + \frac{\partial}{\partial z^k} \otimes \mathrm{d}x^k + \frac{\partial}{\partial w^k} \otimes \mathrm{d}y^k\right),$$

$$J_3 = \sum_{k=1}^n \left(\frac{\partial}{\partial x^k} \otimes \mathrm{d}w^k - \frac{\partial}{\partial y^k} \otimes \mathrm{d}z^k + \frac{\partial}{\partial z^k} \otimes \mathrm{d}y^k - \frac{\partial}{\partial w^k} \otimes \mathrm{d}x^k\right).$$

- Real part of the metric induced by $\hat{h}$ on the Siegel domain model $(E_+, g_{E_+}, \upsilon^3_+)$ of $\mathbb{H}\mathrm{H}(n)$:

$$E_+ = \left\{\big(u^1, \dots, u^n\big) \in \mathbb{H}^n : \mathrm{Re}\big(u^1\big) - \sum_{k=2}^n \big|u^k\big|^2 > 0\right\},$$

$$\begin{aligned}
g_{E_+} = -\frac{4}{c(\mathrm{Re}(u^1) - \sum_{k>1} |u^k|^2)^2} \Bigg(& \frac{1}{4}\big|\mathrm{d}u^1\big|^2 - \sum_{k>1} \mathrm{Re}\big(\bar{u}^k\,\mathrm{d}u^k\,\mathrm{d}\bar{u}^1\big) \\
& + \sum_{k>1} \Bigg(\mathrm{Re}\big(u^1\big) - \sum_{\substack{j>1 \\ j\neq k}} \big|u^j\big|^2\Bigg) \big|\mathrm{d}u^k\big|^2 + \sum_{\substack{k,l>1 \\ k\neq l}} \mathrm{Re}\big(\bar{u}^k u^l\,\mathrm{d}u^k\,\mathrm{d}\bar{u}^l\big)\Bigg).
\end{aligned}$$

- Corresponding Riemannian metric g_{E_+} (with $u^k = a^k + \mathbf{i}b^k + \mathbf{j}c^k + \mathbf{k}e^k$, $1 \le k \le n$):

$$g_{E_+} = -\frac{4}{c(a^1 - \sum_{k>1}|u^k|^2)^2} \sum_{k,l=1}^{n} \big\{ \tilde{A}_{kl}\big(\mathrm{d}a^k\,\mathrm{d}a^l + \mathrm{d}b^k\,\mathrm{d}b^l + \mathrm{d}c^k\,\mathrm{d}c^l + \mathrm{d}e^k\,\mathrm{d}e^l\big)$$
$$+ \tilde{B}_{kl}\big(\mathrm{d}a^k\,\mathrm{d}b^l - \mathrm{d}b^k\,\mathrm{d}a^l + \mathrm{d}c^k\,\mathrm{d}e^l - \mathrm{d}e^k\,\mathrm{d}c^l\big)$$
$$+ \tilde{C}_{kl}\big(\mathrm{d}a^k\,\mathrm{d}c^l - \mathrm{d}b^k\,\mathrm{d}e^l - \mathrm{d}c^k\,\mathrm{d}a^l + \mathrm{d}e^k\,\mathrm{d}b^l\big)$$
$$+ \tilde{D}_{kl}\big(\mathrm{d}a^k\,\mathrm{d}e^l + \mathrm{d}b^k\,\mathrm{d}c^l - \mathrm{d}c^k\,\mathrm{d}b^l - \mathrm{d}e^k\,\mathrm{d}a^l\big)\big\},$$

where

$$\tilde{A}_{kl} = \begin{cases} 1/4 & \text{if } k = l = 1, \\ -a^k/2 & \text{if } k > 1,\ l = 1, \\ -a^l/2 & \text{if } k = 1,\ l > 1, \\ a^1 - \sum_{j>1, j\neq k}|u^j|^2 & \text{if } k = l > 1, \\ a^k a^l + b^k b^l + c^k c^l + e^k e^l & \text{if } k, l > 1, k \neq l, \end{cases}$$

$$\tilde{B}_{kl} = \begin{cases} b^k/2 & \text{if } k > 1,\ l = 1, \\ -b^l/2 & \text{if } k = 1,\ l > 1, \\ a^k b^l - b^k a^l + c^k e^l - e^k c^l & \text{otherwise}, \end{cases}$$

$$\tilde{C}_{kl} = \begin{cases} c^k/2 & \text{if } k > 1,\ l = 1, \\ -c^l/2 & \text{if } k = 1,\ l > 1, \\ a^k c^l - c^k a^l + e^k b^l - b^k e^l & \text{otherwise}, \end{cases}$$

$$\tilde{D}_{kl} = \begin{cases} e^k/2 & \text{if } k > 1,\ l = 1, \\ -e^l/2 & \text{if } k = 1,\ l > 1, \\ a^k e^l - e^k a^l + b^k c^l - c^k b^l & \text{otherwise}. \end{cases}$$

- A basis of the corresponding almost quaternionic structure υ^3_+:

$$(J_1)_+ = \frac{1}{|u^1+1|^2}\big\{\big((a^1+1)^2 + (b^1)^2 - (c^1)^2 - (e^1)^2\big)F$$
$$+ 2\big(b^1c^1 - (a^1+1)e^1\big)G + 2\big(b^1e^1 + (a^1+1)c^1\big)H\big\},$$

$$(J_2)_+ = \frac{1}{|u^1+1|^2}\big\{\big((a^1+1)^2 - (b^1)^2 + (c^1)^2 - (e^1)^2\big)G$$
$$+ 2\big(b^1c^1 + (a^1+1)e^1\big)F + 2\big(c^1e^1 - (a^1+1)b^1\big)H\big\},$$

$$(J_3)_+ = \frac{-1}{|u^1+1|^2}\big\{\big((a^1+1)^2 - (b^1)^2 - (c^1)^2 + (e^1)^2\big)H$$
$$+ 2\big(c^1e^1 + (a^1+1)b^1\big)G + 2\big(b^1e^1 - (a^1+1)c^1\big)F\big\},$$

where

$$F=\sum_{k=1}^{n}\left(\frac{\partial}{\partial b^k}\otimes \mathrm{d}a^k-\frac{\partial}{\partial a^k}\otimes \mathrm{d}b^k-\frac{\partial}{\partial e^k}\otimes \mathrm{d}c^k+\frac{\partial}{\partial c^k}\otimes \mathrm{d}e^k\right),$$

$$G=\sum_{k=1}^{n}\left(\frac{\partial}{\partial c^k}\otimes \mathrm{d}a^k+\frac{\partial}{\partial e^k}\otimes \mathrm{d}b^k-\frac{\partial}{\partial a^k}\otimes \mathrm{d}c^k-\frac{\partial}{\partial b^k}\otimes \mathrm{d}e^k\right),$$

$$H=\sum_{k=1}^{n}\left(\frac{\partial}{\partial e^k}\otimes \mathrm{d}a^k-\frac{\partial}{\partial c^k}\otimes \mathrm{d}b^k+\frac{\partial}{\partial b^k}\otimes \mathrm{d}c^k-\frac{\partial}{\partial a^k}\otimes \mathrm{d}e^k\right).$$

- *Cayley planes*
- Held–Stavrov–van Koten canonical metric g on:
  Cayley projective plane $\mathbb{O}\mathrm{P}^2$:

$$g=\frac{c}{4}\frac{|\mathrm{d}u|^2(1+|v|^2)+|\mathrm{d}v|^2(1+|u|^2)-2\mathrm{Re}((u\bar{v})(\mathrm{d}v\,\mathrm{d}\bar{u}))}{(1+|u|^2+|v|^2)^2},\quad c<0.$$

  The open unit ball model (B^2, g, v^9) on the Cayley hyperbolic plane $\mathbb{O}\mathrm{H}(2)$:

$$B^2=\left\{(u,v)\in\mathbb{O}^2 : |u|^2+|v|^2<1\right\},$$

$$g=\frac{c}{4}\frac{|\mathrm{d}u|^2(1-|v|^2)+|\mathrm{d}v|^2(1-|u|^2)+2\mathrm{Re}((u\bar{v})(\mathrm{d}v\,\mathrm{d}\bar{u}))}{(1-|u|^2-|v|^2)^2},\quad c<0$$

(see Held, Stavrov and van Koten [17]).

Left-Invariant Metrics on Lie Groups

- Koszul formula for the Levi-Civita connection of a left-invariant metric g on a Lie group G with Lie algebra $\mathfrak{g}$ (where $X, Y, Z\in\mathfrak{g}$):

$$2g(\nabla_X Y, Z)=g\big([X,Y],Z\big)-g\big([Y,Z],X\big)+g\big([Z,X],Y\big).$$

- Case of a bi-invariant metric $(g([X,Y],Z)+g(Y,[X,Z])=0$:

$$\nabla_X Y=\frac{1}{2}[X,Y].$$

- Case of a co-biinvariant metric $(g([X,Y],Z)+g([Y,Z],X)+g([Z,X],Y)=0)$:

$$g(\nabla_X Y, Z)=-g\big([Y,Z],X\big).$$

- Levi-Civita connection, curvature tensor field, and sectional curvature (at the identity element e) for a compact Lie group G with a bi-invariant metric g (X, Y, Z, left-invariant vector fields):

$$\nabla_X Y=\frac{1}{2}[X,Y],\qquad R(X,Y)Z=-\frac{1}{4}\big[[X,Y],Z\big],$$

$$K\big(\langle X,Y\rangle\big)=\frac{1}{4}g\big([X,Y],[X,Y]\big),\quad X, Y \text{ here orthonormal.}$$

Basic Differential Operators

- The gradient:

$$\nabla f = (\mathrm{d}f)^\sharp = g^{-1}(\mathrm{d}f) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \qquad f \in C^\infty(M, g).$$

- Divergence of $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(\operatorname{div} X)(p) = \sum_{i=1}^{n} \theta^i (\nabla_{e_i} Z)$$

($\{e_i\}, \{\theta^i\}$ = dual bases for T_pM and T_p^*M, $p \in M$; $i = 1, \dots, n = \dim M$).
- Divergence of $X = \sum_i X^i \partial/\partial x^i$ with respect to (the Levi-Civita connection of) a metric tensor g:

$$\sum_i \frac{1}{\sqrt{\det(g_{jk})}} \frac{\partial \sqrt{\det(g_{jk}) X^i}}{\partial x^i}.$$

- Divergence of a $(0, r)$ tensor α on (M, g):

$$(\operatorname{div} \alpha)_p(v_1, \dots, v_r) = \sum_i (\nabla_{e_i} \alpha)(e_i, v_1, \dots, v_{r-1})$$

(∇ = Levi-Civita connection; $v_i \in T_pM$; $\{e_i\}$ = orthonormal basis for T_pM, $p \in M$).
- The Hessian:

$$H^f(X, Y) = XYf - (\nabla_X Y)f.$$

- Trace of the second covariant derivative:

$$\operatorname{tr} \nabla^2 X = \sum_i (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}) X.$$

- The Laplacian and the Laplacian on functions:

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta f = -\sum_{i,j} g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right).$$

- Weitzenböck's formula for the Laplacian Δ on (M, g) ($n = \dim M$; $\{e_i\}$ = orthonormal basis of T_pM):

$$g(\Delta\alpha, \alpha) = \frac{1}{2} \Delta |\alpha|^2 + |\nabla \alpha|^2 + g(\rho_\alpha, \alpha), \quad \alpha \in \Lambda^r M,$$

where $\rho_\alpha(v_1, \dots, v_r) = \sum_{i=1}^{n} \sum_{j=1}^{r} \big(R(e_i, v_j)\alpha\big)(v_1, \dots, v_{j-1}, e_i, v_{j+1}, \dots, v_r)$.

Case of (M,g) of constant sectional curvature c:

$$g(\Delta\alpha,\alpha)=\frac{1}{2}\Delta|\alpha|^2+|\nabla\alpha|^2+r(n-r)c|\alpha|^2.$$

- For $f,h\in C^\infty M$, $X,Y\in\mathfrak{X}(M)$:

  (i) $\operatorname{grad}(fh)=f\operatorname{grad}h+h\operatorname{grad}f$,
  (ii) $\operatorname{div}(fX)=Xf+f\operatorname{div}X$,
  (iii) $H^{fh}=fH^h+hH^f+\mathrm{d}f\otimes\mathrm{d}h+\mathrm{d}h\otimes\mathrm{d}f$,
  (iv) $\Delta(fh)=f\Delta h+h\Delta f+2g(\operatorname{grad}f,\operatorname{grad}h)$,
  (v) $\operatorname{curl}(\operatorname{grad}f)=0$,
  (vi) $\operatorname{curl}X=\mathrm{d}\alpha$, where α is the 1-form metrically dual to X.

- Hodge star operator $\star$: $\Lambda^rM\to\Lambda^{n-r}M$, $0\leqslant r\leqslant n$, on an oriented pseudo-Riemannian n-manifold ($v=$ volume form):

$$\alpha\wedge\star\beta=g^{-1}(\alpha,\beta)v;$$
$$\alpha_p\wedge(\star\beta_p)=\beta_p\wedge(\star\alpha_p),\quad \alpha,\beta\in\Lambda^r_pM,\ p\in M;$$
$$\star^2=(-1)^{r(n-r)},\qquad \star^{-1}=(-1)^{r(n-r)}\star.$$

Conformal Changes of Riemannian Metrics

- $\tilde{g}=\mathrm{e}^{2f}g$, $f\in C^\infty(M,g)$, $\dim M=n$, $|\mathrm{d}f|^2=g^{-1}(\mathrm{d}f,\mathrm{d}f)$:

Levi-Civita connection:

$$\widetilde{\nabla}_XY=\nabla_XY+\big(\mathrm{d}f(X)\big)Y+\big(\mathrm{d}f(Y)\big)X-g(X,Y)\operatorname{grad}f.$$

Riemann–Christoffel curvature tensor ($\owedge$ = Kulkarni–Nomizu product):

$$\widetilde{R}=\mathrm{e}^{2f}\left(R-g\owedge\left(H^f-\mathrm{d}f\otimes\mathrm{d}f+\frac{1}{2}\big|\mathrm{d}f^2\big|^2g\right)\right).$$

Ricci tensor:

$$\tilde{r}=r-(n-2)\big(H^f-\mathrm{d}f\otimes\mathrm{d}f\big)+\big(\Delta f-(n-2)\big|\mathrm{d}f^2\big|^2\big)g.$$

Scalar curvature:

$$\tilde{s}=\mathrm{e}^{-2f}\big(s+2(n-1)\Delta f-(n-2)(n-1)\big|\mathrm{d}f^2\big|\big).$$

$(1,3)$ Weyl tensor W:

$$\widetilde{W}=W.$$

Volume element:

$$v_{\tilde{g}}=\mathrm{e}^{nf}v_g.$$

Codifferential on r-forms:

$$\tilde{\delta}\alpha = \mathrm{e}^{-2f}\big(\delta\alpha - (n-2)i_{\operatorname{grad} f}\alpha\big).$$

Hodge operator on r-forms (for oriented M):

$$\star_{\tilde{g}} = \mathrm{e}^{(n-2r)f}\star_g.$$

For the relevant theory and a wealth of formulas, see Besse [5].

Some Geometric Vector Fields

- Affine $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(L_X\nabla)(Y,Z) = 0, \quad Y, Z \in \mathfrak{X}(M);$$
$$(\nabla_Y \nabla X)Z = R(Y,X)Z \quad \text{(if } \nabla \text{ is torsionless)}$$

  ($(L_X\nabla)(Y,Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z]$; $(\nabla_Y\nabla X)Z = \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z}X$).
- Projective $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(L_X\nabla)(Y,Z) = \theta(Y)Z + \theta(Z)Y, \quad Y, Z \in \mathfrak{X}(M),\ \theta \in \Lambda^1 M.$$

- Jacobi equation along a geodesic γ:

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_{\gamma'}\big(T\big(Y,\gamma'\big)\big) + R\big(Y,\gamma'\big)\gamma' = 0.$$

Volumes of Spheres, Balls, and Projective Spaces

- Volumes of: sphere with the round metric, closed unit ball $B^{n+1} = \bar{B}(0,1) \in \mathbb{R}^{n+1}$, and projective spaces $\mathbb{K}\mathrm{P}^n$ with the canonical metric ($\operatorname{diam}(\mathbb{K}\mathrm{P}^n) = \pi/2$)):

$$\operatorname{vol}\big(S^{2n+1}\big) = \frac{2\pi^{n+1}}{n!}, \qquad \operatorname{vol}\big(S^{2n}\big) = \frac{(n-1)!(4\pi)^n}{(2n-1)!},$$
$$\operatorname{vol}\big(B^{n+1}\big) = \frac{1}{n+1}\operatorname{vol}\big(S^n\big), \qquad \operatorname{vol}\big(\mathbb{R}\mathrm{P}^{2n+1}\big) = \frac{\pi^{n+1}}{n!},$$
$$\operatorname{vol}\big(\mathbb{R}\mathrm{P}^{2n}\big) = \frac{(2\pi)^n}{(2n-1)(2n-3)\cdot\dots\cdot 3\cdot 1}, \qquad \operatorname{vol}\big(\mathbb{C}\mathrm{P}^n\big) = \frac{\pi^n}{n!},$$
$$\operatorname{vol}\big(\mathbb{H}\mathrm{P}^n\big) = \frac{\pi^{2n}}{(2n+1)!}, \qquad \operatorname{vol}\big(\mathbb{O}\mathrm{P}^1\big) = \frac{\pi^4}{8\cdot 7\cdot 5\cdot 3}, \qquad \operatorname{vol}\big(\mathbb{O}\mathrm{P}^2\big) = \frac{6\pi^8}{11!}.$$

Riemannian Submanifolds $M \hookrightarrow \widetilde{M}$ an immersion; $X, Y, Z, W \in \mathfrak{X}(M)$, $\xi \in \mathfrak{X}(M)^\perp$, ∇, $\widetilde{\nabla}$ = Levi-Civita connections.

- Gauss formula:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y),$$
$$\text{where } \nabla_X Y = \tau \widetilde{\nabla}_X Y,\ \alpha(X, Y) = \nu \widetilde{\nabla}_X Y$$

($\tau =$ tangential part, $\nu =$ normal part; $\alpha =$ second fundamental form of M for the given immersion).

- Weingarten formula:

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where

$$\begin{aligned}
&-A_\xi X = \tau \widetilde{\nabla}_X \xi, \qquad \nabla^\perp_X \xi = \nu \widetilde{\nabla}_X \xi, \qquad g(A_\xi X, Y) = g\big(\alpha(X, Y), \xi\big),\\
&\nabla^\perp_X : \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \to \mathfrak{X}(M)^\perp \quad \text{(the normal connection)},\\
&\nabla^\perp_X W = \nu \widetilde{\nabla}_X W, \quad X \in \mathfrak{X}(M), W \in \mathfrak{X}(M)^\perp.
\end{aligned}$$

- Gauss equation:

$$\begin{aligned}
\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ g\big(\alpha(Z, Y), \alpha(W, X)\big)\\
&- g\big(\alpha(Z, X), \alpha(W, Y)\big).
\end{aligned}$$

- (M, g) m-dimensional Riemannian submanifold of n-dimensional space $(N, \tilde{g})$ of constant curvature K. Gauss equation and length of mean curvature vector $|H|^2$ at $p \in M$ ([30, p. 123]) :

$$\begin{aligned}
R(X, Y, Z, W) = K\big(g(Z, Y)g(W, X) &- g(Z, X)g(W, Y)\big)\\
&+ \tilde{g}\big(\alpha(Z, Y), \alpha(W, X)\big) - \tilde{g}\big(\alpha(Z, X), \alpha(W, Y)\big),\\
&X, Y, Z, W \in \mathfrak{X}(M),\\
|H|^2 = \frac{1}{m^2}\big(\mathbf{s} - Km(m-1) + \ell^2\big)
\end{aligned} \tag{7.9}$$

($\mathbf{s}$ scalar curvature of M; ℓ^2 square of the length of the second fundamental form).

- Codazzi equation:

$$\nu \widetilde{R}_{XY} Z = (\widehat{\nabla}_X \alpha)(Y, Z) - (\widehat{\nabla}_Y \alpha)(X, Z),$$

where

$$(\widehat{\nabla}_X \alpha)(Y, Z) = \nabla^\perp_X\big(\alpha(Y, Z)\big) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

- Ricci equation:

$$\nu \widetilde{R}_{XY} \xi = R^\perp_{XY} \xi - \alpha(A_\xi X, Y) - \alpha(X, A_\xi Y).$$

- Mean curvature normal:

$$\eta = \frac{1}{n}\sum_{i=1}^{r}(\operatorname{tr} A_{\xi_i})\xi_i$$

($M = n$-dimensional Riemannian manifold isometrically immersed in an $(n+r)$-dimensional Riemannian manifold N; $\{\xi_1, \dots, \xi_r\}$ = orthonormal basis in $(T_pM)^{\perp}$).

- Riemann–Christoffel curvature tensor on a complex submanifold M of a Kähler manifold $(\widetilde{M}, g, J)$ (α = second fundamental form; $\widetilde{R}$ = Riemann–Christoffel curvature tensor of $\widetilde{M}$, $X \in \mathfrak{X}(M)$):

$$R(X, JX, X, JX) = \widetilde{R}(X, JX, X, JX) - 2g\big(\alpha(X, X), \alpha(X, X)\big).$$

Hypersurfaces in $\mathbb{R}^{n+1}$ M hypersurface in $\mathbb{R}^{n+1}$; $X, Y, Z \in \mathfrak{X}(M)$; ξ field of unit normal vectors defined locally, or globally if this is the case; ∇' = covariant differentiation in $\mathbb{R}^{n+1}$; $A = A_\xi$ = symmetric transformation of each T_pM corresponding to the symmetric bilinear function h on $T_pM \times T_pM$ defined by $\alpha(X, Y) = h(X, Y)\xi$.

- Gauss formula for hypersurfaces:

$$\nabla'_X Y = \nabla_X Y + h(X, Y)\xi.$$

- Weingarten formula for hypersurfaces:

$$\nabla'_X \xi = -AX.$$

- Gauss equation for hypersurfaces:

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY.$$

- Codazzi equation for hypersurfaces:

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

Surfaces in $\mathbb{R}^3$

- Gauss–Bonnet formula for a compact surface M:

$$\chi(M) = \frac{1}{4\pi}\int_M \mathbf{s}\,\omega_g = \frac{1}{2\pi}\int_M K\omega_g$$

($\chi(M)$ = Euler characteristic; $\mathbf{s}$ = scalar curvature; K = Gauss curvature).

- parametrisation of $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$:

$$x = \sin\theta\cos\varphi, \qquad y = \sin\theta\sin\varphi, \qquad z = \cos\theta, \quad 0 < \theta < \pi,\ 0 < \varphi < 2\pi.$$

(See Remark 1.4.)

- A parametrisation of the torus T^2 (with $R > r$, $\theta, \varphi \in (0, 2\pi)$):

$$x = (R + r\cos\theta)\cos\varphi, \qquad y = (R + r\cos\theta)\sin\varphi, \qquad z = r\sin\theta.$$

(See Remark 1.4.)

- Gauss curvature K of an abstract parametrised surface with metric

$$\begin{aligned} g &= E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 = E\,\mathrm{d}u \otimes \mathrm{d}u + F\,\mathrm{d}u \otimes \mathrm{d}v + F\,\mathrm{d}v \otimes \mathrm{d}u \\ &\quad + G\,\mathrm{d}v \otimes \mathrm{d}v, \end{aligned}$$

$$\begin{aligned} K &= -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & F & G \\ E_u & F_u & G_u \\ E_v & F_v & G_v \end{vmatrix} \\ &\quad - \frac{1}{2\sqrt{EG-F^2}}\left\{\left(\frac{G_u - F_v}{\sqrt{EG-F^2}}\right)_u - \left(\frac{F_u - E_v}{\sqrt{EG-F^2}}\right)_v\right\} \end{aligned}$$

(here a subindex u, v, denotes the derivative with respect to that variable).

Pseudo-Riemannian Manifolds

- Normal coordinates for a pseudo-Riemannian n-manifold (M, g). On a neighbourhood of the origin $p \in M$:

  (i) $g_{ij}(p) = \delta_{ij}\varepsilon_j$, $\varepsilon_j = \pm 1$.
  (ii) Geodesics through p: $y^i = a^i t$, $i = 1, \dots, n$, $a^i = \text{const.}$
  (iii) Christoffel symbols: $\Gamma^i_{jk}(p) = 0$.

- Cartan's structure equations (for a pseudo-Riemannian metric).
  $\sigma = (X_i)$ an orthonormal moving frame; $\varepsilon_i = g(X_i, X_i) = \pm 1$; $(\widetilde{\theta}^i)$ dual moving coframe; $\widetilde{\theta}^i = \sigma^*\theta^i$, with $\theta = (\theta^i)$ the canonical form on the bundle of orthonormal frames; $\widetilde{\omega}^i_j = \sigma^*\omega^i_j$, with ω^i_j the connection forms; $\widetilde{\theta}_i = \varepsilon_i\widetilde{\theta}^i$, $\widetilde{\omega}_{ij} = \varepsilon_i\widetilde{\omega}^i_j$; $\widetilde{\Omega}_{ij} = \varepsilon_i\widetilde{\Omega}^i_j$:

$$\mathrm{d}\widetilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge (\varepsilon_j\widetilde{\theta}_j), \qquad \widetilde{\omega}_{ij} + \widetilde{\omega}_{ji} = 0,$$

$$\mathrm{d}\widetilde{\omega}_{ij} = -\sum_k \varepsilon_k\widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik} + \widetilde{\Omega}_{ij}$$

(in the expression $\varepsilon_j\widetilde{\theta}_j$, no sum in j; in the expression $\varepsilon_k\widetilde{\omega}_{jk}$, no sum in k).
  For the relevant theory and a wealth of related formulas, many of them reproduced here, see Kobayashi and Nomizu [19] and Wolf [31].

- Metric of constant curvature c. There exist coordinate functions x^i on a neighbourhood of $p \in M$ such that

$$g = \frac{\varepsilon_i\,\mathrm{d}x^i \otimes \mathrm{d}x^i}{(1 - \frac{c}{4}\varepsilon_i(x^i)^2)^2}, \quad \varepsilon_i = \pm 1.$$

- Pseudo-Riemannian metric of constant curvature c in normal coordinates x^i with origin q, at $p \neq q$:

$$g = \sum_{i,j} \left(\frac{x^i x^j}{er^2} + \frac{\sin^2(r\sqrt{ec})}{ecr^2} \left((g_{ij})_q - \frac{x^i x^j}{er^2} \right) \right) \mathrm{d}x^i \, \mathrm{d}x^j$$

(signature of $g = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i = \pm 1$; $\sum_j (g_{ij})_q = \varepsilon_i \delta_{ij}$; $x_i = \sum_j (g_{ij})_p x^j$; $e = \sum_i x_i x^i / |x_i x^i|$ if $\sum_i x_i x^i \neq 0$ and $e = 0$ if $\sum_i x_i x^i = 0$; $r = \sqrt{e \sum_i x_i x^i}$).

For the relevant theory, see Cartan [9] and Ruse, Walker and Willmore [25].

- First variation formula for a piecewise C^∞ curve segment $\sigma \colon [a,b] \to M$ with constant speed $c > 0$ and sign ε:

$$L'(0) = \frac{\varepsilon}{c} \left\{ -\int_a^b g\big(\sigma'', V\big) \, \mathrm{d}u - \sum_{i=1}^k g\big(\Delta\sigma'(u_i), V(u_i)\big) + \big[g\big(\sigma', V\big)\big]_a^b \right\}$$

($V = V(u) =$ variation vector field; $u_1 < \cdots < u_k$ breaks of σ and its variation; $\Delta\sigma'(u_i) = \sigma'(u_i^+) - \sigma'(u_i^-)$).

- Synge's formula for the second variation of the arc of a geodesic segment $\sigma \colon [a,b] \to M$ of speed $c > 0$ and sign ε:

$$L''(0) = \frac{\varepsilon}{c} \left\{ \int_a^b \left\{ g\big(V'^\perp, V'^\perp\big) + g\big(R\big(V, \sigma'\big)V, \sigma'\big) \right\} \mathrm{d}u + \big[g\big(\sigma', A\big)\big]_a^b \right\}$$

($V'^\perp =$ component of V' perpendicular to γ; $A =$ transverse acceleration vector field).

For the relevant theory of these and the next formulas, see O'Neill [23].

- Einstein field equations:

$$\mathbf{r} - \frac{1}{2} \mathbf{s} g = T$$

($g =$ metric tensor; $\mathbf{r} =$ Ricci tensor; $\mathbf{s} =$ scalar curvature; $T =$ stress-energy tensor).

- Schwarzschild metric:

$$g = -\left(1 - \frac{2m}{R}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{R}\right)^{-1} \mathrm{d}R^2 + R^2\big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\big),$$
$$\theta \in (0, \pi), \varphi \in (0, 2\pi).$$

- Kerr metric for a fast rotating planet (cylindrically symmetric gravitational field; $a =$ angular momentum):

$$g = -\mathrm{d}t^2 + \big(r^2 + a^2\big) \sin^2\theta \, \mathrm{d}\varphi^2 + \frac{2Mr(\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\varphi)^2}{r^2 + a^2 \cos^2\theta}$$
$$+ \big(r^2 + a^2 \cos^2\theta\big) \left(\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2 - 2Mr + a^2} \right).$$

- de Sitter metric on S^4:

$$g = \frac{1}{(1+(\frac{r}{2R})^2)^2}\big(\mathrm{d}r^2 + r^2(\sigma_x^2+\sigma_y^2+\sigma_z^2)\big),$$

where $R = $ radius of S^4;

$$\sigma_x = \frac{1}{r^2}(y\,\mathrm{d}z - z\,\mathrm{d}y + x\,\mathrm{d}t - t\,\mathrm{d}x), \qquad \sigma_y = \frac{1}{r^2}(z\,\mathrm{d}x - x\,\mathrm{d}z + y\,\mathrm{d}t - t\,\mathrm{d}y),$$

$$\sigma_z = \frac{1}{r^2}(x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z).$$

- Robertson–Walker metric:

$$g = -\mathrm{d}t^2 + f^2(t)\frac{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2}{(1+\frac{k}{4}(x^2+y^2+z^2))^2}$$

(three-dimensional space is fully isotropic; $f(t)$ = (increasing) distance between two neighbouring galaxies in space; $k = -1, 0, +1$).

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Some Notations

P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*, Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7,

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P.M. Gadea et al., *Analysis and Algebra on Differentiable Manifolds*,
Problem Books in Mathematics, DOI 10.1007/978-94-007-5952-7,

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